Discrete Mathematics 310 (2010) 3120–3125

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Regular parallelisms in kinematic spaces

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ARTICLE INFO

Article history:
Received 29 September 2008
Received in revised form 28 April 2009
Accepted 10 June 2009
Available online 2 July 2009

Dedicated to Helmut Karzel on the occasion of his 80th birthday

Keywords:
Linear spaces
Regular parallelism
Kinematic spaces

ABSTRACT

Here we propose a definition of *regular parallelism* in a linear space not necessarily embedded onto a projective space and we investigate its properties in the particular case of kinematic spaces. We prove that the kinematic parallelisms are always regular in that sense and we deduce some results on the group of translations acting transitively on the pointset.

1. Introduction and motivations

In an earlier paper [3] we extended the classical definition of *Clifford parallelisms* in the real 3-dimensional projective space PG(3, $\mathbb{R}$) to the case of a general commutative field $\mathbb{F}$ of characteristic different from 2. One of our main tools in this investigation is the hyperbolic quadrics of the projective space PG(3, $\mathbb{K}$), where $\mathbb{K}$ is a suitable quadratic extension of $\mathbb{F}$ and, among other results, we can prove that the Clifford parallelisms we build are *regular*, in the sense that their spreads contain with any three pairwise skew lines also all the lines of the (unique) regulus defined by those lines.

Since it is well known that PG(3, $\mathbb{R}$) endowed with the two Clifford parallelisms is an example of a *kinematic space*, our aim here is to generalize the notions of regulus and regular parallelism of a 3-dimensional projective space to the more general situation of kinematic spaces (or even of linear spaces) not necessarily embedded into a projective space. A first step in this direction has been made by considering those kinematic spaces which are obtained from projective spaces by removing a prescribed set of points, namely the *porous kinematic spaces* (see e.g. [10,8]). In particular Herzer in [6] introduces two axioms which define a "regular" parallelism in a projective space deprived of some points, and then shows that the particular class of kinematic spaces he is interested in (those arising from *kinematic algebras*) fulfills those axioms. In this way he can answer (implicitly) for that class of kinematic spaces the long time outstanding question whether it is possible that the group of translations acting transitively on the pointset is not a normal subgroup of the group of all collineations of the geometric structure: this cannot happen (see Theorem 10).

Here we are interested in extending both the definition of "classical" regular parallelism and of Herzer's regular parallelism dropping the assumption that the kinematic space is projective, in order to obtain some results similar to Herzer's. Since our framework is heavily poorer, our results are weaker than those known for projective kinematic spaces, in particular in order to prove the normality of the group of translations we need to assume further hypotheses on the kinematic space we are considering, which restricts the range of kinematic spaces for which our results are feasible.

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doi:10.1016/j.disc.2009.06.011

2. Setting and known results

In this section to fix the notation we recall the main results known for regular parallelisms and kinematic spaces for future reference.

2.1. Regular parallelisms in projective spaces

Let $(\mathcal{P}, \mathcal{L}) = \mathrm{PG}(3, \mathbb{F})$ denote the 3-dimensional projective space over a commutative field $\mathbb{F}$ that we always assume of characteristic different from 2. Recall that for a subset $\mathcal{M} \subseteq \mathcal{L}$ a line $L \in \mathcal{L}$ is said to be *transversal to* $\mathcal{M}$ if L meets each line $M \in \mathcal{M}$ in exactly one point.

Definition 1 (*[14, Ch. 18]*)**.** A *regulus* of $\mathrm{PG}(3, \mathbb{F})$ is the set of all lines of $\mathcal{L}$ transversal to three pairwise skew lines $T_1, T_2, T_3 \in \mathcal{L}$.

A *spread* (or *parallel class*) $\mathrm{S} \subseteq \mathcal{L}$ is a partition of the pointset $\mathcal{P}$ into lines, thus it has the property that each point $p \in \mathcal{P}$ lies on exactly one line $L_p \in \mathrm{S}$. A *parallelism* of $\mathrm{PG}(3, \mathbb{F})$ is a set of mutually disjoint spreads which covers the whole lineset $\mathcal{L}$. Once a parallelism is given, we say that two lines $L, M \in \mathcal{L}$ are *parallel* ($L \parallel M$) if they belong to the same spread.

Definition 2. A spread in a Pappian 3-space is said to be *regular* if whenever it contains three pairwise skew lines L_1, L_2, L_3, then it contains also all the lines of the unique regulus through L_1, L_2, L_3. A *regular parallelism* is a parallelism consisting of regular spreads only.

Regular spreads and regular parallelisms arise in many different contexts and have been the target of many investigations (see e.g. [5,4]). In particular the classical Clifford parallelisms in the real projective space $\mathrm{PG}(3, \mathbb{R})$ and their generalization introduced in [3] turn out to be regular parallelisms (see [3, Cor. 4.8]).

2.2. Kinematic spaces and kinematic algebras

For the results here collected we refer e.g. to [1,9,7,12].

Definition 3. Let $\mathcal{P}$ be a nonempty set whose elements we call *points*, $\mathcal{L}$ a family of subsets of $\mathcal{P}$ we call *lines* and "$\parallel$" a binary relation on $\mathcal{L}$ called *parallelism*. The triple $(\mathcal{P}, \mathcal{L}, \parallel)$ is an *affine parallel structure (ap-structure)* if it fulfills the following properties:

(AP1) for any pair of distinct points p and q there exists exactly one line $\overline{p, q} \in \mathcal{L}$ such that $p, q \in \overline{p, q}$ (hence $(\mathcal{P}, \mathcal{L})$ is a *linear space*);
(AP2) "$\parallel$" is an equivalence relation on $\mathcal{L}$ such that, for any line $R \in \mathcal{L}$ and for any point $p \in \mathcal{P}$ there exists exactly one line $S \in \mathcal{L}$ such that $p \in S$ and $S \parallel R$ (*euclidean parallel axiom*); we will denote the line S by $\mathcal{L}(p \parallel R)$;
(AP3) in $\mathcal{P}$ there exist at least three non collinear points and every line contains at least two points.

Definition 4. Two ap-structures $(\mathcal{P}, \mathcal{L}, \parallel)$ and $(\mathcal{P}', \mathcal{L}', \parallel')$ are *isomorphic* if there exists a bijection $\phi : \mathcal{P} \longrightarrow \mathcal{P}'$ which maps lines onto lines and preserves the parallelism. If $(\mathcal{P}, \mathcal{L}, \parallel)$ and $(\mathcal{P}', \mathcal{L}', \parallel')$ coincide, the isomorphism ϕ is called an *automorphism* or a *collineation*. A *dilatation* of $(\mathcal{P}, \mathcal{L}, \parallel)$ is a collineation ϕ such that, for any $R \in \mathcal{L}$, $R \parallel \phi(R)$; a *translation* is the identity or a dilatation without fixed points.

Definition 5. An ap-structure $(\mathcal{P}, \mathcal{L}, \parallel)$ is an *affine parallel translation structure* (*apt-structure* or *André structure*) if there exists a group T of translations of $(\mathcal{P}, \mathcal{L}, \parallel)$ which acts transitively (and hence regularly) on $\mathcal{P}$. A *kinematic space* is a quadruple $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ where $(\mathcal{P}, \mathcal{L}, \parallel_\ell)$ and $(\mathcal{P}, \mathcal{L}, \parallel_r)$ are apt-structures and moreover the following axioms hold for all $a, b, c, a_1, a_2, a_3, b_1, b_2, b_3 \in \mathcal{P}$ distinct:

(K1) $\mathcal{L}(b \parallel_\ell \overline{a, c}) \cap \mathcal{L}(c \parallel_r \overline{a, b}) \neq \emptyset$;
(K2) If $\overline{a_1, b_1} \parallel_\ell \overline{a_2, b_2} \parallel_\ell \overline{a_3, b_3}$, $\overline{b_1, b_2} \parallel_r \overline{a_1, a_2}$ and $\overline{b_1, b_3} \parallel_r \overline{a_1, a_3}$, then also $\overline{b_2, b_3} \parallel_r \overline{a_2, a_3}$;
(K3) If $\overline{a_1, b_1} \parallel_r \overline{a_2, b_2} \parallel_r \overline{a_3, b_3}$, $\overline{b_1, b_2} \parallel_\ell \overline{a_1, a_2}$ and $\overline{b_1, b_3} \parallel_\ell \overline{a_1, a_3}$, then also $\overline{b_2, b_3} \parallel_\ell \overline{a_2, a_3}$.

Note that, by [9], the quadruple $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ is already a kinematic space if $(\mathcal{P}, \mathcal{L}, \parallel_\ell)$ and $(\mathcal{P}, \mathcal{L}, \parallel_r)$ are ap-structures satisfying (K1), (K2) and (K3).

Recall that a *fibration* (or *partition*) of a group $(G, \cdot, 1_G)$ is a family $\mathcal{F}$ of non-trivial subgroups of G such that for all $X, Y \in \mathcal{F}$, if $X \neq Y$ then $X \cap Y = \{1_G\}$ and for all $g \in G$ there exists $X \in \mathcal{F}$ such that $g \in X$; a fibration $\mathcal{F}$ of the group G is said to be *normal* if, for every $F \in \mathcal{F}$ and for every $g \in G$, $F^g := gFg^{-1} \in \mathcal{F}$. It is well known that apt-structures and kinematic spaces are related to groups with a fibration.

Theorem 6 (*André [1], Karzel–Kroll–Sörensen [9]*)**.** *Let G be a group and $\mathcal{F}$ a normal fibration of G. The quadruple $(G, \mathcal{L}(\mathcal{F}), \parallel_\ell, \parallel_r)$ made up of:*

(1) *the set of elements of* G*;*
(2) *the set* $\mathcal{L}(\mathcal{F})$ *of the (left) cosets of the elements of* $\mathcal{F}$*;*
(3) *the binary relations "* $\parallel_\ell$ *" and "* $\parallel_r$ *" defined as follows:*

$$\forall\, a, b \in G,\ \forall\, F_1, F_2 \in \mathcal{F}: \quad \begin{array}{l} aF_1 \parallel_\ell bF_2 \iff F_1 = F_2 \\ aF_1 \parallel_r bF_2 \iff F_1^a = F_2^b \end{array}$$

is a kinematic space, denoted by $[G, \mathcal{F}]$*, with transitive translation group isomorphic to* G*.*
Conversely if $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ *is a kinematic space,* $o \in \mathcal{P}$ *and* T *the group of translations with respect to* $\parallel_\ell$ *regular on the point-set* $\mathcal{P}$*, then the set* $\mathcal{P}$ *can be endowed with an operation which makes it into a group isomorphic to* T *and such that the set* $\mathcal{L}_o := \{L \in \mathcal{L} \mid o \in L\}$ *is a normal fibration of* $\mathcal{P}$ *and, moreover,* $[\mathcal{P}, \mathcal{L}_o] \cong (\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$*.*

Note that, differently from the case of affine translation spaces (which is a special case of apt-structures), in a general apt-structure the regular group T of translations need not be a normal subgroup of the group of all collineations (see [2] for an example). To the best of our knowledge, however, it is not known whether it is possible to provide an example of a kinematic space such that T is not normal. In [12] the following result is proved.

Theorem 7 *([12, Thm 5]). Let* $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ *be a kinematic space. If* $\parallel_r$ *is the only parallelism of* $(\mathcal{P}, \mathcal{L})$ *which fulfills along with* $\parallel_\ell$ *the axioms* (K1), (K2) *and* (K3)*, then the group* T *of left translations acting regularly on* $\mathcal{P}$ *is normal in the group of all collineations.*

In the following we will refer to a situation like the one described in the statement of Theorem 7 by saying that $\parallel_r$ is uniquely determined by $\parallel_\ell$.

In the paper [6] of 1980 A. Herzer succeeded in proving that $\parallel_r$ is uniquely determined by $\parallel_\ell$ in the particular situation that the kinematic space is built starting from a *kinematic algebra*.

Definition 8. A *kinematic algebra* (or *quadratic algebra*) A over a commutative field $\mathbb{F}$ is a $\mathbb{F}$-algebra such that for all $a \in A$: $a^2 \in \mathbb{F} + \mathbb{F}a$.

Starting from a kinematic algebra and the multiplicative subgroup of its invertible elements in [7] is described how to construct in a canonical way a kinematic space which turns out to be the projective space PG$(3, \mathbb{F})$ deprived of a suitable set $\mathcal{Q}$ of *singular points* in such a way that on every line at most 2 points are removed.

Theorem 9 *([6, (1) and (2)]). If* $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ *is the kinematic space obtained from a kinematic algebra, then the parallelism* $\parallel_\ell$ *fulfills the following properties:*

(H1) *If* H_1, H_2, H_3 *are distinct parallel lines intersecting a line* L*, then* $H_1 \leqslant H_2 + H_3$*, where by* $H_2 + H_3$ *we denote the smallest projective space containing both* H_2 *and* H_3 *deprived of its singular points;*
(H2) *If* $L, L' \in \mathcal{L}$ *meet three distinct parallel lines* H_1, H_2, H_3 *and* $T, T' \in \mathcal{L}$ *are parallel lines such that* T *meets both* L *and* L' *and* T' *meets* L*, then* T' *and* L' *are coplanar. Moreover if* L *and* L' *are skew, then* $T' \cap L' \neq \emptyset$*.*

In the following we will refer to a parallelism fulfilling properties (H1) and (H2) as *H-regular*. One of the main results of the aforementioned paper (even if it is not explicitly stated in this form) is the following.

Theorem 10. *If* $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ *is a kinematic space obtained from a kinematic algebra, then* $\parallel_r$ *is uniquely determined by* $\parallel_\ell$*, thus the group* T *of left translations acting regularly on the pointset is normal in the group of all collineations.*

In Herzer's proof of Theorem 10 the hypothesis that $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ is obtained from a 3-dimensional projective space by removing a prescribed set of points is heavily used both to prove that $\parallel_\ell$ is H-regular and to prove that $\parallel_r$ is the only parallelism consistent with $\parallel_\ell$ with respect to (K1), (K2) and (K3). The definition of H-regular parallelism however, once opportunely restated, can be sensitive also in the more general context of linear spaces, thus our interest in investigating it also in a non-projective setting.

3. Regular parallelisms in linear spaces

In this section we extend the definition of regular parallelism to the case of a linear space not necessarily embedded into a projective space and we compare the definition with that of H-regular parallelism introduced in Theorem 9.

Before starting recall (see e.g. [15, Ch. 4]) that, if $(\mathcal{P}, \mathcal{L})$ is a linear space, a *linear subspace* of $(\mathcal{P}, \mathcal{L})$ is a linear space $(\mathcal{P}', \mathcal{L}')$ such that $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{L}' \subseteq \mathcal{L}$. If $X \subseteq \mathcal{P}$ is a set of points of $(\mathcal{P}, \mathcal{L})$, then the *closure* of X, denoted by $\mathcal{C}(X)$, is the intersection of all the linear subspaces of $(\mathcal{P}, \mathcal{L})$ which contain X; the points of the set X are called *generators* of $\mathcal{C}(X)$. If $X = \{p_1, p_2, \ldots\} \subseteq \mathcal{P}$ we say that the points $p_1, p_2, \ldots$ are *independent* if none of the $p_i \in X$ belongs to the closure of the remaining ones. A linear space $(\mathcal{P}, \mathcal{L})$ is said to be an *exchange space* if it fulfills the following axiom, namely the *exchange axiom*:

(E) for any pair of points $p, q \in \mathcal{P}$ and for any subset X of $\mathcal{P}$ such that $p \notin \mathcal{C}(X)$, if $p \in \mathcal{C}(\{q\} \cup X)$, then $q \in \mathcal{C}(\{p\} \cup X)$.

If $(\mathcal{P}, \mathcal{L})$ is an exchange space, the *dimension* of a linear subspace $(\mathcal{P}', \mathcal{L}')$ of $(\mathcal{P}, \mathcal{L})$, denoted by $\dim(\mathcal{P}', \mathcal{L}')$, is the number of independent generators of $(\mathcal{P}', \mathcal{L}')$ diminished by 1; a *plane* is a linear space of dimension 2.

From now on let $(\mathcal{P}, \mathcal{L})$ be an exchange linear space.

Definition 11. Two lines $L_1, L_2 \in \mathcal{L}$ are said to be *skew* if there exists no plane containing both L_1 and L_2. A *regulus* is the set of all lines intersecting three pairwise skew lines. A parallelism of $(\mathcal{P}, \mathcal{L})$ is said to be *regular* if whenever three distinct skew lines are contained in the same parallel class, then all the reguli containing those lines belong to the parallel class.

Note that, of course, if the linear space $(\mathcal{P}, \mathcal{L})$ is a projective space of dimension at least three, then this definition is consistent with the classical Definition 2. Note also that, in general, we can have different reguli all containing three fixed skew lines. On the other hand three pairwise skew lines could have less than three distinct common transversals: in this case we agree that the regulus defined by those lines is empty.

Remark 12. Definition 11 above is not the most general possible. In fact, in general, in a linear space with parallelism a situation which has no corresponding in the projective or affine setting arises: there can be coplanar distinct lines that are neither parallel nor incident in a point (see e.g. [13], but be careful: there the name "skew" is reserved to this kind of lines). One would like to have a definition of regulus and regular parallelism which somehow involves also lines in this situation, but this seems to be rather difficult to obtain.

From now on we assume that $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ is an exchange kinematic space.

Lemma 13. *Let $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ be a kinematic space, $L, H_1, H_2, H_3 \in \mathcal{L}$ such that $H_i \parallel_\ell H_j$, $L \cap H_i \neq \emptyset$ and H_i are pairwise skew, $i, j = 1, 2, 3$. If $h_1 \in H_1 \setminus L$, then L' is a line through h_1 intersecting both H_2 and H_3 if and only if $L' = \mathcal{L}(h_1 \parallel_r L)$. The situation is analogous with $\parallel_\ell$ and $\parallel_r$ interchanged.*

Proof. One implication is obvious by (K1), thus assume that L' is a line through h_1 intersecting both H_2 and H_3, write $L'' = \mathcal{L}(h_1 \parallel_r L)$ and by contradiction assume $L' \neq L''$. Of course L' and L'' are coplanar since $L' \cap L'' = \{h_1\}$, moreover by (K1), L'' intersects both H_2 and H_3, thus the lines H_2 and H_3 are coplanar as well, a contradiction. □

Theorem 14. *Let $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ be a kinematic space. Then both $\parallel_\ell$ and $\parallel_r$ are regular parallelisms.*

Proof. The proof is analogous for $\parallel_\ell$ and $\parallel_r$ thus we will prove the statement only for $\parallel_\ell$. Let L_1, L_2 and L_3 be pairwise skew left parallel lines and consider any three distinct transversals H_1, H_2 and H_3 to these lines. Then by Lemma 13 above $H_1 \parallel_r H_2 \parallel_r H_3$, thus, since these transversals are obviously pairwise skew, a line L intersecting all of them must be left parallel to L_i. □

We would like now to compare this new definition of regular parallelism with Herzer's definition of H-regularity. In order to make the definition of H-regular parallelism sensitive in this new context, for any pair of linear subspaces H_1, H_2 of $(\mathcal{P}, \mathcal{L})$ let us agree to write $H_1 + H_2$ for the linear subspace $\mathcal{C}(H_1 \cup H_2)$.

Theorem 15. *In a kinematic space $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ property* (H1) *is always fulfilled with respect both to $\parallel_\ell$ and $\parallel_r$. If, moreover, we assume for the assumptions of* (H2) *also one of the following conditions:*

(1) *the lines L, L' are skew*
(2) *it holds $T \parallel_\ell H_i$ ($T \parallel_r H_i$ respectively),*

then also (H2) *holds true for $\parallel_\ell$ ($\parallel_r$ respectively).*

Proof. Let H_1, H_2 and $H_3 \in \mathcal{L}$ be distinct left parallel lines intersecting a further line L and let h_2 be a point on $H_2 \setminus L$. Then the line $L' \in \mathcal{L}$ right parallel with L through h_2 intersects both H_1 and H_3 by (K1), thus, in particular, the line H_1 has two points in $H_2 + H_3$, and hence it is completely contained in $H_2 + H_3$.

Now let the assumptions of (H2) be satisfied with $L \neq L'$. If we assume L and L' skew, then also the lines H_1, H_2 and H_3 are pairwise skew, thus by Lemma 13 $L' \parallel_r L$, and by (K1) $T' \cap L' \neq \emptyset$ (and T' and L' are coplanar). If we assume $T \parallel_\ell H_i$ (see condition (2)) and L, L' coplanar, then H_1 and H_2 are coplanar too, and since all the lines H_1, H_2, H_3, T, T' meet the line L, the lines T and T' are contained in $H_1 + H_2$ and in particular T' is coplanar with L'. The proof in the case of right parallel lines is analogous. □

Definition 16. Let L, H be two distinct lines of $\mathcal{L}$. We define the sets

$$S_\ell(L, H) := \left\{R \in \mathcal{L} \mid R \parallel_\ell H \text{ and } R \cap L \neq \emptyset\right\},$$
$$S_r(L, H) := \left\{R \in \mathcal{L} \mid R \parallel_r H \text{ and } R \cap L \neq \emptyset\right\}$$

and we refer to their linear closure as the *left and right local linear subspaces defined by L and H*. We will write also $s_\ell(L, H) := \dim \mathcal{C}\,(S_\ell(L, H))$ and $s_r(L, H) := \dim \mathcal{C}\,(S_r(L, H))$.

Remark 17. Note that, as a consequence of (H1) and Theorem 15 the linear dimensions $s_\ell(L, H)$ and $s_r(L, H)$ are either 2 or 3. As it often happens, the case of dimension 2 is the most difficult to deal with, and in fact we do not have completely satisfying results for it.

Section 4 will be dedicated to investigate some properties of these local linear subspaces, in particular concerning their dimensions. Here we show how, by assuming a quite strong hypothesis on the dimension of the local linear subspace of any pair of distinct lines, we can prove a result similar to Theorem 10.

Theorem 18. *Let $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ be a kinematic space such that for any pair of distinct lines $L, H \in \mathcal{L}$ the dimension $s_\ell(L, H)$ is 3 and let T be its group of left translations acting transitively on the pointset. Then $\parallel_r$ is uniquely determined by $\parallel_\ell$ and the group T is normal in the group of all collineations.*

Proof. Let L be a line in $\mathcal{L}$ and $p \in \mathcal{P} \setminus L$. Consider any line H through p that intersects L and two lines H_2, H_3 intersecting L and left parallel with H. By our assumption $\mathcal{C}(S_\ell(L, H))$ has dimension 3, thus the lines H, H_2 and H_3 are pairwise skew, hence, by Lemma 13, the only line through p which can fulfill axiom (K1) with respect to H_i is $\mathcal{L}(p \parallel_r L)$, showing that the right parallelism is uniquely determined. The second part of the statement is now a consequence of Theorem 7. □

4. Local linear subspaces

In this section we investigate some properties of the local linear spaces introduced in Definition 16. Since the situation is symmetric with respect to $\parallel_\ell$ and $\parallel_r$ we concentrate only on $\parallel_\ell$, but we stress that analogous results can be proved for $\parallel_r$ as well.

First of all we investigate how the dimension of the local linear spaces is related to two pairs of distinct lines L, H and L', H'. It is easy to show an example of a kinematic space where, for a suitable choice of the lines, the two linear spaces $\mathcal{C}(S_\ell(L, H))$ and $\mathcal{C}(S_\ell(L', H'))$ have different dimension.

Example 19 (*Motions of the Hyperbolic Plane*)**.** Consider the 4-dimensional algebra $\mathrm{Mat}_2(\mathbb{R})$ of 2×2 matrices with coefficients in $\mathbb{R}$ and identify each element of $\mathrm{Mat}_2(\mathbb{R})$ with the quadruple of its coordinates with respect to the (canonical) base. Then it is well known (see [7]) that the subgroup $\mathrm{GL}(2, \mathbb{R})$ of invertible matrices gives rise to a kinematic space $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ obtained from the 3-dimensional projective space $\mathrm{PG}(3, \mathbb{R})$ by removing the hyperbolic quadric $\mathcal{Q}$ of equation $x_1x_4 - x_2x_3 = 0$. The kinematic parallelisms are defined as follows. Write $\mathcal{R}_\ell$ and $\mathcal{R}_r$ for the two reguli of $\mathcal{Q}$ and fix a line $H \in \mathcal{L}$. If this line is secant to the quadric $\mathcal{Q}$ in the distinct points h_1 and h_2 then through each of these points there are two distinct lines, one belonging to each regulus; denote e.g. by I_1, I_2 the lines of $\mathcal{R}_\ell$. Then, for every point $p \in \mathcal{P}$ the left parallel line with H through p is the (unique) line of $\mathcal{L}$ through p which intersects both I_1 and I_2, namely the unique line of the hyperbolic linear congruence of axes I_1 and I_2 through p. If instead the line H is tangent to $\mathcal{Q}$ the lines I_1 and I_2 collapse to a unique line $I \in \mathcal{R}_\ell$, thus the left parallel lines with H are the lines of the parabolic congruence with axis I. If, in the end, the line H intersects $\mathcal{Q}$ in two complex points, again through these points there exist two complex lines J_1 and J_2 of $\mathcal{R}_\ell$ considered as a regulus in $\mathrm{PG}(3, \mathbb{C})$, and the left parallel class is made up with the lines of the elliptic congruence having J_1 and J_2 as axes.

It is now easy to note that if H is tangent to $\mathcal{Q}$ and we fix a line L in the tangent plane π to $\mathcal{Q}$ through H, then $S_\ell(L, H)$ is contained in π, thus $s_\ell(L, H) = 2$, while if H is secant to $\mathcal{Q}$ and L is not coplanar with any of the lines I_1 and I_2 of $\mathcal{R}_\ell$, then $s_\ell(L, H) = 3$.

We have the following.

Proposition 20. *Let $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r) = [G, \mathcal{F}]$ be a kinematic space and let $H, L \in \mathcal{L}$ and $y \in \mathcal{P}$. Then*

(1) $s_\ell(L, H) = s_\ell(yL, H) = s_\ell(L, yH)$*, thus* $s_\ell(yL, yH) = s_\ell(L, H)$*;*
(2) $s_\ell(Ly, H) = s_\ell(L, yHy^{-1})$ *and* $s_\ell(L, Hy) = s_\ell(Ly^{-1}, H)$*;*
(3) $s_\ell(Ly, Hy) = s_\ell(L, H)$*; if moreover* $y \in L \cup H$*, then* $s_\ell(L, Hy) = s_\ell(L, H) = s_\ell(Ly, H)$*;*
(4) *if for all* $F \in \mathcal{F}$ $s_\ell(L, F) = d$*, then* $s_\ell(Ly, F) = d$*.*

Proof. Consider the maps $\{{}_y\tau : x \mapsto yx\}$ and $\{\tau_y : x \mapsto xy\}$ and let H_1 be a line left parallel with H intersecting L. It is well known that ${}_y\tau$ and τ_y are translations preserving $\parallel_\ell$ and $\parallel_r$ respectively, hence ${}_y\tau(H_1)$ is a line left parallel with H and intersecting yL, and thus $S_\ell(yL, H) = {}_y\tau(S_\ell(L, H)) = y\,S_\ell(L, H)$, which shows (1). Write now $H_1 = xH$ for a suitable $x \in \mathcal{P}$ and $H \in \mathcal{F}$, then $\tau_y(H_1) = xHy = xy(y^{-1}Hy)$ which shows that $\tau_y(H_1) \parallel_\ell y^{-1}Hy$, thus $\tau_{y^{-1}}(S_\ell(Ly, H)) = S_\ell(L, yHy^{-1})$, which proves (2). The first part of claim (3) follows from $S_\ell(L, H)y = \tau_y(S_\ell(L, H)) = S_\ell(Ly, Hy)$, while the second part is obvious. Claim (4) is a straightforward consequence of (2). □

The next proposition tries to answer the question whether the local linear subspace defined by two distinct lines $L, H \in \mathcal{L}$ is also a *kinematic subspace*, meaning that it is closed with respect to both $\parallel_\ell$ and $\parallel_r$ (see [12,11] for further details).

Proposition 21. *Let $(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r)$ be a kinematic space and let $(\mathcal{P}', \mathcal{L}')$ be a linear subspace. Then the following facts are equivalent:*

(1) *$(\mathcal{P}', \mathcal{L}')$ is closed with respect to $\parallel_\ell$;*
(2) *for any line $L \in \mathcal{L}'$ and for any $p \in \mathcal{P}'$ there exists a line $L' \in \mathcal{L}'$ such that $L' \parallel_\ell L$ and for which a common transversal to L and L' through p exists.*

Proof. Assume $(\mathcal{P}', \mathcal{L}')$ is closed with respect to $\parallel_\ell$. Then for any line $L \in \mathcal{L}'$ its left parallel line through p is still a line of $\mathcal{L}'$ and obviously has a common transversal with L through p.

Conversely assume (2) holds and let L be any line of $\mathcal{L}'$, $p \in \mathcal{P}'$ and $L'' := \mathcal{L}(p \parallel_\ell L)$. Let $L' \parallel_\ell L$ be a line of $\mathcal{L}'$ admitting a common transversal $T \in \mathcal{L}'$ with L through p. If $L'' = L' \in \mathcal{L}'$ we are done; if this is not the case consider a point $q \in L \setminus T$ and the line $T' = \mathcal{L}(q \parallel_r T)$. Then by (K1) T' meets L', thus it is a line of $\mathcal{L}'$, and again by (K1) it meets also L'', which consequently has two points in $\mathcal{P}'$, and thus belongs to $\mathcal{L}'$ as well. □

Remark 22. The property expressed by (2) of Proposition 21 above is a weakening of a statement which characterizes projective spaces with sufficiently many points on every line. It is in fact possible to prove the following Proposition; we cannot provide a useful reference, thus for the sake of completeness we propose here the following proof.

Proposition 23. *Let $(\mathcal{P}, \mathcal{L})$ be a linear space of dimension at least 3 and such that any line has at least four points. Then for any couple of skew lines L_1 and L_2 and for any point p not on these lines there exists a common transversal T to L_1 and L_2 through p if and only if $(\mathcal{P}, \mathcal{L})$ is a 3-dimensional projective space.*

Proof. One implication is obvious. Let $a, b, c, d \in \mathcal{P}$ be distinct points no three of which are collinear and assume that $\overline{a,b} \cap \overline{c,d} \neq \emptyset$, we claim that $\overline{ac} \cap \overline{bd} \neq \emptyset$, namely that the Veblen axiom holds true. For, consider two distinct points p, q of $\overline{a,c} \setminus \{a, c\}$ and let L be a line through p not contained in the plane π defined by the lines $\overline{a,b}$ and $\overline{c,d}$. Then, by our assumption there exists a common transversal T to L and $\overline{b,d}$ through q. The line T is contained in π, thus necessarily intersect L in the point p, hence coincides with $\overline{a,c}$, which concludes the proof. □

Acknowledgements

The author was partially supported by MIUR (Italy) and by Fondazione Giuseppe Tovini of Brescia (Italy).

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