Hilbert Modular Threefolds of Arithmetic Genus One

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In 1981, Weisser proved that there are exactly four Galois cubic number fields with Hilbert modular threefolds of arithmetic genus one. In this paper, we extend Weisser’s work to cover all cubic number fields. Our main result is that there are exactly 33 fields with Hilbert modular threefolds of arithmetic genus one. These fields are enumerated explicitly. © 2002 Elsevier Science (USA)

Key Words: threefolds; varieties; modular; arithmetic genus; Hilbert; rational.

1. INTRODUCTION

Hilbert modular varieties were first studied by Blumenthal [1] in 1903. The varieties of dimension two received extensive attention in the 1970s culminating in a complete classification of these surfaces by Hirzebruch and van de Ven [6] and Hirzebruch and Zagier [7]. Of interest here is their result that exactly ten Hilbert modular surfaces have arithmetic genus one. In 1981, Weisser [10] considered Hilbert modular varieties of Galois cubic fields and proved that exactly four of these threefolds have arithmetic genus equal to one. In this paper, the restriction to Galois fields is dropped. It was shown previously [2,10] that there are at least 28 Hilbert modular threefolds of arithmetic genus one. Here we show that there are exactly 33 such threefolds.

Let \( k \) be a totally real algebraic number field of degree \( n \) over \( \mathbb{Q} \) with ring of integers \( \mathcal{O}_k \). For \( i = 1, \ldots, n \) and \( a \in k \), let \( a \mapsto a^{(i)} \) be the \( i \)th embedding of \( k \) into the real numbers. The Hilbert modular group of the field \( k \) is defined by \( \Gamma_k = \text{PSL}_2(\mathcal{O}_k) = \text{SL}_2(\mathcal{O}_k)/\{ \pm I \} \). The group \( \text{SL}_2(\mathcal{O}_k) \) acts on \( \mathcal{H}^n \), the product of \( n \) copies of the complex upper half-plane, by

\[
M_z = \left( \frac{a^{(1)}z_1 + b^{(1)}}{c^{(1)}z_1 + d^{(1)}}, \frac{a^{(2)}z_2 + b^{(2)}}{c^{(2)}z_2 + d^{(2)}}, \ldots, \frac{a^{(n)}z_n + b^{(n)}}{c^{(n)}z_n + d^{(n)}} \right),
\]

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where \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathcal{O}_k), \) and \( z = (z_1, z_2, \ldots, z_n) \in \mathcal{H}^n. \) This gives a well-defined, effective action of \( \Gamma_k \) on \( \mathcal{H}^n. \)

The quasi-projective complex variety \( \Gamma_k \backslash \mathcal{H}^n \) is compactified in a natural way by adding \( h_k \) points where \( h_k \) is the class number of \( k. \) The resulting compact variety can be desingularized to form a compact, non-singular complex variety called the Hilbert modular variety of \( k. \) See [4, 8] for a more detailed discussion.

The arithmetic genus, in the sense of Hirzebruch [5], is a birational invariant useful in the classification of non-singular varieties. For a variety to be rational (birationally equivalent to complex projective space), it must have arithmetic genus equal to one. So the Hilbert modular varieties with arithmetic genus one are the only possible candidates for rationality.

The following theorem, proved in [3], provides an upper bound on the discriminant of a cubic field (Galois or non-Galois) with arithmetic genus of zero or one.

**Theorem 1.** If \( k \) is a totally real cubic number field with discriminant at least 75,125, the Hilbert modular variety of \( k \) has arithmetic genus less than zero.

Hence it suffices to compute the arithmetic genus for each cubic field of discriminant less than or equal to 75,125. A list of these fields along with relevant information about their rings of integers, etc., can be found in the database established by the Computational Number Theory Group in Bordeaux (ftp://megrez.math.u-bordeaux.fr/pub/numberfields). In the following section, we describe the details of the computations done on each of these fields.

2. **COMPUTATIONS**

The starting point for the computation is Lemma 2, below, which is derived from the work of Hirzebruch [4] and Vignéras [9]. Let \( \zeta_k \) be the Dedekind zeta function of \( k \) and \( a_r \) be the number of equivalence classes of elliptic fixed points in \( \mathcal{H}^n \) under the action of \( \Gamma_k \) with isotropy subgroup of order \( r. \)

**Lemma 2.** For a totally real algebraic number field \( k \) of degree \( n \) over \( \mathbb{Q} \) containing a unit of norm \(-1\), the arithmetic genus of the Hilbert modular variety of the field \( k \) is given by

\[
\chi(k) = 2^{-n} \left( 2^{\zeta_k(-1)} + \sum_{r \geq 2} \frac{(r - 1)}{r} a_r \right).
\]
This formula clearly applies to every cubic number field since $-1$ is a unit of norm $-1$.

The values of the $a_r$’s can be determined using the following theorem proved in [2]. Further, the theorem shows that, except for two special fields for which the values are already known, the only non-zero values are $a_2$ and $a_3$.

Let $h(\ell)$ be the relative class number, $h(\ell) = h_k\left(\sqrt{\ell}\right) / h_k$, $\ell = -1, -3$. For a prime ideal $p$ in $\mathcal{O}_k$, let $(k'/p)$ be the Artin symbol, that is, $1$ if $p$ splits in $k'$, $0$ if $p$ ramifies in $k'$, and $-1$ if $p$ remains prime in $k'$.

**Theorem 3.** Let $k$ be a totally real cubic number field with discriminant $d_k$.
If $d_k \neq 49$ or $81$, then

$$If \ 2 \nmid d_k, \ then \ a_2 = 4h(-1).$$

$$If \ 2 \mid d_k, \ let \ p \ be \ the \ prime \ in \ \mathcal{O}_k \ ramified \ over \ (2), \ then$$

$$a_2 = \begin{cases} 
16h(-1) & \text{if } (k(i)/p) = 1; \\
12h(-1) & \text{if } (k(i)/p) = 0; \\
40h(-1) & \text{if } (k(i)/p) = -1.
\end{cases}$$

$$If \ 3 \nmid d_k, \ then \ a_3 = 4h(-3).$$
$$If \ 3 \mid d_k, \ let \ q \ be \ the \ prime \ in \ \mathcal{O}_k \ ramified \ over \ (3), \ then$$

$$a_3 = \begin{cases} 
12h(-3) & \text{if } (k(\sqrt{-3})/q) = 1; \\
16h(-3) & \text{if } (k(\sqrt{-3})/q) = 0; \\
20h(-3) & \text{if } (k(\sqrt{-3})/q) = -1.
\end{cases}$$

If $d_k = 49$, then $a_2 = a_3 = a_7 = 4$.
If $d_k = 81$, then $a_2 = a_3 = a_9 = 4$.
For all other values of $r$, $a_r = 0$.

Theorem 3 reduces the calculation of a value of $a_2$ or $a_3$ to a factorization in $k'$ and a calculation of a relative class number. Both of these were accomplished using KASH (http://www.math.tu-berlin.de/~kant/kash.html). The values of the Dedekind zeta function were calculated using PARI-GP (http://www.parigp-home.de/, ftp://megrez.math.u-bordeaux.fr/pub/pari/). These results were then combined as in Lemma 2 to produce the arithmetic genera.
3. NUMERICAL RESULTS

Table I gives the arithmetic genus and other constants for the Hilbert modular varieties of the 50 totally real cubic fields of smallest discriminant. For each field \( k \), the table supplies the discriminant, the value of the Dedekind zeta function of the field evaluated at \(-1\), the number of equivalence classes of elliptic fixed points of orders 2 and 3, and finally the arithmetic genus of the Hilbert modular variety over \( k \). It should be noted that the first two fields are special cases in which additional \( a_r \)'s are non-zero. Values of these can be found in Theorem 3.

As seen in Table I, exactly 33 of these 50 varieties have arithmetic genus equal to one. The computations described in the previous section completes the proof that these are in fact all Hilbert modular threefolds of arithmetic genus one.

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