Arf numerical semigroups

J.C. Rosales, a, ✤ P.A. García-Sánchez, a J.I. García-García, a and M.B. Branco b, 1

a Departamento de Álgebra, Universidad de Granada, E-18071 Granada, Spain
b Departamento de Matemática, Universidade de Évora, 7000 Évora, Portugal

Received 5 September 2001
Communicated by Michel Broué

Abstract

We introduce and study the concept of Arf system of generators for an Arf numerical semigroup. This study allows us to arrange the set of all Arf numerical semigroups in a binary tree and enables us to compute the Arf closure of a given numerical semigroup.

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Introduction

A numerical semigroup \( S \) is a subset of \( \mathbb{N} \) that is closed under addition, \( 0 \in S \) and it generates \( \mathbb{Z} \) as a group (\( \mathbb{Z} \) and \( \mathbb{N} \) denote the set of integers and nonnegative integers, respectively). It is well known (see, for instance, [2,8,13]) that if \( S \) is a numerical semigroup, then the set \( \mathbb{N} \setminus S \) has finitely many elements. The greatest integer not belonging to \( S \) is called the Frobenius number of \( S \), usually denoted by \( g(S) \). Moreover, \( S \) admits a unique minimal system of generators \( \{ n_1 < \cdots < n_p \} \) (that is, \( S = \{ \sum_{i=1}^{p} a_i n_i \mid a_1, \ldots, a_p \in \mathbb{N} \} \) and no proper subset of \( \{ n_1, \ldots, n_p \} \) generates \( S \). The integers \( n_1 \) and \( p \) are known as the multiplicity and embedding dimension of \( S \), and they are denoted by \( m(S) \) and \( \mu(S) \), respectively. Observe that \( m(S) \) is the minimum nonzero element of \( S \). For a given \( A \subseteq \mathbb{N} \),
we will denote by \(\langle A \rangle\) the submonoid of \(\mathbb{N}\) generated by \(A\). The monoid \(\langle A \rangle\) is a numerical semigroup if and only if \(\gcd(A) = 1\) (\(\gcd\) stands for greatest common divisor).

There is a large amount of literature concerning the study of one-dimensional analytically unramified domains via their valuation semigroups (see, for instance, [3,5–7, 10,15,16]). One of the properties studied for this kind of rings using this approach has been the Arf property. From [1], Lipman in [11] introduces and motivates the study of Arf rings; the characterization given in that paper of Arf rings in terms of its semigroup of values gives rise to the notion of Arf semigroup (see also [2] for the connection between the Arf property of a one-dimensional analytically irreducible domain and the Arf property of its semigroup of values). In [4,14] it is studied the relationship between the Pythagorean property of a real curve germ and the Arf property of its value numerical semigroup. The reader can find an extensive set of different characterizations of Arf numerical semigroups in [2].

For describing and working with Arf numerical semigroups one can use their systems of generators (usually this is the case). In this way one does not take advantage of the extra structure that Arf numerical semigroups have over general numerical semigroups. In this paper we introduce the concept of Arf system of generators for an Arf numerical semigroup. Using this tool, one can actually find all Arf numerical semigroups, and once we are given a set of nonnegative integers, the Arf numerical semigroup generated by this set (regarding this set as an Arf system of generators) is easy to compute. In fact, it is easier to enumerate all the elements in an Arf numerical semigroup generated by a set of nonnegative integers, than computing the numerical semigroup generated by the same set. As we will see, every system of generators of an Arf numerical semigroup \(S\) is an Arf system of generators of \(S\). In this way, the results appearing here for Arf systems of generators apply for classical systems of generators.

The contents of this paper are organized as follows. We start in Section 1 by observing that the intersection of two Arf numerical semigroups is again an Arf numerical semigroup. This allows us to introduce the concept of Arf system of generators of an Arf numerical semigroup. As we pointed out before, every numerical semigroup admits a unique minimal system of generators (as a semigroup), and what we prove next in this section is that every Arf numerical semigroup admits a unique minimal Arf system of generators (in general, minimal Arf systems of generators have less elements than minimal systems of generators). In Section 2 and as a consequence of this last result, we show that if \(S\) is an Arf numerical semigroup and \(A\) is its minimal Arf system of generators, then \(a \in A\) if and only if \(S \setminus \{a\}\) is an Arf numerical semigroup. We also observe that if \(A\) is an Arf numerical semigroup, then so is \(S \cup \{g(S)\}\). These two facts allow us to show that the set of all Arf numerical semigroups can be arranged in a binary tree with root \(\mathbb{N}\). Finally, Theorems 16 and 18 of Section 3 give a procedure for computing the elements of an Arf numerical semigroup once one of its Arf systems of generators is given.

1. Arf systems of generators

A numerical semigroup \(S\) is an **Arf numerical semigroup** if for every \(x, y, z \in S\) such that \(x \geq y \geq z\), we have that \(x + y - z \in S\) (see [2, Theorem I.3.4] for fifteen alternative characterizations of this property).
Observe that if \( S \) is a numerical semigroup, then \( \mathbb{N} \setminus S \) is finite, whence there are finitely many numerical semigroups containing \( S \). For \( A \subseteq \mathbb{N} \) with \( \gcd(A) = 1 \), if \( T \) is an Arf numerical semigroup containing \( A \), then clearly \( T \) must contain \( S = \langle A \rangle \). A candidate for the smallest (with respect to set inclusion) Arf numerical semigroup containing \( A \) is the intersection of all Arf numerical semigroups containing \( S \), provided that the intersection of a finite set of Arf numerical semigroups is Arf. Actually this is ensured by the next result, which follows easily from the definition.

**Proposition 1.** If \( S_1, \ldots, S_n \) are Arf numerical semigroups, then \( S = S_1 \cap \cdots \cap S_n \) is also Arf.

This enables us to define the Arf numerical semigroup generated by \( A \) (\( \gcd(A) = 1 \)) as the intersection of all Arf numerical semigroups containing \( A \) (and thus \( \langle A \rangle \)), and will be denoted by \( \operatorname{Arf}(A) \). Observe that in view of Proposition 1, \( \operatorname{Arf}(A) \) is the smallest Arf numerical semigroup containing \( A \). Note also that if \( S \) is an Arf semigroup, then clearly \( \operatorname{Arf}(S) = S \). If \( S = \operatorname{Arf}(A) \), we say that \( A \) is an Arf system of generators of \( S \), and we will say that \( A \) is minimal if no proper subset of \( A \) is an Arf system of generators of \( S \). For a numerical semigroup \( S \), \( \operatorname{Arf}(S) \) will be also called the Arf closure of \( S \).

Our principal aim in this section is to show that every Arf numerical semigroup has a unique minimal Arf system of generators (Theorem 6). First, we give a description of \( \operatorname{Arf}(A) \). Observe that if we are given \( A \subseteq \mathbb{N} \) with \( \gcd(A) = 1 \), then \( \operatorname{Arf}(A) \) must contain the set of all the elements of the form \( x + y - z \) with \( x, y, z \in \langle A \rangle \) and \( x \geq y \geq z \). It must also contain the set of elements that are derived from those obtained above using the same rule and so on. This motivates the following results and definitions.

**Lemma 2.** Let \( S \) be a submonoid of \( \mathbb{N} \). Then

\[
S' = \{ x + y - z \mid x, y, z \in S, \ x \geq y \geq z \}
\]

is a submonoid of \( \mathbb{N} \) and \( S \subseteq S' \).

**Proof.** Let \( x \in S \). Then \( x + x - x \in S' \), whence \( S \subseteq S' \). Clearly \( S' \subseteq \mathbb{N} \). Now take \( a, b \in S' \) and let us prove that \( a + b \in S' \). By the definition of \( S' \), there exist \( x_1, x_2, y_1, y_2, z_1, z_2 \in S \), such that \( x_i \geq y_i \geq z_i \), \( i \in \{1, 2\} \), and \( a = x_1 + y_1 - z_1 \), \( b = x_2 + y_2 - z_2 \). Hence, \( a + b = (x_1 + x_2) + (y_1 + y_2) - (z_1 + z_2) \). Clearly \( x_1 + x_2, y_1 + y_2, z_1 + z_2 \in S \) and \( x_1 + x_2 \geq y_1 + y_2 \geq z_1 + z_2 \). Therefore \( a + b \in S' \). \( \square \)

For a given submonoid \( S \) of \( \mathbb{N} \) and \( n \in \mathbb{N} \), define \( S^n \) recurrently as follows:

- \( S^0 = S \),
- \( S^{n+1} = (S^n)' \).

**Lemma 3.** Let \( S \) be a numerical semigroup. Then there exists \( k \in \mathbb{N} \) such that \( S^k = \operatorname{Arf}(S) \).

**Proof.** Using induction on \( n \), it can be easily proved that \( S^0 \subseteq \operatorname{Arf}(S) \) for all \( n \in \mathbb{N} \). By Lemma 2, \( S^n \subseteq S^{n+1} \) and \( S^k \subseteq S^n \) for all \( n \in \mathbb{N} \). As we pointed out before, the number of
For proving that minimal Arf systems of generators are unique, we first prove that every Arf system of generators must contain the multiplicity of the semigroup.

**Lemma 4.** Let S be an Arf numerical semigroup and let A be an Arf system of generators of S. Then m(S) ∈ A.

**Proof.** For x, y, z ∈ S \ {m(S)} with x ≥ y ≥ z, we get that x + y − z ∈ S \ {m(S)}, whence S \ {m(S)} is an Arf numerical semigroup. If m(S) ∉ A, then Arf(A) ⊆ Arf(S \ {m(S)}) = S \ {m(S)} ⊈ S, which contradicts Arf(A) = S. □

We already know that for a given numerical semigroup S = ⟨A⟩, there exists k ∈ N such that Sk = Arf(A). This in particular implies that every element in Arf(A) can be expressed as a linear combination with integer coefficients of the elements in A. What we basically prove next is that for s ∈ Arf(A) the generators that appear in any of the expressions of s must be smaller than s.

**Lemma 5.** Let S be an Arf numerical semigroup and let A be an Arf system of generators of S. For every s ∈ S, set B(s) = {a ∈ A | a ≤ s}. If s ∈ ⟨A⟩n, then s ∈ ⟨B(s)⟩ n.

**Proof.** We use induction on n. For n = 0, the result is clear by the definition of B(s). Now assume that the result is true for n ∈ N and let us prove it for n + 1. Take s ∈ ⟨A⟩ n+1. Then there exist x, y, z ∈ ⟨A⟩ n with x ≥ y ≥ z and such that s = x + y − z. By induction hypothesis, x ∈ ⟨B(x)⟩ n, y ∈ ⟨B(y)⟩ n, and z ∈ ⟨B(z)⟩ n. Since s = x + y − z and x ≥ y ≥ z, we have that z ≤ y ≤ x ≤ s, whence B(z) ⊆ B(y) ⊆ B(x) ⊆ B(s). It follows that x, y, z ∈ ⟨B(s)⟩ n and this leads to s = x + y − z ∈ ⟨B(s)⟩ n+1. □

**Theorem 6.** Let A and B be two minimal Arf systems of generators of an Arf numerical semigroup S. Then A = B.

**Proof.** Assume that A = {n1 < · · · < np} and B = {m1 < · · · < mq} and set A = {m1 = m(S)}. If A ∉ B, then let r be the least integer such that mr ∉ m(S). Assume without loss of generality that mr < nr. As mr ∈ S, we can apply Lemma 3 and obtain that mr ∈ ⟨A⟩ n for some nonnegative integer n. Using Lemma 5, we deduce that mr ∈ ⟨n1, . . . , nr−1⟩ n. Since being an Arf system of generators.

This result allows us to define the Arf rank of an Arf numerical semigroup S as the cardinality of its minimal Arf system of generators. This amount will be denoted by Arf-rank(S). If S is an Arf numerical semigroup and S = {n1, . . . , np}, then
as we pointed out before $S = \text{Arf}(n_1, \ldots, n_p)$ (we write $\text{Arf}(n_1, \ldots, n_p)$ instead of $\text{Arf}([n_1, \ldots, n_p])$) in the same way as one writes $\langle n_1, \ldots, n_p \rangle$ instead of $\langle [n_1, \ldots, n_p] \rangle$).

Hence $\text{Arf-rank}(S) \leq \mu(S)$, that is, the Arf rank of $S$ is smaller than or equal to its embedding dimension. Moreover, it is well known (see, for instance, [2]) that every Arf numerical semigroup has maximal embedding dimension, that is, $\mu(S) = m(S)$. It follows that for an Arf numerical semigroup $S$,

$$\text{Arf-rank}(S) \leq \mu(S) = m(S) = \min \{S \setminus \{0\}\}.$$ 

2. The binary tree of Arf numerical semigroups

A binary tree is a rooted tree such that every vertex has at most two sons (see [9]). Our goal in this section is to describe a recursive procedure that arranges the set of all Arf numerical semigroups in a binary tree whose root is $\mathbb{N}$. The idea is to learn how to construct new Arf numerical semigroups by adding or removing an element from a given Arf numerical semigroup. We will show first that adding the Frobenius number to an Arf numerical semigroup yields a new Arf numerical semigroup, and this operation will enable us to move from one vertex in the tree to its parent. The process of generating the sons of a vertex will be by removing certain elements from the minimal Arf system of generators of the semigroup.

**Lemma 7.** Let $S$ be an Arf numerical semigroup, $S \neq \mathbb{N}$. Then $S \cup \{g(S)\}$ is again an Arf numerical semigroup (recall that $g(S)$ denotes the Frobenius number of $S$, that is, $\max \{\mathbb{N} \setminus S\}$).

**Proof.** By [12, Lemma 0.2], we already know that $S \cup \{g(S)\}$ is a numerical semigroup. Take $x, y, z \in S \cup \{g(S)\}$ such that $x \geq y \geq z$, and let us prove that $x + y - z \in S \cup \{g(S)\}$.

- If $x, y, z \in S$, then as $S$ is Arf, we obtain that $x + y - z \in S \subseteq S \cup \{g(S)\}$.
- If $g(S) \in \{x, y, z\}$, then $x + y - z \geq g(S)$ and thus $x + y - z \in S \cup \{g(S)\}$. \qed

Given a numerical semigroup $S$, for $n \in \mathbb{N}$, define recursively the semigroup $S_n$ as:

- $S_0 = S$,
- $S_{n+1} = S_n \cup \{g(S_n)\}$, if $S_n \neq \mathbb{N}$; $S_{n+1} = \mathbb{N}$, otherwise.

Clearly, for every numerical semigroup there exists $k \in \mathbb{N}$ such that $S_k = \mathbb{N}$. Note also that if $S$ is an Arf numerical semigroup, then by Lemma 7, the chain $S = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k = \mathbb{N}$ is a chain of Arf numerical semigroups, and $S_i = S_{i+1} \setminus \{a\}$ for some $a \in S_{i+1}$. The following result studies the condition that we must impose to an element $a$ in an Arf numerical semigroup $S$ for $S \setminus \{a\}$ to be Arf.

**Lemma 8.** Let $S$ be an Arf numerical semigroup and let $a \in S$. The following conditions are equivalent:

- $S_0 = S$, $S_{n+1} = S_n \cup \{g(S_n)\}$, if $S_n \neq \mathbb{N}$; $S_{n+1} = \mathbb{N}$, otherwise.
(1) $a$ belongs to the minimal Arf system of generators of $S$,
(2) $S \setminus \{a\}$ is an Arf numerical semigroup.

**Proof.** (1) implies (2). Since $a$ belongs to the minimal Arf system of generators of $S$, we have that $\text{Arf}(S \setminus \{a\})$ is strictly contained in $S$. Hence $S \setminus \{a\} \subseteq \text{Arf}(S \setminus \{a\}) \subseteq S$, and $S \neq \text{Arf}(S \setminus \{a\})$ yields $\text{Arf}(S \setminus \{a\}) = S \setminus \{a\}$, which means that $S \setminus \{a\}$ is an Arf numerical semigroup.

(2) implies (1). If $a$ does not belong to the minimal Arf system of generators of $S$, then $\text{Arf}(S \setminus \{a\}) = S$, and this in particular implies that $S \setminus \{a\}$ does not have the Arf property.

With the following result we can detect when an Arf numerical semigroup has been constructed by using the procedure described in Lemma 7.

**Proposition 9.** Let $S$ be an Arf numerical semigroup. The following conditions are equivalent:

(1) $S = \overline{S} \cup \{g(S)\}$, with $\overline{S}$ an Arf numerical semigroup,
(2) the minimal Arf system of generators of $S$ contains at least one element greater than $g(S)$.

**Proof.** (1) implies (2). Clearly, if $S = \overline{S} \cup \{g(S)\}$, then $\overline{S} = S \setminus \{g(S)\}$. Using Lemma 8, we obtain that $g(S)$ must belong to the minimal Arf system of generators of $S$, and since $\overline{S} \subseteq S$ and $g(S) \in S$, we get that $g(S) > g(S)$.

(2) implies (1). If $a$ is an element of the minimal Arf system of generators of $S$, then by Lemma 8, we know that $\overline{S} = S \setminus \{a\}$ is an Arf numerical semigroup. If in addition $a > g(S)$, then $a = g(\overline{S})$, whence $S = \overline{S} \cup \{g(\overline{S})\}$ with $\overline{S}$ an Arf numerical semigroup.

Proposition 9 together with the remark given just after Lemma 7 allow us to construct recursively from the Arf numerical semigroup $\mathbb{N}$ the set of all Arf numerical semigroups (see Fig. 1). This construction arranges them all in a tree ordering shape. It is also clear that as we move “downwards” the branches of this tree, we encounter semigroups with larger Frobenius numbers.

Let $S$ and $T$ be two Arf numerical semigroups such that there exists $r \in \mathbb{N}$ and $S_0, \ldots, S_r$ Abf numerical semigroups with $S_r = S$, $S_0 = T$, and $S_{i+1} = S_i \cup \{g(S_i)\}$. Then $S$ is said to be an ancestor of $T$ and $T$ is a descendant of $S$. If $S \neq T$, then $S$ is a proper ancestor of $T$ and $T$ a proper descendant of $S$. If $r = 1$ and $S \neq T$, then we say that $T$ is a son of $S$. An Arf numerical semigroup having no sons is a leaf. As a consequence of Proposition 9 we get the following result.

**Corollary 10.** Let $S$ be an Arf numerical semigroup. Then $S$ is a leaf if and only if the minimal Arf system of generators of $S$ does not contain elements greater than $g(S)$.

Finally we show that the tree of Arf numerical semigroups is binary. To this end we need a couple of technical lemmas. The idea is to prove that in a minimal Arf system of
Lemma 11. Let \( x \in \mathbb{N} \) and \( X \subseteq \mathbb{N} \) with \( \{x, x+1\} \subseteq X \). Then \( \{a \in \mathbb{N} \mid a \geq x\} \subseteq \text{Arf}(X) \).

Proof. We use induction to prove that \( x+n \in \text{Arf}(X) \) for all \( n \in \mathbb{N} \). For \( n = 0 \), we get \( x \in X \subseteq \text{Arf}(X) \). Now assume that \( x+n \in \text{Arf}(X) \). Then \( x+n+1 \in \text{Arf}(X) \), since \( x+n+1 = (x+n) + (x+1) - x, x+n, x+1, x \in \text{Arf}(X) \), and \( x+n+1 \geq x+1 \geq x \).

Lemma 12. Let \( S \) be an Arf numerical semigroup and let \( A \) be its minimal Arf system of generators. Then \( \{a \in A \mid g(S) < a\} \) has at most two elements.

Proof. Let \( \{a_1, \ldots, a_r\} = \{a \in A \mid a < g(S)\} \). Using Lemmas 5 and 11, we deduce that \( \text{Arf}(a_1, \ldots, a_r, g(S)+1, g(S)+2) = S \), whence \( \{a_1, \ldots, a_r, g(S)+1, g(S)+2\} \) is an Arf system of generators of \( S \). Applying now Theorem 6, we get that \( \{a \in A \mid g(S) < a\} \subseteq \{g(S)+1, g(S)+2\} \).

Proposition 13. The tree of Arf numerical semigroups is binary.

Proof. It suffices to observe, by Lemma 12 and Proposition 9, that if \( T \) is a son of \( S \), then either \( T = S \setminus \{g(S)+1\} \) or \( T = S \setminus \{g(S)+2\} \). Therefore, every vertex in the tree has at most two sons.

3. Computing the Arf closure of a numerical semigroup

The aim of this section is to present an algorithmic procedure for computing, from a finite subset \( X \) of \( \mathbb{N} \) with \( \gcd(X) = 1 \), the elements of \( \text{Arf}(X) \). The reader will find
a similitude between the algorithm described here and Euclid’s algorithm for computing gcd’s. It turns out that finding the elements of Arf(X) is much easier than computing (X).

Lemma 14. Let $S$ be an Arf numerical semigroup and take $m \in S$. Then $(m + S) \cup \{0\}$ is also an Arf numerical semigroup.

**Proof.** It is clear that $(m + S) \cup \{0\}$ is a numerical semigroup. Now take $m + s_1, m + s_2, m + s_3 \in m + S$ with $m + s_1 \geq m + s_2 \geq m + s_3$. Then $s_1 \geq s_2 \geq s_3$ and since $S$ is Arf, we get $s_1 + s_2 - s_3 \in S$. It follows that $(m + s_1) + (m + s_2) - (m + s_3) = m + (s_1 + s_2 - s_3) \in m + S$. The reader can check that this proves that $(m + S) \cup \{0\}$ is Arf. \[square\]

Lemma 15. Let $m, r_1, \ldots, r_p \in \mathbb{N}$ such that $\gcd\{(m, r_1, \ldots, r_p)\} = 1$. Then

$$m + \langle m, r_1, \ldots, r_p \rangle \subseteq \operatorname{Arf}(m, m + r_1, \ldots, m + r_p).$$

**Proof.** We use once more induction on $n$. For $n = 0$ we have to prove that $m + \langle m, r_1, \ldots, r_p \rangle \subseteq \operatorname{Arf}(m, m + r_1, \ldots, m + r_p)$. Let $i, j \in \{1, \ldots, p\}$. Then $m, m + r_i, m + r_j \in \operatorname{Arf}(m, m + r_1, \ldots, m + r_p)$, whence

$$m + r_i + r_j = (m + r_i) + (m + r_j) - m \in \operatorname{Arf}(m, m + r_1, \ldots, m + r_p).$$

Now for $k \in \{1, \ldots, p\}$, $m, m + r_i + r_j, m + r_k \in \operatorname{Arf}(m, m + r_1, \ldots, m + r_k)$ and therefore

$$m + r_i + r_j + r_k = (m + r_i + r_j) + (m + r_k) - m \in \operatorname{Arf}(m, m + r_1, \ldots, m + r_k).$$

Using this idea, we obtain that for every $a, a_1, \ldots, a_p \in \mathbb{N}$, we have that $(a + 1)m + a_1r_1 + \cdots + a_pr_p \in \operatorname{Arf}(m, m + r_1, \ldots, m + r_p)$ and thus $m + \langle m, r_1, \ldots, r_p \rangle \subseteq \operatorname{Arf}(m, m + r_1, \ldots, m + r_p)$.

Now assume that $m + \langle m, r_1, \ldots, r_p \rangle^n \subseteq \operatorname{Arf}(m, m + r_1, \ldots, m + r_p)$ and let us prove that $m + \langle m, r_1, \ldots, r_p \rangle^{n+1} \subseteq \operatorname{Arf}(m, m + r_1, \ldots, m + r_p)$. Let $a \in m + \langle m, r_1, \ldots, r_p \rangle^n$. Then $a = m + b$ with $b \in \langle m, r_1, \ldots, r_p \rangle^n$. Hence there exist $x, y, z \in \langle m, r_1, \ldots, r_p \rangle^n$ such that $x \geq y \geq z$ and $x + y - z = b$. In this way

$$a = m + b = m + x + y - z = (m + x) + (m + y) - (m + z) \in \operatorname{Arf}(m, m + r_1, \ldots, m + r_p),$$

since by induction hypothesis, $m + x, m + y, m + z \in \langle m, r_1, \ldots, r_p \rangle^n \subseteq \operatorname{Arf}(m, m + r_1, \ldots, m + r_p)$. \[square\]

Theorem 16. Let $m, r_1, \ldots, r_p$ be nonnegative integers with greatest common divisor one. Then

$$\operatorname{Arf}(m, m + r_1, \ldots, m + r_p) = \langle m + \operatorname{Arf}(m, r_1, \ldots, r_p) \rangle \cup \{0\}.$$
Proof. Using Lemmas 3 and 15, we obtain that \( (m + \text{Arf}(m, r_1, \ldots, r_p)) \cup \{0\} \subseteq \text{Arf}(m, m + r_1, \ldots, m + r_p) \). For the other inclusion observe that \( m, m + r_1, \ldots, m + r_p \in (m + \text{Arf}(m, r_1, \ldots, r_p)) \cup \{0\} \), and since by Lemma 14, \( (m + \text{Arf}(m, r_1, \ldots, r_p)) \cup \{0\} \) is an Arf numerical semigroup, we get that \( \text{Arf}(m, m + r_1, \ldots, m + r_p) \subseteq (m + \text{Arf}(m, r_1, \ldots, r_p)) \cup \{0\} \). □

As an immediate consequence of Theorem 16 we obtain the following result.

Corollary 17. Let \( m, r_1, \ldots, r_p \) be nonnegative integers with greatest common divisor one. Then

\[
g(\text{Arf}(m, m + r_1, \ldots, m + r_p)) = m + g(\text{Arf}(m, r_1, \ldots, r_p)).
\]

Let \( X \subseteq \mathbb{N} \setminus \{0\} \) be such that \( \gcd(X) = 1 \). Define recursively the following sequence of subsets of \( \mathbb{N} \):

- \( A_1 = X \),
- \( A_{n+1} = \{x - \min \leq A_n \mid x \in A_n \} \cup \{\min \leq A_n\} \).

As a consequence of Euclid’s algorithm for the computation of \( \gcd(X) \), we obtain that there exists \( q = \min \leq \{k \in \mathbb{N} \mid 1 \in A_k\} \).

Theorem 18. Under the standing notation, we have that

\[
0, \min \leq A_1, \min \leq A_1 + \min \leq A_2, \ldots, \min \leq A_1 + \cdots + \min \leq A_{q-1}
\]

are the elements in \( \text{Arf}(X) \) that are less than or equal to \( g(\text{Arf}(X)) + 1 \).

Proof. Since \( 1 \in A_q, \text{Arf}(A_q) = \mathbb{N} \). Hence applying Theorem 16, we get that \( \text{Arf}(A_{q-1}) = (\min \leq A_{q-1} + \mathbb{N}) \cup \{0\} \). This implies that the elements \( 0, \min \leq A_{q-1} \) are the elements that are less than or equal to \( g(\text{Arf}(A_{q-1})) + 1 \). Assume as induction hypothesis that \( 0, \min \leq A_{q-i}, \min \leq A_{q-i} + \min \leq A_{q-i+1}, \ldots, \min \leq A_{q-i} + \cdots + \min \leq A_{q-1} \) are the elements of \( \text{Arf}(A_{q-i}) \) less than or equal to \( g(\text{Arf}(A_{q-i})) + 1 \). We must prove now that \( 0, \min \leq A_{q-i-1}, \min \leq A_{q-i-1} + \min \leq A_{q-i-1} + \cdots + \min \leq A_{q-1} \) are the elements of \( \text{Arf}(A_{q-i-1}) \) less than or equal to \( g(\text{Arf}(A_{q-i-1})) + 1 \). By Theorem 16, we know that \( \text{Arf}(A_{q-i-1}) = (\min \leq A_{q-i-1} + \text{Arf}(A_{q-i-1})) \cup \{0\} \). Using now the induction hypothesis and Corollary 17, we obtain the desired result. □

Example 19. Let us compute \( \text{Arf}(7, 24, 33) \).

\[
A_1 = \{7, 24, 33\}, \quad \min \leq A_1 = 7, \quad A_2 = \{7, 17, 26\}, \quad \min \leq A_2 = 7, \\
A_3 = \{7, 10, 19\}, \quad \min \leq A_3 = 7, \quad A_4 = \{7, 3, 12\}, \quad \min \leq A_4 = 3, \\
A_5 = \{4, 3, 9\}, \quad \min \leq A_5 = 3, \quad A_6 = \{1, 3, 6\}.
\]

whence \( \text{Arf}(7, 24, 33) = \{0, 7, 14, 21, 24, 27, \ldots\} \).
References