Differential Geometry and its Applications 11 (1999) 1–12
North-Holland
www.elsevier.com/locate/difgeo

A Kaehler structure on the nonzero tangent bundle of a space form*

V. Oproiu [1]
Faculty of Mathematics, University "Al. I. Cuza," Iaşi, România

N. Papaghiuc
Department of Mathematics, Technical University, Iaşi, România

Communicated by I. Kolář
Received 8 September 1997

Abstract: We obtain a Kaehler structure on the bundle of nonzero tangent vectors to a Riemannian manifold of constant positive sectional curvature. This Kaehler structure is determined by a Lagrangian depending on the density energy only.

Keywords: Tangent bundle, Kaehler manifold.

MS classification: 53C07, 53C15, 53C55.

Introduction

It is well known (see [2, 9, 15]) that the tangent bundle TM of a Riemannian manifold (M, g) has a structure of almost Kaehlerian manifold with an almost complex structure determined by the isomorphic vertical and horizontal distributions VTM, HTM on TM (the last one being determined by the Levi-Civita connection on M) and the Sasaki metric on TM (see also [14, 16]). However, this structure is Kaehler only in the case where the base manifold is locally Euclidean. On the other hand, Calabi (see [1]) defined a new Riemannian metric on the cotangent bundle of a Kaehler manifold, by using a special Lagrangian defined by a smooth real valued function depending on the density energy only and has obtained a new almost complex structure, which together with the original one determines a structure of hyper-Kaehler manifold on the cotangent bundle of a Kaehler manifold of holomorphic constant positive sectional curvature.

In the present paper we have been inspired by the idea of Calabi to consider a regular Lagrangian on a Riemannian manifold (M, g) defined by a smooth function L depending on the energy density only. An interesting result is that the usual nonlinear connection determined by the Euler–Lagrange equations associated to L (see [4, 10, 11]) does coincide with the nonlinear connection defined by the Levi-Civita connection of g, thus the horizontal distribution HTM used in this paper is the standard one. Then we have obtained a Riemannian metric G on the tangent bundle TM such that the vertical and horizontal distributions VTM, HTM are

[1] E-mail: voproiu@uaic.ro.

* Partially supported by the Grant 644/1997, Ministerul Invăţământului, România.

PII S0926-2245(99)00013-3

orthogonal to each other but they are no longer isometric. Then we have considered an almost complex structure J on TM related to the above Riemannian metric G such that (TM, J, G) is an almost Kaehlerian manifold (Theorem 2). From the integrability conditions of the almost complex structure J we have obtained our main result: If (M, g) has positive constant sectional curvature then we may obtain a certain smooth function L on the subset T_0M of the nonzero tangent vectors to M such that the structure (T_0M, J, G) is Kaehlerian (Theorem 3). Next we have obtained the Levi-Civita connection $\tilde{\nabla}$ of G and its curvature tensor field showing that the Kaehlerian manifold (T_0M, J, G) cannot be an Einstein manifold and cannot have constant holomorphic sectional curvature (see [2, 6, 9, 15], for the expression of the Levi-Civita connection of the Sasaki metric and that of its curvature tensor field). Next, the covariant derivative of the curvature tensor field K with respect to the Levi-Civita connection $\tilde{\nabla}$ of G is studied and we obtain that the components of the covariant derivative of K with respect to $\tilde{\nabla}$ are expressed as linear combinations of the components K^h_{ijk} ans S^h_{ijk} of the curvature tensor field K.

The manifolds, tensor fields and geometric objects we consider in this paper are assumed to be differentiable of class C^∞ (i.e., smooth). We use computations in local coordinates but many results may be expressed in an invariant form. The well-known summation convention is used throughout this paper, the range for the indices i, j, k, l, h, s, r being always $\{1, \dots, n\}$ (see [5, 3, 12, 13]). We shall denote by $\Gamma(TM)$ the module of smooth vector fields on TM.

1. The tangent bundle and special Lagrangians.

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that TM has a structure of $2n$-dimensional smooth manifold induced from the smooth manifold structure of M. A local chart $(U, \varphi) = (U, x^1, \dots, x^n)$ on M induces a local chart $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots x^n, y^1, \dots, y^n)$ on TM where the local coordinate $x^i, y^i; i = 1, \dots, n$ are defined as follows. The first n local coordinates $x^i = x^i \circ \tau; i = 1, \dots, n$ on TM are the local coordinates in the local chart (U, φ) of the base point of a tangent vector from $\tau^{-1}(U)$. The last n local coordinates $y^i; i = 1, \dots, n$ are the vector space coordinates of the same tangent vector, with respect to the natural local basis in the corresponding tangent space defined by the local chart (U, φ).

This special structure of TM allows us to introduce the notion of M-tensor field on it (see [7]). An M-tensor field of type (p, q) on TM is defined by sets of functions

$$T^{i_1 \dots i_p}_{j_1 \dots j_q}; \quad i_1, \dots, i_p, j_1, \dots, j_q = 1, \dots, n$$

assigned to any induced local chart $(\tau^{-1}(U), \Phi)$ on TM, such that the change rule is that of the components of a tensor field of type (p, q) on the base manifold, when a change of local charts on the base manifold is performed. Remark that any M-tensor field on TM may be thought of as an ordinary tensor field T with the expression

$$T = T^{i_1 \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial y^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}.$$

However, there are many other posibilities to interpret an M-tensor field as an ordinary tensor field on TM. Remark also that any ordinary tensor field on the base manifold may be thought of as an M-tensor field on TM, having the same type and with the components in the induced local chart on TM, equal to the local coordinate components of the given tensor field in the chosen local chart on the base manifold. In the case of a covariant tensor field on the base manifold M, the corresponding M-tensor field on the tangent bundle TM may be thought of as the pull back of the initial tensor field defined on the base manifold, by the smooth submersion $\tau : TM \to M$.

The tangent bundle TM of a Riemannian manifold (M, g) can be organized as a Riemannian or a pseudo-Riemannian manifold in many ways. The most known such structures are given by the Sasaki metric on TM defined by g (see [14,2]) and the complete lift type pseudo-Riemannian metric defined by g (see [16,15,10,11]). Recall that the Levi-Civita connection of g defines a direct sum decomposition

$$TTM = VTM \oplus HTM \tag{1}$$

of the tangent bundle to TM into the vertical distribution $VTM = \operatorname{Ker} \tau_*$ and the horizontal distribution HTM. The vector fields $(\partial/\partial y^1, \ldots, \partial/\partial y^n)$ define a local frame field for VTM and for HTM we have the local frame field $(\delta/\delta x^1, \ldots, \delta/\delta x^n)$ where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{i0}\frac{\partial}{\partial y^h}; \qquad \Gamma^h_{i0} = \Gamma^h_{ik} y^k$$

and $\Gamma^h_{ik}(x)$ are the Christoffel symbols defined by the Riemannian metric g.

The distributions VTM and HTM are isomorphic to each other and it is possible to derive an almost complex structure on TM which, together with the Sasaki metric, determines a structure of almost Kaehlerian manifold on TM (see [2]). Consider now the density energy (the kinetic energy or "forza viva," according to the terminology used by Levi-Civita)

$$t = \tfrac{1}{2} g_{ik}(x) y^i y^k \tag{2}$$

defined on TM by the Riemannian metric g of M. We shall find some interesting properties of TM by using a Lagrangian function defined as the antiderivative (indefinite integral) of a real smooth function depending on kinetic energy only, i.e.:

$$L = \int u(t)\, dt = \int u\left(\tfrac{1}{2} g_{ik}(x) y^i y^k\right) \tfrac{1}{2} d\left(g_{ik} y^i y^k\right), \tag{3}$$

where $u : \mathbb{R}_+ = [0, \infty) \longrightarrow \mathbb{R}$ is a smooth function. In the sequel it will be necessary to make some supplementary assumptions concerning the function u, in order to assure the regularity of the Lagrangian L. As usual in Lagrange geometry (see [4,10,11]), we may consider the symmetric M-tensor field of type $(0, 2)$ on TM, defined by the components

$$G_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j} = u g_{ij} + u' g_{0i} g_{0j}, \tag{4}$$

where $g_{0i} = g_{hi} y^h$. The matrix (G_{ij}) has the inverse with the entries

$$G^{jk} = \frac{1}{u} g^{jk} + w y^j y^k, \tag{5}$$

where $w = -u'/(u^2 + 2tuu')$. We shall assume that $u(t) > 0, u'(t) > 0$ for $t \geqslant 0$ so that the functions G^{jk}: $j, k = 1, \ldots, n$ do always exist. The components G^{jk} define a symmetric

M-tensor field of type (2, 0) on TM and the symmetric matrix G_{ij} is positive definite. It follows that, under these conditions, the Lagrangian $L = \int u(t)\,dt$ is regular.

A regular Lagrangian L defines a nonlinear connection on TM given by the horizontal distribution $H'TM$ spanned, locally, by the vector fields

$$\left(\frac{\delta}{\delta x^i}\right)' = \frac{\partial}{\partial x^i} - N_i^k(x, y)\frac{\partial}{\partial y^k}; \qquad i = 1, \dots, n,$$

where

$$N_i^k = \frac{1}{2}\frac{\partial}{\partial y^i}\left(G^{kl}\left(\frac{\partial^2 L}{\partial y^l \partial x^h} y^h - \frac{\partial L}{\partial x^l}\right)\right)$$

(see [4, 10, 11]). We have:

Proposition 1. *If the regular Lagrangian L is given by* (3) *then $H'TM = HTM$.*

Proof. We have

$$\frac{\partial L}{\partial x^i} = \frac{1}{2} u \frac{\partial g_{kl}}{\partial x^i} y^k y^l = u g_{0h} \Gamma_{i0}^h,$$

where $g_{0h} = g_{jh} y^j$, $\Gamma_{i0}^h = \Gamma_{ij}^h y^j$, and

$$\frac{\partial^2 L}{\partial y^l \partial x^h} = u\left(g_{lk}\Gamma_{h0}^k + g_{0k}\Gamma_{hl}^k\right) + u' g_{0l} g_{0k} \Gamma_{h0}^k.$$

Then

$$y^h \frac{\partial^2 L}{\partial y^l \partial x^h} = \left(u g_{lh} + u' g_{0l} g_{0h}\right)\Gamma_{00}^h + u g_{0h} \Gamma_{l0}^h,$$

where $\Gamma_{00}^h = \Gamma_{ij}^h y^i y^j$, and by using (5) we get

$$G^{kl}\left(\frac{\partial^2 L}{\partial y^l \partial x^h} y^h - \frac{\partial L}{\partial x^l}\right) = \Gamma_{00}^k.$$

Then $N_i^k = \Gamma_{i0}^k$, showing that $H'TM = HTM$ and $(\delta/\delta x^i)' = \delta/\delta x^i$.

Hence the horizontal distribution HTM defined by the Levi-Civita connection ∇ of g may be used in the study of the Lagrange geometry of M, defined by the Lagrangian (3).

2. A Riemannian metric on the tangent bundle

Consider the symmetric M-tensor field of type (0, 2) on TM, defined by the components

$$H_{ij} = g_{ik} G^{kl} g_{lj} = \frac{1}{u} g_{ij} + w g_{0i} g_{0j}. \tag{6}$$

Then the following Riemannian metric may be considered on TM:

$$G = G_{ij}\, dx^i\, dx^j + H_{ij} \nabla y^i \nabla y^j, \tag{7}$$

where $\nabla y^i = dy^i + \Gamma^i_{j0} dx^j$ is the absolute differential of y^i with respect to the Levi-Civita connection ∇ of g. Equivalently, we have

$$G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = G_{ij}, \quad G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = H_{ij}, \quad G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = G\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = 0.$$

Note that HTM, VTM are orthogonal to each other with respect to G but the Riemannian metrics induced from G on HTM, VTM are not the same, so the considered metric G on TM is no longer a metric of Sasaki type. Note also that the system of 1-forms $(dx^1, \dots, dx^n, \nabla y^1, \dots, \nabla y^n)$ defines a local frame of T^*TM, dual to the local frame $(\delta/\delta x^1, \dots, \delta/\delta x^n, \partial/\partial y^1, \dots, \partial/\partial y^n)$ adapted to the direct sum decomposition (1).

An almost complex structure J may be defined on TM by

$$J\frac{\delta}{\delta x^i} = J^k_i \frac{\partial}{\partial y^k}, \qquad J\frac{\partial}{\partial y^i} = H^k_i \frac{\delta}{\delta x^k}, \tag{8}$$

where

$$\begin{aligned} J^k_i &= G_{ij} g^{jk} = u\delta^k_i + u' y^k g_{0i}, \\ H^k_i &= -H_{ij} g^{jk} = -G^{kh} g_{hi} = -\frac{1}{u}\delta^k_i - w y^k g_{0i}. \end{aligned} \tag{9}$$

Theorem 2. *(TM, J, G) is an almost Kaehlerian manifold.*

Proof. First of all we may check easily that $J^2(\delta/\delta x^i) = -\delta/\delta x^i$, $J^2(\partial/\partial y^i) = -\partial/\partial y^i$; thus J really defines an almost complex structure on TM. Then we have

$$\begin{aligned} G\left(J\frac{\delta}{\delta x^i}, J\frac{\delta}{\delta x^j}\right) &= G_{ik} g^{ka} G_{jh} g^{hb} G\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right) \\ &= G_{ik} g^{ka} G_{jh} g^{hb} g_{ac} G^{cd} g_{db} = G_{ij} = G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right). \end{aligned}$$

The relations

$$\begin{aligned} G\left(J\frac{\partial}{\partial y^i}, J\frac{\partial}{\partial y^j}\right) &= G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right), \\ G\left(J\frac{\partial}{\partial y^i}, J\frac{\delta}{\delta x^j}\right) &= G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = 0 \end{aligned}$$

may be obtained in a similar way, thus G is almost Hermitian with respect to J. The associated 2-form Ω is given by

$$\begin{aligned} \Omega\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) &= G\left(\frac{\delta}{\delta x^i}, J\frac{\delta}{\delta x^j}\right) = G\left(\frac{\delta}{\delta x^i}, J^k_j\frac{\partial}{\partial y^k}\right) = 0, \\ \Omega\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) &= G\left(\frac{\partial}{\partial y^i}, J\frac{\partial}{\partial y^j}\right) = G\left(\frac{\partial}{\partial y^i}, H^k_j\frac{\delta}{\delta x^k}\right) = 0, \\ \Omega\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) &= G\left(\frac{\partial}{\partial y^i}, J\frac{\delta}{\delta x^j}\right) = J^k_j G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^k}\right) = J^k_j H_{ik} = g_{ij}. \end{aligned}$$

Hence we have

$$\Omega = g_{ij}\nabla y^i \wedge dx^j \tag{10}$$

and Ω is closed since it does coincide with the 2-form associated to the Sasaki metric on TM (see [2]).

3. The Kaehlerian structure on T_0M

In order to study the integrability of the almost complex structure defined by J on TM we need the following well-known formulas for the brackets of the vector fields $\partial/\partial y^i$, $\delta/\delta x^i$; $i = 1, \ldots, n$:

$$\left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0, \quad \left[\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right] = -\Gamma^h_{ij}\frac{\partial}{\partial y^h}, \quad \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -R^h_{0ij}\frac{\partial}{\partial y^h}, \tag{11}$$

where $R^h_{0ij} = R^h_{kij}y^k$ and R^h_{kij} are the local coordinate components of the curvature tensor field of ∇ on M.

Theorem 3. *The almost complex structure J on TM is integrable if and only if (M, g) has positive constant sectional curvature c and the function $u(t)$ satisfies the ordinary differential equation*

$$2t(u')^2 = c. \tag{12}$$

Proof. First of all, the following formulas can be checked by straightforward computation:

$$\nabla_i G_{jk} = \frac{\delta}{\delta x^i} G_{jk} - \Gamma^h_{ij}G_{hk} - \Gamma^h_{ik}G_{jh} = 0,$$
$$\nabla_i H_{jk} = \frac{\delta}{\delta x^i} H_{jk} - \Gamma^h_{ij}H_{hk} - \Gamma^h_{ik}H_{jh} = 0,$$
$$\nabla_i J^j_k = \frac{\delta}{\delta x^i} J^j_k + \Gamma^j_{ih}J^h_k - \Gamma^h_{ik}J^j_h = 0,$$
$$\nabla_i H^j_k = \frac{\delta}{\delta x^i} H^j_k + \Gamma^j_{ih}H^h_k - \Gamma^h_{ik}H^j_h = 0.$$

Then, by using the definition of the Nijenhuis tensor field N_J of J we have

$$\begin{aligned} N_J\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) &= \left[J\frac{\delta}{\delta x^i}, J\frac{\delta}{\delta x^j}\right] - J\left[\frac{\delta}{\delta x^i}, J\frac{\delta}{\delta x^j}\right] - J\left[J\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] - \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] \\ &= \left(J^h_i\frac{\partial J^k_j}{\partial y^h} - J^h_j\frac{\partial J^k_i}{\partial y^h} + R^k_{0ij}\right)\frac{\partial}{\partial y^k} \\ &\quad + \left(\frac{\delta}{\delta x^j}J^k_i - \frac{\delta}{\delta x^i}J^k_j + J^h_i\Gamma^k_{hj} - J^h_j\Gamma^k_{ih}\right)H^l_k\frac{\delta}{\delta x^l}. \end{aligned}$$

The coefficient of $H_k^l(\delta/\delta x^l)$ is just $\nabla_j J_i^k - \nabla_i J_j^k = 0$ so, we have to study the vanishing of the coefficient of $\partial/\partial y^k$. By using the expression (9) of J_i^k we get:

$$2t(u')^2 g_{0i}\delta_j^k - 2t(u')^2 g_{0j}\delta_i^k + R_{0ij}^k = 0.$$

It follows that the curvature tensor field of ∇ must have the expression

$$R_{hij}^k = c(\delta_i^k g_{hj} - \delta_j^k g_{hi}),$$

where c is a constant and the function $u(t)$ must satisfy the condition (12). Next, we have

$$N_J\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = H_i^h\left(\frac{\delta}{\delta x^h}J_j^k - \frac{\delta}{\delta x^j}J_h^k + \Gamma_{hl}^k J_j^l - \Gamma_{jl}^k J_h^l\right)\frac{\partial}{\partial y^k}$$
$$+ H_h^k H_i^l\left(-R_{0jl}^h + J_l^r\frac{\partial J_j^h}{\partial y^r} - J_j^r\frac{\partial J_l^h}{\partial y^r}\right)\frac{\delta}{\delta x^k}$$

and

$$N_J\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = H_i^k H_j^r\left(\frac{\delta}{\delta x^k}J_r^l - \frac{\delta}{\delta x^r}J_k^l + J_r^s\Gamma_{sk}^l - \Gamma_{rs}^l J_k^s\right)H_l^h\frac{\delta}{\delta x^h}$$
$$+ H_i^k H_j^l\left(-R_{0kl}^h - \frac{\partial J_l^h}{\partial y^r}J_k^r + \frac{\partial J_k^h}{\partial y^r}J_l^r\right)\frac{\partial}{\partial y^h}.$$

Hence, the condition $N_J(\delta/\delta x^i, \delta/\delta x^j) = 0$ implies that $N_J(\partial/\partial y^i, \delta/\delta x^j) = 0$ and $N_J(\partial/\partial y^i, \partial/\partial y^j) = 0$.

It follows that (M, g) must have constant positive sectional curvature and the function u must be a solution of the differential equation (12). The general solution of the differential equation (12) may be obtained easily. Since we look for a solution u defined for $t > 0$, for which $u > 0$, $u' > 0$, we may take

$$u' = \sqrt{\frac{c}{2t}}, \qquad u = \sqrt{2ct}. \tag{13}$$

It follows

$$L = \int u(t)\,dt = \frac{2t}{3}\sqrt{2ct}, \qquad w = -\frac{1}{4t\sqrt{2ct}}.$$

Remark that the Lagrangian $L = \int u(t)\,dt$ is smooth only on the nonzero tangent vectors of M. Hence we obtain, in fact, a Kaehler structure only on the manifold T_0M = the tangent bundle to M minus the null section.

4. The Levi-Civita connection of the metric G and its curvature tensor field

It is well known that in the case of the Kaehler manifolds (M, J, g) the almost complex structure operator J is parallel with respect to the Levi-Civita connection ∇ of the corresponding Riemannian metric g. We shall obtain the explicite expression of the Levi-Civita connection $\tilde{\nabla}$ of the metric G on TM in the general case where G is given by (7), then we shall consider the particular case where (T_0M, J, G) is Kaehler, i.e., M is a space form having positive constant sectional curvature c and the function $u(t)$ is a solution of the differential equation (12).

Recall that the Levi-Civita connection ∇ on a Riemannian manifold (M, g) is obtained from the formula

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X),$$
$$\forall X, Y, Z \in \Gamma(M).$$

We shall use this formula in order to obtain the expression of the Levi-Civita connection $\tilde{\nabla}$ on TM, determined by the conditions

$$\tilde{\nabla} G = 0, \quad \tilde{T} = 0,$$

where $\tilde{T}$ is the torsion tensor of $\tilde{\nabla}$.

Theorem 4. *The Levi-Civita connection $\tilde{\nabla}$ of G has the following expression in the local adapted frame* $(\partial/\partial y^1, \ldots, \partial/\partial y^n, \delta/\delta x^1, \ldots, \delta/\delta x^n)$:

$$\tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = \frac{1}{2} H^{hk}\left(\frac{\partial H_{jk}}{\partial y^i} + \frac{\partial H_{ik}}{\partial y^j} - \frac{\partial H_{ij}}{\partial y^k}\right)\frac{\partial}{\partial y^h},$$
$$\tilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \Gamma^h_{ij}\frac{\partial}{\partial y^h} + \frac{1}{2}\left(\frac{\partial G_{ik}}{\partial y^j} + H_{jl}R^l_{0ik}\right)G^{kh}\frac{\delta}{\delta x^h},$$
$$\tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = \frac{1}{2}\left(\frac{\partial G_{jk}}{\partial y^i} + H_{il}R^l_{0jk}\right)G^{kh}\frac{\delta}{\delta x^h},$$
$$\tilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma^h_{ij}\frac{\delta}{\delta x^h} + \frac{1}{2}\left(-R^h_{0ij} - \frac{\partial G_{ij}}{\partial y^k}H^{kh}\right)\frac{\partial}{\partial y^h}.$$

Consider now the case where (T_0M, J, G) has a structure of Kaehler manifold, i.e., M has positive constant sectional curvature c and the function u is given by (13). Introduce, for convenience, the following M-tensor fields on T_0M:

$$a_{ij} = g_{ij} - \frac{1}{2t} g_{0i}g_{0j}, \quad a^k_i = \delta^k_i - \frac{1}{2t} g_{0i}y^k. \tag{14}$$

Remark that we have $a_{i0} = a_{0i} = a_{ij}y^j = 0$ and $a_{ij} = a^k_i g_{kj} = a^k_i a_{kj}$. Then we obtain

$$\begin{aligned}
G_{ij} &= \sqrt{2ct}\, g_{ij} + \sqrt{\frac{c}{2t}}\, g_{0i}g_{0j} = \sqrt{2ct}\left(a_{ij} + \frac{1}{t}\, g_{0i}g_{0j}\right),\\
H_{ij} &= \frac{1}{\sqrt{2ct}} g_{ij} - \frac{1}{4t\sqrt{2ct}} g_{0i}g_{0j} = \frac{1}{\sqrt{2ct}}\left(a_{ij} + \frac{1}{4t}\, g_{0i}g_{0j}\right),\\
G^{jk} &= \frac{1}{\sqrt{2ct}} g^{jk} - \frac{1}{4t\sqrt{2ct}} y^j y^k,\\
H^{jk} &= \sqrt{2ct}\, g^{jk} + \sqrt{\frac{c}{2t}}\, y^j y^k,
\end{aligned} \tag{15}$$

and

Theorem 5. *The Levi-Civita connection of the Kaehler manifold (T_0M, J, G) is given by*

$$\tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = \left(-\frac{1}{4t} g_{0i}\delta_j^h - \frac{1}{4t} g_{0j}\delta_i^h + \frac{1}{8t^2} g_{0i}g_{0j}y^h\right)\frac{\partial}{\partial y^h},$$

$$\tilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \Gamma_{ij}^h \frac{\partial}{\partial y^h} + \left(\frac{1}{4t} g_{ij}y^h + \frac{1}{4t} g_{0j}\delta_i^h - \frac{1}{8t^2} g_{0i}g_{0j}y^h\right)\frac{\delta}{\delta x^h}$$

$$= \Gamma_{ij}^h \frac{\partial}{\partial y^h} + \left(\frac{1}{4t} a_{ij}y^h + \frac{1}{4t} \delta_i^h g_{0j}\right)\frac{\delta}{\delta x^h},$$

$$\tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = \left(\frac{1}{4t} g_{ij}y^h + \frac{1}{4t} g_{0i}\delta_j^h - \frac{1}{8t^2} g_{0i}g_{0j}y^h\right)\frac{\delta}{\delta x^h}$$

$$= \left(\frac{1}{4t} a_{ij}y^h + \frac{1}{4t} \delta_j^h g_{0i}\right)\frac{\delta}{\delta x^h},$$

$$\tilde{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma_{ij}^h \frac{\delta}{\delta x^h} - c\left(g_{ij}y^h + \delta_i^h g_{0j}\right)\frac{\partial}{\partial y^h}.$$

Then the expression of the operator J is given by

$$J\frac{\delta}{\delta x^i} = \left(\sqrt{2ct}\,\delta_i^k + \sqrt{\frac{c}{2t}} g_{0i}y^k\right)\frac{\partial}{\partial y^k}, \qquad J\frac{\partial}{\partial y^i} = \left(-\frac{1}{\sqrt{2ct}}\delta_i^k + \frac{1}{4t\sqrt{2ct}} g_{0i}y^k\right)\frac{\delta}{\delta x^k},$$

and it can be checked easily that $\tilde{\nabla} J = 0$.

Denote by K the curvature tensor field of the Levi-Civita connection $\tilde{\nabla}$ of the Riemannian metric G on T_0M, when (M, g) has positive constant sectional curvature c and the function $u(t)$ is given by (13). Then we get by a straightforward computation

Theorem 6. *The local coordinate expression of the curvature tensor field K of the Kaehler manifold (T_0M, J, G) is given in the adapted local frame $(\partial/\partial y^i, \delta/\delta x^i)$ by*

$$K\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k} = \frac{1}{4t}K_{kij}^h \frac{\partial}{\partial y^h}, \qquad K\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^k} = \frac{1}{4t}K_{kij}^h \frac{\delta}{\delta x^h},$$

$$K\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} = \frac{1}{4t}S_{kij}^h \frac{\delta}{\delta x^h}, \qquad K\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} = -\frac{c}{2}S_{kij}^h \frac{\partial}{\partial y^h}, \tag{16}$$

$$K\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} = \frac{c}{2}K_{kij}^h \frac{\partial}{\partial y^h}, \qquad K\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} = \frac{c}{2}K_{kij}^h \frac{\delta}{\delta x^h},$$

where we have denoted

$$K_{kij}^h = \frac{1}{c}\left\{R_{kij}^h - \frac{1}{2t}g_{0k}R_{0ij}^h + \frac{1}{2t}g_{lk}R_{0ij}^l y^h\right\} = a_i^h a_{jk} - a_j^h a_{ik},$$

$$\begin{aligned} S_{kij}^h &= g_{ik}\delta_j^h + g_{jk}\delta_i^h - \frac{1}{2t}\left[g_{0i}g_{jk}y^h + g_{0j}g_{ik}y^h + g_{0i}g_{0k}\delta_j^h + g_{0j}g_{0k}\delta_i^h\right] \\ &\quad + \frac{1}{2t^2}g_{0i}g_{0j}g_{0k}y^h \\ &= a_i^h a_{jk} + a_j^h a_{ik}. \end{aligned} \tag{17}$$

From the above formulas, we get by a straightforward computation that the local coordinate expression of the Ricci tensor $S(Y, Z) = \text{trace}(X \longrightarrow K(X, Y)Z)$ in the local frame adapted to the direct sum decomposition (1) is given by

$$\begin{aligned} &S\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = -\frac{1}{2t}\left[g_{ij} - \frac{1}{2t} g_{0i} g_{0j}\right] = -\frac{1}{2t} a_{ij}, \\ &S\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = -c\left[g_{ij} - \frac{1}{2t} g_{0i} g_{0j}\right] = -c a_{ij}, \\ &S\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = 0. \end{aligned} \tag{18}$$

Comparing the obtained expressions with the expressions (15) of the components of G we obtain:

Proposition 7. *The Kaehlerian manifold (T_0M, J, G) cannot be an Einstein manifold.*

From the expression (16) of K it follows also:

Proposition 8. *The Kaehlerian manifold (T_0M, J, G) cannot have constant holomorphic sectional curvature.*

Finally, we should like to study the covariant derivative of the curvature tensor field K with respect to $\tilde{\nabla}$. To do this, it is useful to compute the expressions of the derivatives of the M-tensor fields a_{ij} and a^i_k. The following formulas are obtained by a straightforward computation:

$$\frac{\delta}{\delta x^i} a_{jk} = \Gamma^h_{ij} a_{hk} + \Gamma^h_{ik} a_{jh}, \qquad \frac{\delta}{\delta x^i} a^k_j = \Gamma^h_{ij} a^k_h - \Gamma^k_{ih} a^h_j,$$
$$\frac{\partial}{\partial y^i} a_{jk} = -\frac{1}{2t} g_{0j} a_{ik} - \frac{1}{2t} g_{0k} a_{ij}, \qquad \frac{\partial}{\partial y^i} a^k_j = -\frac{1}{2t} a_{ij} y^k - \frac{1}{2t} a^k_i g_{0j}.$$

Then we have:

$$\begin{aligned} &\frac{\delta}{\delta x^i} K^h_{ljk} = -\Gamma^h_{ir} K^r_{ljk} + \Gamma^r_{il} K^h_{rjk} + \Gamma^r_{ij} K^h_{lrk} + \Gamma^r_{ik} K^h_{ljr}, \\ &\frac{\delta}{\delta x^i} S^h_{ljk} = -\Gamma^h_{ir} S^r_{ljk} + \Gamma^r_{il} S^h_{rjk} + \Gamma^r_{ij} S^h_{lrk} + \Gamma^r_{ik} S^h_{ljr}, \\ &\frac{\partial}{\partial y^i} K^h_{ljk} = -\frac{1}{2t}(a_{ir} K^r_{ljk} y^h + g_{0j} K^h_{lik} + g_{0k} K^h_{lji} + g_{0l} K^h_{ijk}), \\ &\frac{\partial}{\partial y^i} S^h_{ljk} = -\frac{1}{2t}(a_{ir} S^r_{ljk} y^h + g_{0j} S^h_{lik} + g_{0k} S^h_{lji} + g_{0l} S^h_{ijk}), \end{aligned}$$

and it follows that the components of the covariant derivative of K with respect to $\tilde{\nabla}$ are expressed as linear combinations of the components K^h_{ljk} and S^h_{ljk}. In fact, if we denote, for convenience

$$\frac{\delta}{\delta x^i} = \delta_i, \qquad \frac{\partial}{\partial y^i} = \partial_i,$$

we have

$$(\tilde{\nabla}_{\delta_i} K)(\delta_j, \delta_k)\delta_l = \frac{c^2}{2}\left\{-g_{0j}S^h_{lki} + g_{0k}S^h_{lji} + g_{0l}K^h_{ijk} - a_{ir}K^r_{ljk}y^h\right\}\partial_h,$$

$$(\tilde{\nabla}_{\partial_i} K)(\delta_j, \delta_k)\delta_l = \frac{c}{8t}\left\{-2g_{0i}K^h_{ljk} + 2g_{0j}K^h_{lki} - 2g_{0k}K^h_{lji} - 2g_{0l}K^h_{ijk} - a_{ir}K^r_{ljk}y^h\right\}\delta_h,$$

$$(\tilde{\nabla}_{\delta_i} K)(\delta_j, \delta_k)\partial_l = \frac{c}{8t}\left\{2g_{0j}S^h_{lki} - 2g_{0k}S^h_{lji} - g_{0l}K^h_{ijk} + a_{ir}K^r_{ljk}y^h\right\}\delta_h,$$

$$(\tilde{\nabla}_{\partial_i} K)(\delta_j, \delta_k)\partial_l = \frac{c}{8t}\left\{-2g_{0i}K^h_{ljk} + 2g_{0j}K^h_{lki} - 2g_{0k}K^h_{lji} - g_{0l}K^h_{ijk} - 2a_{ir}K^r_{ljk}y^h\right\}\partial_h,$$

$$(\tilde{\nabla}_{\delta_i} K)(\partial_j, \delta_k)\delta_l = \frac{c}{8t}\left\{g_{0j}K^h_{lki} + 2g_{0k}K^h_{lji} + 2g_{0l}S^h_{ijk} - a_{ir}S^r_{ljk}y^h\right\}\delta_h,$$

$$(\tilde{\nabla}_{\partial_i} K)(\partial_j, \delta_k)\delta_l = \frac{c}{8t}\left\{2g_{0i}S^h_{ljk} + g_{0j}S^h_{lki} + 2g_{0k}S^h_{lji} + 2g_{0l}S^h_{ijk} + 2a_{ir}S^r_{ljk}y^h\right\}\partial_h,$$

$$(\tilde{\nabla}_{\delta_i} K)(\partial_j, \delta_k)\partial_l = \frac{c}{8t}\left\{g_{0j}K^h_{lki} + 2g_{0k}K^h_{lji} + g_{0l}S^h_{ijk} - 2a_{ir}S^r_{ljk}y^h\right\}\partial_h,$$

$$(\tilde{\nabla}_{\partial_i} K)(\partial_j, \delta_k)\partial_l = \frac{1}{16t^2}\left\{-2g_{0i}S^h_{ljk} - g_{0j}S^h_{lki} - 2g_{0k}S^h_{lji} - g_{0l}S^h_{ijk} - a_{ir}S^r_{ljk}y^h\right\}\delta_h,$$

$$(\tilde{\nabla}_{\delta_i} K)(\partial_j, \partial_k)\delta_l = \frac{c}{8t}\left\{-g_{0j}S^h_{lki} + g_{0k}S^h_{lji} + 2g_{0l}K^h_{ijk} - a_{ir}K^r_{ljk}y^h\right\}\partial_h,$$

$$(\tilde{\nabla}_{\partial_i} K)(\partial_j, \partial_k)\delta_l = \frac{1}{16t^2}\left\{-2g_{0i}K^h_{ljk} + g_{0j}K^h_{lki} - g_{0k}K^h_{lji} - 2g_{0l}K^h_{ijk} - a_{ir}K^r_{ljk}y^h\right\}\delta_h,$$

$$(\tilde{\nabla}_{\delta_i} K)(\partial_j, \partial_k)\partial_l = \frac{1}{16t^2}\left\{g_{0j}S^h_{lki} - g_{0k}S^h_{lji} - g_{0l}K^h_{ijk} + a_{ir}K^r_{ljk}y^h\right\}\delta_h,$$

$$(\tilde{\nabla}_{\partial_i} K)(\partial_j, \partial_k)\partial_l = \frac{1}{16t^2}\left\{-2g_{0i}K^h_{ljk} + g_{0j}K^h_{lki} - g_{0k}K^h_{lji} - g_{0l}K^h_{ijk} - 2a_{ir}K^r_{ljk}y^h\right\}\partial_h.$$

Similar results are obtained for the components of the covariant derivative of the Ricci tensor field S:

$$(\tilde{\nabla}_{\delta_i} S)(\delta_j, \delta_k) = 0, \quad (\tilde{\nabla}_{\partial_i} S)(\delta_j, \delta_k) = \frac{c}{2t}\left(g_{0i}a_{jk} + g_{0j}a_{ik} + g_{0k}a_{ij}\right),$$

$$(\tilde{\nabla}_{\delta_i} S)(\partial_j, \delta_k) = \frac{c}{4t}\left(g_{0j}a_{ik} - 2g_{0k}a_{ij}\right), \quad (\tilde{\nabla}_{\partial_i} S)(\partial_j, \delta_k) = 0,$$

$$(\tilde{\nabla}_{\delta_i} S)(\partial_j, \partial_k) = 0, \quad (\tilde{\nabla}_{\partial_i} S)(\partial_j, \partial_k) = \frac{1}{8t^2}\left(2g_{0i}a_{jk} + g_{0j}a_{ik} + g_{0k}a_{ij}\right).$$

References

[1] E. Calabi, Métriques Kaehlériennes et fibrés holomorphes, *Ann. Scient. Ec. Norm. Sup.* **12** (1979) 269–294.

[2] P. Dombrowski, On the geometry of the tangent bundle, *J. Reine Angew. Mathematik* **210** (1962) 73–88.

[3] Gh. Gheorghiev and V. Oproiu, *Varietăţi Diferenţiabile Finit şi Infinit Dimensionale I, II* (Ed. Academiei Rom., 1976, 1979).

[4] J. Kern, Lagrange geometry, *Archiv. Math.* **25** (1974) 438–443.

[5] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry I, II* (Interscience, New York, 1963, 1969).

[6] O. Kowalski, Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold, *J. Reine Angew. Math.* (1970) (250) 124–129.
[7] K.P. Mok, E.M. Patterson and Y.C. Wong, Structure of symmetric tensors of type (0, 2) and tensors of type (1, 1) on the tangent bundle, *Trans. Am. Math. Soc.* **234** (1977) 253–278.
[8] V. Oproiu, On the differential geometry of the tangent bundles, *Rev. Roum. Math. Pures et Appl.* **13** (1968) 847–856.
[9] V. Oproiu, Some remarkable structures and connections defined on the tangent bundle, *Rend. Mat. Roma* **6** (1973) 503–540.
[10] V. Oproiu, A Riemannian structure in Lagrange geometry, *Rend. Sem. Fac. Sci. Univ. Cagliari* **55** (1985) (2) 1–20.
[11] V. Oproiu, A pseudo-Riemannian structure in Lagrange geometry, *An. St. Univ. "Al. I. Cuza" Iaşi* **33** (1987) 239–254.
[12] V. Oproiu and N. Papaghiuc, A pseudo-Riemannian metric on the cotangent bundle, *An. St. Univ. "Al. I. Cuza" Iaşi* **36** (1990) 265–276.
[13] V. Oproiu and N. Papaghiuc, Some examples of almost complex manifolds with Norden metric, *Publ. Math. Debrecen* **41** (1992) 199–212.
[14] S. Sasaki, On the differential geometry of the tangent bundle of Riemannian manifolds, *Tohoku Math. J.* **10** (1958) 238–354.
[15] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles* (M. Dekker, New York, 1973).
[16] K. Yano and S. Kobayashi, Prolongations of tensor fields and connections to tangent bundles, I. General theory, *Jour. Math. Soc. Japan* **18** (1966) 194–210.