Journal of Computational and Applied Mathematics 236 (2011) 916–923

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Quading triangular meshes with certain topological constraints[☆]

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ARTICLE INFO

Keywords:
Computational topology
Fundamental group
Tight orthogonal homotopic basis
Mesh quading
Quasi-regular quadrangulation
T-shirt

ABSTRACT

In computer graphics and geometric modeling, shapes are often represented by triangular meshes (also called 3D meshes or manifold triangulations). The quadrangulation of a triangular mesh has wide applications. In this paper, we present a novel method of quading a closed orientable triangular mesh into a quasi-regular quadrangulation, i.e., a quadrangulation that only contains vertices of degree four or five. The quasi-regular quadrangulation produced by our method also has the property that the number of quads of the quadrangulation is the smallest among all the quasi-regular quadrangulations. In addition, by constructing the so-called orthogonal system of cycles our method is more effective to control the quality of the quadrangulation.

1. Introduction

In computer graphics and geometric modeling, shapes are often represented by triangular meshes (also called 3D meshes or manifold triangulations). A *triangular mesh* is a surface (2-manifold) that is composed of triangles such that the intersection between any two triangles is their common vertex or their common edge if the intersection is not empty. A *quad* is a surface being homeomorphic to the unit squire $I_2 = [0, 1] \times [0, 1]$. For a quad S and a homeomorphism $\psi : S \to I_2$, $\psi^{-1}(v)$ and $\psi^{-1}(e)$ are called a vertex and an edge of S if v and e are a vertex and an edge of I_2, respectively. A *mesh quad* is a triangular mesh quad, i.e., a quad that is also a triangular mesh. A mesh quad is also called a quad if there is no confusion. Quading a surface is a process of decomposing the surface into a family of quads such that

i. The intersection of any two quads is the common boundaries, the common vertices, or empty.
ii. Any quad has four different vertices.
iii. Any two quads have at most two common edges and the two common edges are not adjacent if they have.

A quaded triangular mesh is called a *triangular mesh quadrangulation* or a quadrangulation if there is no confusion.

A triangular mesh is a standard surface representation and is widely supported by the graphics hardware. However, the original meshes are usually notoriously expensive to store, transmit and operate (such as mesh compression, remeshing, morphing, multi-resolution analysis, etc.). The quadrangulation of a triangular mesh is an effective method to overcome

☆ This paper is partially funded by ARO (P-54412-CI-ISP), DEPSCoR (W911NF-07-1-0422), and NSFC-Guangdong Joint Fund (U0735001, U0935004).
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0377-0427/$ – see front matter
doi:10.1016/j.cam.2011.05.009

these problems. Moreover, quaded meshes are preferred for many applications, such as numerical PDE, most CAD/CAM production softwares (which are based on tensor-product Non-Uniform Rational *B*-Splines (NURBS)), Catmull–Clark subdivision, surface parameterization and simplification, hierarchical representations of surfaces, texture mapping, etc.

The existing approaches for quading triangular meshes can be divided into three categories.

1. Advancing Frontier. Eck and Hoppe [1] proposed a scheme to construct a quadrangular base mesh by computing a maximal pairing over a triangular mesh. Shimada et al. [2] developed this method for planar finite element mesh generation. Owen et al. [3] enhanced it to the *Q*-Morph algorithm with an advancing front traversal.
2. Differential Geometry Based Methods. Alliez et al. [4] proposed an approach to generate quad-dominant meshes by computing integral lines of the two principal direction fields of the surface. Marikov and Kobbelt [5] developed a non-parametric setting. Dong et al. [6] used a harmonic scalar field over the surface to trace curves. Ray et al. [7] developed a similar quad-dominant method by using parametric functions to approximate the principal direction fields.
3. Topology Based Methods. Hetroy and Attali [8] proposed a quadrangulating scheme by using the Reeb graph of the surface. Bremer et al. [9] computed the minima, maxima and saddles of some Morse function on a surface to construct the quadrangulation called the "Morse–Smale complex".

One common disadvantage of the above methods is that they couldn not control the number of quads of the quadrangulation. Another disadvantage is that the *degree* of some vertices could be very high, where the degree of vertex of a quadrangulation is the number of quads sharing this vertex if this vertex is inner; otherwise the degree is the number of the quads plus one. The high degree vertices can result in unexpected geometric properties for the surfaces produced based on the quadrangulation of triangular meshes. For example, the geometric quality of a NURBS surface is heavily dependent on the degrees of vertices of the quadrangulation. Generally speaking, the higher the degree is, the more unexpected the geometric properties results, especially on high curvature locations. In addition, the above methods are also sensitive to noise. For example, the quadrangulation produced by the Morse–Smale complex method usually contains many unexpected quads, due to the fact that saddles of the Morse function are sensitive to noise.

In this paper, we present a novel approach to quad triangular meshes. Our method is distinguished from the existing ones in the following three features:

1. The quadrangulations we constructed are "almost regular" or quasi-regular, that is, for closed orientable surfaces with genus greater than 1, the degree of each vertex of the quadrangulations is four or five. Euler's formula shows that for a quadrangulation with a general genus, the maximum degree of its vertices cannot be smaller than five. Thus, our method is optimal from the degree point of view.
2. The number of the quads of the quadrangulation obtained by our method is the smallest among all the quasi-regular quadrangulations. This number is uniquely determined by the genius of the surface and not sensitive to noise.
3. By constructing the so-called orthogonal system of cycles our method is more effective to control the quality of the quadrangulations.

The meaning of the second property above is that in application we can first construct a rough quadrangulation of a triangular mesh with the number of quads as small as possible and then refine it if necessary. This way is very different from the current methods. The current methods of quading a triangular mesh usually produce unnecessarily many quads. In most cases, one have to merge those quads to fit the applications. It is obvious that refining a quadrangulation is much easier than merging a quadrangulation.

We would like to point out that in this paper the theoretical results hold for general surfaces, but the algorithms are only implemented for triangular mess surfaces.

2. Preliminaries

Throughout the remainder of the paper, we denote by M an orientable closed surface. The closure, interior and boundary of a set S are denoted by $\overline{S}$, $\mathrm{Int}(S)$ and ∂S, respectively.

A map $f : A \longrightarrow B$ between manifolds A and B is called an *interior homeomorphism*, if f is continuous and $f|_{\mathrm{Int}(A)} : \mathrm{Int}(A) \longrightarrow \mathrm{Int}(B)$ is homeomorphic. Two manifolds A and B are said to be *interiorly homeomorphic* if there is an interior homeomorphism between them.

An *arc* of M is a subset of M which is homeomorphic to the closed unit interval $[0, 1] \subset \mathbb{R}$. A *cycle* of M is a subset of M which is homeomorphic to the unit circle $S^1 = \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$. An open set R of M is called a *Jordan region*, or a *topological disk*, if R is homeomorphic to the unit disk $\mathbb{D} = \{(x, y) : x^2 + y^2 < 1\} \subset \mathbb{R}^2$. $\overline{R}$ is called a *closed disk* if $R \subset M$ is a disk. An open set Σ of M is called a *handle* of M, if it is homeomorphic to the cylinder surface $\{(x, y, z) : x^2 + y^2 = 1, -1 < z < 1\} \subset \mathbb{R}^3$. An open set Ψ of M is called a pair of *pants* of M, if it is homeomorphic to a sphere of cutting off three disjoint closed disks.

An open set Ψ of M is called a *T-shirt* of M, if it is homeomorphic to a sphere of cutting off four disjoint closed disks.

An *M-graph* G on M is a graph embedded on M without crossing edges. For convenience, we also call an M-graph a graph if it is not confusing. The set of all vertices and the set of all edges of G are denoted by $\mathcal{V}(G)$ and $\mathcal{E}(G)$, respectively. Each edge of G is either an arc or a cycle of M. G is called a *cutting graph*, if $M \setminus G$ is a Jordan region. For an edge e of an M-graph with end points v_1 and v_2, we define $\overrightarrow{e} = [v_1, v_2]$ a directed version of e, where v_1 is the start point and v_2 is the end point.

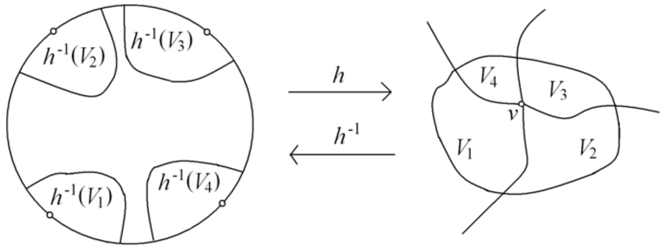


Fig. 1. The mapping of a vertex's Jordan region neighborhood.

The opposite direction is defined by $\overleftarrow{e} = [v_2, v_1]$. For the case that e is close, the directions of e can be defined similarly. If there is no confusion, we write $\overrightarrow{e} = [v_1, v_2]$ and $\overleftarrow{e} = [v_2, v_1]$ by e and e^{-1}, respectively.

We denote by $H(g)$ a g-torus, or a sphere with g handles. According to the classification theorem of surfaces, M is homeomorphic to $H(g)$, if g is the genus of M.

We denote by P_n the open regular n-gon in $\mathbb{R}^2$ with vertices $p_k = \left(\cos\frac{2k\pi}{n}, \sin\frac{2k\pi}{n}\right),\ k = 1, 2, \ldots, n$. For each pair of points $p_1, p_2 \in \mathbb{R}^2$, we denote by $\overline{p_1p_2}$ the closed line segment from p_1 to p_2.

Let G be a cutting graph of M with edges $e_1, \ldots, e_n$ and $h : \mathbb{D} \longrightarrow M \setminus G$ be a homeomorphism. We define $\widetilde{\mathbb{D}} = \overline{\mathbb{D}}/\sim$ the identification topology of $\overline{\mathbb{D}}$ by stitching the boundary of $\overline{\mathbb{D}}$ under the equivalence $\sim$, where for $v, w \in \partial\overline{\mathbb{D}}$, $v \sim w$ means that they corresponding to the same point of G. Then, $\widetilde{\mathbb{D}}$ is topologically equivalent to M. We denote by $\widetilde{h} : \widetilde{\mathbb{D}} \longrightarrow M$ the homeomorphism from $\widetilde{\mathbb{D}}$ to M induced by h. Finally, we denote by $\overline{h} : \overline{\mathbb{D}} \longrightarrow M$ the function defined by $\overline{h}(v) = \widetilde{h}(\{v\})$, where $\{v\} = \{w \in \overline{\mathbb{D}}; w \sim v\} \in \widetilde{\mathbb{D}}$ is the equivalent class of v.

An edge e of G is an *incident* edge of a vertex v of G if v is an end point of e.

Definition 1. Let G be a graph on M and v be a vertex of G. Let all the incident edges of v be $e_1, \ldots, e_h, e_{h+1}, \ldots, e_{h+k}$ where $e_1, \ldots, e_h$ are arcs and $e_{h+1}, \ldots, e_{h+k}$ are cycles. The *degree* of v with respect to G, denoted by $\deg(v) = \deg_G(v)$, is defined to be the number $h + 2k$.

Lemma 1. *Let G be a cutting graph of M with edges $e_1, \ldots, e_n$ and $h : \mathbb{D} \longrightarrow M \setminus G$ be a homeomorphism. Then the following statements hold:*

(1) *If $v \in \mathcal{V}(G)$ is a vertex of G. Let $d = \deg(v)$. Then, there are exactly d points $x_1, \ldots, x_d \in \partial\mathbb{D}$ such that $\overline{h}(x_k) = v,\ k = 1, \ldots, d$;*

(2) *If $u \in G$ is not a vertex of G. Then, there are exactly 2 points $x_1, x_2 \in \partial\mathbb{D}$ such that $\overline{h}(x_k) = u,\ k = 1, 2$.*

Proof. (1) For any point $v \in G$ and any Jordan region neighborhood V of v in M, V is divided by G into d components $V_1, \ldots, V_d$, where $d = \deg(v)$ if v is a vertex of G and $d = 2$ if v is not a vertex of G (see Fig. 1 for $d = 4$). Choose V small enough such that $\overline{h^{-1}(V_k)} \cap \overline{h^{-1}(V_h)} = \emptyset$ if $k \neq h$. For each $k \in \{1, \ldots, d\}$ we can select a sequence of points $\{u_n^k\}_{n=1}^{\infty} \subset V_k$ such that $u_n^k \to v$ as $n \to \infty$.

Set $x_n^k = h^{-1}(u_n^k) \in h^{-1}(V_k),\ n = 1, 2, \ldots$. Suppose that $\{y_n\}_{n=1}^{\infty}$ is a convergent subsequence of the bounded sequence $\{x_n^k\}_{n=1}^{\infty}$ and x^k is the limit of $\{y_n\}_{n=1}^{\infty}$. Then $x^k \in \partial h^{-1}(V_k) \cap \partial\mathbb{D}$. By definition,

$$\lim_{n\to\infty} \overline{h}(x_n^k) = \lim_{n\to\infty} h(h^{-1}(u_n^k)) = \lim_{n\to\infty} u_n^k = v.$$

On the other hand, by continuity,

$$\lim_{n\to\infty} \overline{h}(y_n) = \overline{h}(\lim_{n\to\infty} y_n) = \overline{h}(x^k).$$

Thus $\overline{h}(x^k) = v$. Since $\overline{h^{-1}(V_k)} \cap \overline{h^{-1}(V_h)} = \emptyset$, $x^k \neq x^h$ if $k \neq h$.

Denote by $\mathbb{D}_V = \bigcup_{i=1}^{d} \overline{h^{-1}(V_k)}$. Because $\widetilde{h}|_{\mathbb{D}_V} : \mathbb{D}_V/\sim \longrightarrow \overline{V}$ is a homeomorphism and $\overline{h^{-1}(V_k)},\ 1 \le k \le d$, are disjoint, $\overline{h}|_{\overline{h^{-1}(V_k)}} : \overline{h^{-1}(V_k)} \longrightarrow \overline{V}_k$ is also a homeomorphism.

If there is another point $x_{d+1} \in \overline{\mathbb{D}}$ such that $\overline{h}(x_{d+1}) = v$. Choose a sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{D}$ converging to x_{d+1}. Then

$$\lim_{n\to\infty} \overline{h}(z_n) = \overline{h}(x_{d+1}) = v.$$

Hence $h(z_n)$ is contained in the neighborhood V of v for sufficiently large n. This means that there is some $j \in \{1, \ldots, d\}$ such that there are infinitely many items in $\{h(z_n)\}_{n=1}^{\infty}$ contained in the component V_j, i.e., there is a subsequence $\{w_n\}_{n=1}^{\infty}$ of $\{h(z_n)\}_{n=1}^{\infty}$ contained in V_j. Hence $h^{-1}(w_n) \in h^{-1}(V_j),\ n = 1, 2, \ldots$. This yields $x_{d+1} = x_j$ and completes the proof of the Lemma. □

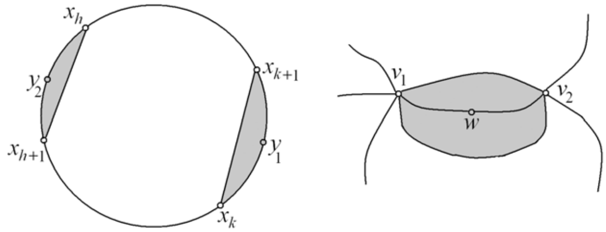


Fig. 2. Interior homeomorphism.

Lemma 2. *Suppose that G is a cutting graph of M with n edges $e_1, \ldots, e_n$. Let P_{2n} be a regular $2n$-gon and let $p_k,\ 1 \le k \le 2n$, be the vertices of P_{2n}. Then, there is an interior homeomorphism $\phi : \overline{P_{2n}} \longrightarrow M$ satisfying the following conditions:*

(1) *For each $k \in \{1, 2, \ldots, 2n\}$, there is an edge e of G such that $\phi|_{\overline{p_kp_{k+1}}} : \overline{p_kp_{k+1}} \longrightarrow e$ is interiorly homeomorphic, where $p_{2n+1} = p_1$;*
(2) *For any directed edge $[v_1, v_2]$ of G, there are two and only two segments $\overline{p_kp_{k+1}}$ and $\overline{p_hp_{h+1}}$ such that $\phi(\overline{p_kp_{k+1}}) = [v_1, v_2]$ and $\phi(\overline{p_hp_{h+1}}) = [v_2, v_1]$.*

Proof. Choose a homeomorphism $h : \mathbb{D} \longrightarrow M \setminus G$. h can be uniquely extended to a continuous map $\tilde{h} : \overline{\mathbb{D}} \longrightarrow M$ and $\tilde{h}$ is interiorly homeomorphic by definition.

Suppose that all the preimages of the vertices of G with respect to $\tilde{h}$ are labeled clockwise as $x_1, x_2, \ldots, x_m$. According to (1) of Lemma 1.1 we have

$$m = \sum_{v \in \mathcal{V}(G)} \deg(v) = 2n.$$

Denote by $\widehat{x_kx_{k+1}}$ the closed arc on $\partial\mathbb{D}$ from x_k to $x_{k+1},\ k = 1, 2, \ldots, 2n$, where $x_{2n+1} = x_1$. For any k given, there are $v_1, v_2 \in \mathcal{V}(G)$ such that $\tilde{h}(x_k) = v_1$ and $\tilde{h}(x_{k+1}) = v_2$. Since $\tilde{h}$ is continuous, $\tilde{h}(\widehat{x_kx_{k+1}})$ is compact and connected. Moreover, there is no vertex of G locating on $\tilde{h}(\widehat{x_kx_{k+1}})$ except v_1 and v_2. Thus $\tilde{h}(\widehat{x_kx_{k+1}})$ is an edge of G. Note that the compactness of $\widehat{x_kx_{k+1}}$ implies that $\tilde{h}|_{\widehat{x_kx_{k+1}}}$ is an open map. $\tilde{h}|_{\mathrm{Int}(\widehat{x_kx_{k+1}})} : \mathrm{Int}(\widehat{x_kx_{k+1}}) \to \tilde{h}(\mathrm{Int}(\widehat{x_kx_{k+1}}))$ is injective and hence a homeomorphism.

Given a directed edge $[v_1, v_2]$, select an interior point w on this edge. By (2) of Lemma 1.1 we can find two distinct points y_1 and y_2 of $\mathbb{D}$ such that $\tilde{h}(y_1) = \tilde{h}(y_2) = w$. Suppose that $y_1 \in \widehat{x_kx_{k+1}}$ and $y_2 \in \widehat{x_hx_{h+1}}$, as shown in Fig. 2. Since w is not a vertex of G, $\tilde{h}(\widehat{x_kx_{k+1}})$ and $\tilde{h}(\widehat{x_hx_{h+1}})$ must be the same edge of G. By symmetry we can assume that $\tilde{h}(x_k) = v_1$ and $\tilde{h}(x_{k+1}) = v_2$. Noting that $\tilde{h}(\overline{x_{k+1}x_k})$ and $\tilde{h}(\overline{x_hx_{h+1}})$ compose a loop on M, we must have $\tilde{h}(x_h) = v_2$ and $\tilde{h}(x_{h+1}) = v_1$.

It is easy to find a homeomorphism $g : \overline{P_{2n}} \to \overline{\mathbb{D}}$ with $g(p_k) = x_k,\ k = 1, 2, \ldots, 2n$. The composition $\phi = g \circ \tilde{h}$ is the map required. □

Definition 2. The directed edge sequence $\phi(\overline{p_kp_{k+1}}),\ k = 1, 2, \ldots, 2n$, denoted by B_G, is called a *virtual boundary* of M with respect to G, where ϕ and p_k are defined as in Lemma 1.2. The vertices of G are also called the vertices of B_G.

Since M is orientable, it is easy to see that B_G is composed, reordering if necessary, of $2n$ directed edges $e_1, e_2, \ldots, e_n, e_1^{-1}, e_2^{-1}, \ldots, e_n^{-1}$, where $\{e_1, e_2, \ldots, e_n\}$ is the set of all edges of G. We should note that e_k of B_G is directed and e_k of G is undirected. In fact, e_k of B_G is obtained from e_k of G by assigning a direction.

Definition 3. The continuous mapping $\phi : \overline{P_{2n}} \longrightarrow M$ defined in Lemma 1.2 is called a *generalized canonical mapping*, or a *generalized polygonal schema*.

Definition 4. A cycle Γ of M is *nontrivial*, if $M \setminus \Gamma$ is connected.

Definition 5. Let G_0 and G_1 be two graphs on M such that $p \in G_0 \cap G_1$. G_0 and G_1 are said to be *orthogonal* to p, if there is a homeomorphism $\phi : U \to \mathbb{D} := \{(x, y) : x^2 + y^2 < 1\}$, where U is a small enough neighborhood of p, such that the following statements hold:

(1) $\phi(G_0 \bigcap U) = \{(x, 0)| - 1 < x < 1\}$;
(2) $\phi(G_1 \bigcap U) = \{(0, y)| - 1 < y < 1\}$.

Note that (1) and (2) imply $\phi(p) = (0, 0)$.

Definition 6. A finite set of nontrivial cycles $\Gamma_1, \ldots, \Gamma_k$ is called an *orthogonal system* of cycles on M, if the following statements hold:

(1) there are points $p_1, \ldots, p_{k-1}$ such that $\Gamma_i \bigcap \Gamma_{i+1} = \{p_i\}$;
(2) $\Gamma_i \bigcap \Gamma_j = \emptyset$ if $|i - j| > 1$;
(3) Γ_i and Γ_{i+1} are orthogonal at p_i for $i = 1, \ldots, k - 1$;
(4) $M \setminus \bigcup_{i=1}^{k} \Gamma_i$ is connected.

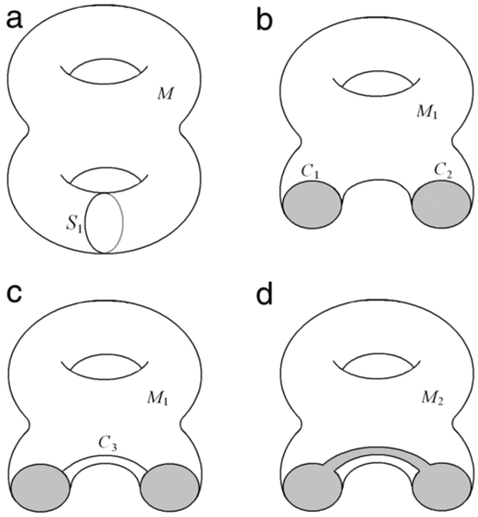


Fig. 3. A cutting graph of an orthogonal system.

Lemma 3. *Suppose M is an orientable closed surface with genus $g > 0$ and $\mathcal{S}_k = \{\Gamma_1, \ldots, \Gamma_k\}$ is an orthogonal system of cycles on M and $S_k = \Gamma_1 \cup \cdots \cup \Gamma_k$, where $k \le 2g$.*

(1) *If $k = 2h - 1$ is odd, then $M \setminus S_k$ is homeomorphic to $H(g - h)$ by resecting two closed disjoint disks.*
(2) *If $k = 2h$ is even, then $M \setminus S_k$ is homeomorphic to $H(g - h)$ by resecting a closed disk.*

Proof. We use induction to h. If $h = 1$ and $k = 1$, then $S_k = \Gamma_1$. It is easy to see that $M \setminus S_1$ is homeomorphic to a surface M_1, which is orientable and with two cycles C_1 and C_2 as its boundary, as shown in (b) of Fig. 3. We denote by $\widetilde{M}_1$ the surface obtained from M_1 by gluing two disks along C_1 and C_2. Then M is homeomorphic to the connected sum of $\widetilde{M}_1$ and a torus, where the connected sum of two surfaces is the surface that is formed by taking a disk out of each surface and connecting the two holes with a tube. Since M is homeomorphic to $H(g)$, $\widetilde{M}_1$ must be homeomorphic to $H(g - 1)$. Thus $M \setminus S_1$ is homeomorphic to $H(g - 1)$ of cutting two disks off.

If $h = 1$ and $k = 2$, then $S_k = \Gamma_1 \cup \Gamma_2$. By the definition of the orthogonality of two cycles, Γ_1 crosses Γ_2 to the point $\Gamma_1 \bigcap \Gamma_2$. This means that for a homeomorphism $\phi : M \setminus S_1 \to M_1$, Γ_2 is mapped to an arc C_3 connecting C_1 and C_2 as shown in (c) of Fig. 3. Thus, $M_1 \setminus C_3$ is homeomorphic to a surface M_2 which is homeomorphic to $H(g - 1)$ of cutting off a closed disk.

Now suppose that statements (1) and (2) of the lemma hold for some $h < g$.

For $k = 2h + 1$, according to the induction assumption $M \setminus S_{2h}$ is homeomorphic to a surface M_a which is homeomorphic to $H(g - h)$ by resecting a closed disk whose boundary is a cycle, say S. Let $\phi : M \setminus S_{2h} \to M_a$ be a homeomorphism. Then, ϕ maps Γ_{2h+1} to an arc C whose two end points, say p and q, are on S, as shown in (a) of Fig. 4.

We glue a disk along S to M_a and obtain a closed surface M_b. Then M_b is homeomorphic to $H(g - h)$. We choose an arc L connecting p and q in the disk, as shown in (b) of Fig. 4. It is clear that M_a is homeomorphic to $M_b \setminus L$. Thus $M \setminus S_{2h+1}$ is homeomorphic to $M_a \setminus C$, which is homeomorphic to $M_b \setminus (L \cup C)$. Note that M_b is homeomorphic to $H(g - h)$ and $L \cup C$ is a cycle on M_b. Using the statement (1) of the lemma for a surface with $g - h$ genus and $k = 1$, it is clear that $M_b \setminus (L \cup C)$ is homeomorphic to $H(g - h - 1)$ by cutting off two closed disjoint disks. Therefore, we conclude that the statement (1) of the lemma holds for $k = 2h + 1 = 2(h + 1) - 1$.

For $k = 2(h + 1)$, according to the mathematical induction, $M \setminus S_{2h+1}$ is homeomorphic to a surface M_c which is homeomorphic to $H(g - h - 1)$ by resecting two closed disjoint disks. Let $\phi : M \setminus S_{2h+1} \to M_c$ be a homeomorphism. Then ϕ maps Γ_{2h+2} to an arc C_3 connecting the two boundary cycles C_1 and C_2, as shown in (c) of Fig. 4. Thus $M_1 \setminus S_{2(h+1)}$ is homeomorphic to a surface M_d which is homeomorphic to $H(g - h - 1)$ by cutting one disk off, as shown in (d) of Fig. 4. Thus the statement (2) of the lemma holds for $k = 2h + 2 = 2(h + 1)$. □

Corollary 1. *Suppose $\mathcal{S} = \{\Gamma_1, \ldots, \Gamma_{2g}\}$ is an orthogonal system on M, where g is the genus of M. Then $\Gamma_1 \cup \cdots \cup \Gamma_{2g}$ is a cutting graph of M.*

Definition 7. A cutting graph G of M is said to be *orthogonal*, if there is an orthogonal system $\{\Gamma_1, \ldots, \Gamma_{2g}\}$ such that $G = \Gamma_1 \cup \cdots \cup \Gamma_{2g}$.

In the remainder of the paper, we always assume that the g-torus $H(g)$ $(g > 0)$ is a smooth surface embedded in $\mathbb{R}^3$.

Theorem 1. *Every close oriented surface with positive genus has an orthogonal cutting graph.*

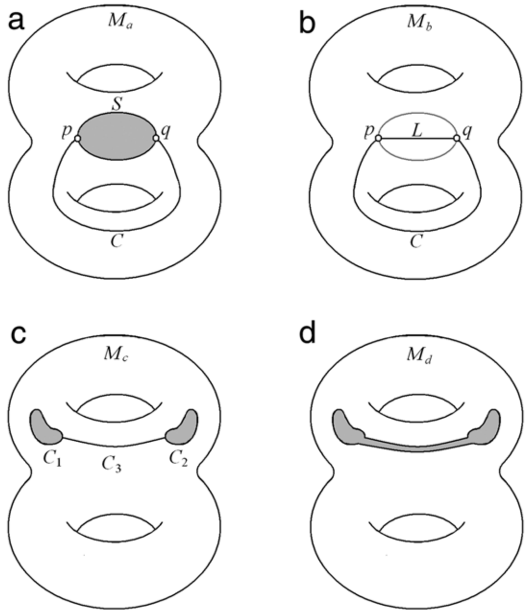


Fig. 4. Some pictures used in the proof of Lemma 3.

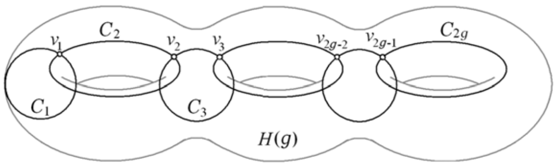


Fig. 5. Orthogonal cutting graph.

Proof. Suppose that M has genus g. It is easy to see that there is an orthogonal system $\{C_1, C_2, \ldots, C_{2g}\}$ of $H(g)$ such that $C_i \cap C_{i+1} = \{v_i\}$, as shown in Fig. 5.

Since M has genus g, M is homeomorphic to $H(g)$. Therefore, there exists a homeomorphism $\phi : H(g) \to M$. Setting $\Gamma_i = \phi(C_i),\ i = 1, 2, \ldots, 2g$, $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_{2g}\}$ is an orthogonal system on M and $B = \Gamma_1 \cup \cdots \cup \Gamma_{2g}$ is by definition an orthogonal virtual boundary of M. □

3. Quasi-regular quadrangulations

Definition 8. A quadrangulation $\mathcal{Q}$ of M is *quasi-regular* if all its vertices are of degree four or five.

Lemma 4. *Let $\mathcal{Q}$ be a quasi-regular quadrangulation of a closed surface M with genus $g > 0$. Then $\mathcal{Q}$ has exactly* $8(g-1)$ *vertices of degree five.*

Proof. Denote by V, E and Q the number of vertices, the number of edges and the number of quadrangles of $\mathcal{Q}$, respectively. Since each quadrangle has four edges and each edge is shared by two adjacent quadrangles, we have

$$E = 2Q.$$

Denote by V_4 and V_5 the number of vertices of degree four and the number of vertices of degree five, respectively. Then

$$V_4 + V_5 = V \quad \text{and} \quad 4V_4 + 5V_5 = 2E.$$

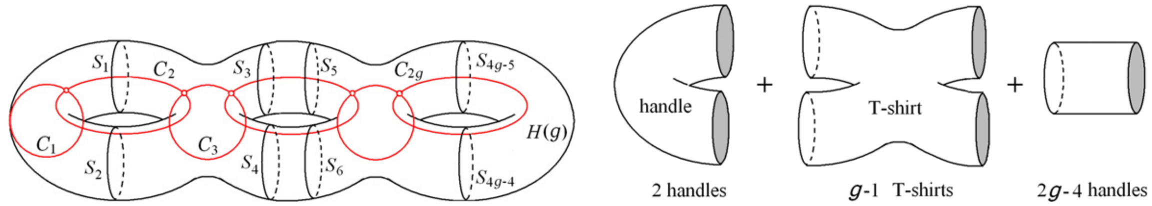
According to Euler's formula, we have

$$4(V_4 + V_5) - 2(4V_4 + 5V_5) + (4V_4 + 5V_5) = 8(1-g),$$

or

$$V_5 = 8(g-1). \quad \square$$

Corollary 2. *Every quasi-regular quadrangulation of a closed surface M contains at least* $10g - 10$ *quads, where $g > 1$ is the genus of M.*



(a) Cut the surface along S_is. (b) The surface is divided into handles and T-shirts.

Fig. 6. Handles and T-shirts.

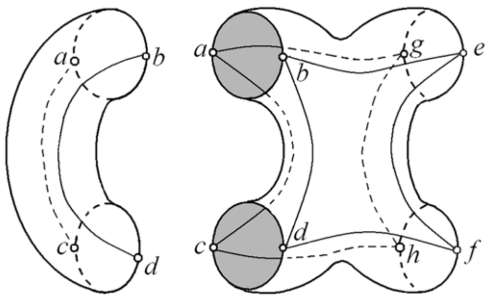


Fig. 7. Cutting handles and T-shirts into quads.

Proof. From $E = 2Q$, it holds that

$$Q = -(V - E + Q) + V = 2(g-1) + V \geq 2(g-1) + V_5 = 10g - 10. \quad \square$$

Lemma 5. *For $g > 1$, there exists a quasi-regular quadrangulation $\mathcal{Q}$ of $H(g)$ with $10g - 10$ quads.*

Proof. Let $\{C_1, C_2, \ldots, C_{2g}\}$ be an orthogonal system of $H(g)$. $H(g)$ can be divided into $2g-2$ handles and $g-1$ T-shirts, see Fig. 6.

Each T-shirt can be divided into 6 quads and each handle can be divided into two quads, as shown in Fig. 7. Hence, we obtain $2(2g-2) + 6(g-1) = 10g - 10$ quads. $\square$

Theorem 2. *Suppose M is a connected closed orientable surface with positive genus $g > 1$. Then, there is a quasi-regular quadrangulation $\mathcal{Q}$ of M with $10g - 10$ quadrangles.*

Proof. Since M is homeomorphic to $H(g)$, there exists a homeomorphism $h : H(g) \to M$. Thus, a quasi-regular quadrangulation $\mathcal{Q}$ of $H(g)$ induces a quasi-regular quadrangulation $h(\mathcal{Q})$ of M and the quads of $h(\mathcal{Q})$ are the images of the quads of $\mathcal{Q}$. $\square$

Remark 1. If $g = 1$, then it holds $V = Q$. Since any quadrangulation contains at least one quad with four different vertices, then any quadrangulation of M contains at least four quads if the genus of M is one. This kind of quadrangulation can be constructed by cutting M into four pieces.

4. Algorithms and implementations

In this section we give an algorithm for constructing quasi-regular quadrangulations as described in Theorem 2.5. To this end, we first use the method presented in [10] to construct an orthogonal cutting graph on M. Then, we use the following algorithm cutting M into handles and T-shirts.

Algorithm 1. Suppose that there is a given orthogonal system $\Gamma_1, \ldots, \Gamma_{2g}$ of M. Denote $v_i \in \Gamma_i \cap \Gamma_{i+1}$, $i = 1, \ldots, 2g-1$ and $G = \Gamma_1 \cup \cdots \cup \Gamma_{2g}$. Then, Γ_{2h}, $h = 1, \ldots, g$, is divided into two arcs by v_{2h-1} and v_{2h}, say $L_1 = L_{h,1}$ and $L_2 = L_{h,2}$, where $v_{2g} \in \Gamma_{2g}$ is additionally selected such that v_{2g-1} and v_{2g} evenly divided the length of Γ_{2g}.

(1) For $h = 1$, by choosing a point $u_i \in L_i (u_i \neq v_1, v_2,\ i = 1, 2)$, we respectively construct a cycle S_i such that $S_i \cap G = \{u_i\}$ and S_i is orthogonal with L_i to u_i. In addition, we ask $S_1 \cap S_2 = \emptyset$.
(2) Similarly, for $h = g$, let $u_i \in L_i (u_i \neq v_{2g-1}, v_{2g},\ i = 1, 2)$. Construct a cycle S_{4g-6+i} such that $S_{4g-6+i} \cap G = \{u_i\}$ and S_{4g-6+i} is orthogonal with L_i to u_i, $i = 1, 2$. Also, we ask $S_{4g-5} \cap S_{4g-4} = \emptyset$.
(3) For $1 < h < g$, by choosing two points $u_{4h-5}, u_{4h-3} \in L_1$ other then v_{2h-1} and v_{2h}, we construct two cycles S_i, $i = 4h-5, 4h-3$, such that $S_i \cap G = \{u_i\}$ and S_i is orthogonal with L_1 to u_i. Similarly, we choose $u_i \in L_2$, $i = 4h-4, 4h-2$ and construct two cycles S_i such that $S_i \cap G = \{u_i\}$ and S_i is orthogonal to L_2 at u_i. We still ask that $S_i \cap S_j = \emptyset$ if $i \neq j$.

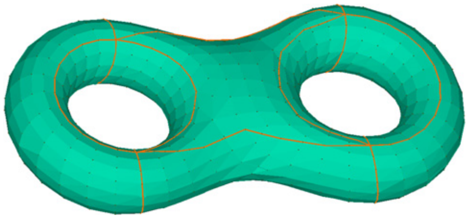


Fig. 8. Quadrangulation.

Now we have $4g-4$ cycles S_i, $i = 1, \ldots, 4g-4$ on M. These cycles divide M into $2g-2$ handles and $g-1$ T-shirts. To complete the quadrangulation we divide each handle into two quads and each T-shirts into six quads as shown in Fig. 7.

We implement the algorithm for triangular meshes with a half-linked edge list data structure. The quadrangulation can be obtained in $O(n)$ time, where n is the numbers of the triangles of the surface mesh. Fig. 8 shows a numerical result.

Remark 2. When constructing an orthogonal cutting graph with Algorithm 4.1, we have already considered some geometric optimizations to construct the quadrangulation.

References

[1] M. Eck, H. Hoppe, Automatic reconstruction of *B*-spline surfaces of arbitrary topological type, in: SIGGRAPH'96 Proc., 1996, pp. 325–333.
[2] K. Shimada, J.H. Liao, T. Itoh, Quadrilateral meshing with directionality control through the packing of square cells, in: Proc. 7th Int'l Meshing Roundtable, 1998, pp. 61–76.
[3] S. Owen, M. Staten, S. Canann, et al., *Q*-Morph: an indirect approach to advancing front quad meshing, Internat. J. Numer. Methods Engrg. 44 (1999) 1317–1340.
[4] P. Alliez, D. Cohen-Steiner, O. Devillers, B. Lévy, M. Desbrun, Anisotropic polugonal remeshing, ACM Trans. Graph. (2003) 485–493.
[5] M. Marinov, L. Kobbelt, Direct anisotropic quad-dominant remeshing, in: Proc. Pacific Graphics 2004, 2004, pp. 207–216.
[6] S. Dong, S. Kircher, M. Galand, Harmonic functions for quadrilateral remeshing of arbitrary manifolds, Comput. Aided Geom. Design 22 (5) (2005) 392–423.
[7] N. Ray, W. Li, B. Lévy, A. Sheffer, P. Alliez, Periodic global parameterization, ACM Trans. Graph. 25 (4) (2006) 1460–1485.
[8] F. Hétroy, D. Attali, Topological quadrangulations of closed triangulated surfaces using the Reeb graph, Graph. Models 65 (1–3) (2003) 131–148.
[9] P. Bremer, H. Edelsbrunner, B. Hamann, V. Pascucci, A multi-resolution data structure for two-dimensional Morse functions, in: Proc. IEEE Conf. Visualization 03', 2003, pp. 139–146.
[10] Shugao Xia, Xiquan Shi, Fengshan Liu, Zhixun Su, The tight orthogonal homotopic bases of closed oriented triangulated surfaces and their computing, Manuscript.