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Tight Subgroups in Almost Completely Decomposable Groups

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1. INTRODUCTION

A *completely decomposable* group is a direct sum of groups isomorphic to subgroups of the additive group of rational numbers. An *almost completely decomposable group* X is a finite essential (abelian) extension of a completely decomposable group A of finite rank. In 1974, Lady [Lad74] initiated a systematic theory of such groups based on the fundamental concept of *regulating subgroup*. The regulating subgroups can be defined as the completely decomposable subgroups of least index in an almost completely decomposable group. Details on the subsequent developments can be found in the survey article [Mad95] or in the monograph [Mad99].

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The present paper deals with the class TSgps(X) of all completely decomposable subgroups in an almost completely decomposable group X that are maximal with respect to inclusion. These will be called *tight subgroups* of X adopting a term coined by Faticoni and Vinsonhaler. Regulating subgroups are examples of tight subgroups. Among the major results of this paper is a method for producing all completely decomposable subgroups of finite index in X from a given regulating subgroup (Proposition 2.4); a characterization of tight subgroups (Proposition 2.7); examples of groups with tight subgroups that are not regulating (Example 2.13); a proof that an almost completely decomposable group without non-regulating tight subgroups must have a regulating regulator (Corollary 3.4); a determination of the intersection of all tight subgroups (Corollary 3.3), a proof that all prime divisors of the index of a tight subgroup divide the regulating index (Corollary 2.10). We further show that tight subgroups with elementary quotient are necessarily regulating (Theorem 4.5), and use this result to give a new proof of a theorem of Burkhardt (Theorem 4.7).

The purification of a subgroup H in a torsion-free group G is denoted by H_*^G for emphasis. We take it for granted that the reader is familiar with the usual type subgroups $G[\tau]$, $G(\tau)$, $G^*(\tau)$, and $G^\sharp(\tau) = G^*(\tau)_*$. If A is a completely decomposable group, then $A = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(A)} A_\rho$ is assumed to be a decomposition of A into ρ-homogeneous components $A_\rho \neq 0$, so that $\mathrm{T}_{\mathrm{cr}}(A)$ is the *critical typeset* of A. For background on torsion-free abelian groups and almost completely decomposable groups see [Arn82], [Mad95], and [Mad99]. Maps are written on the right. The set inclusion symbol $\subset$ allows equality.

2. TIGHT SUBGROUPS OF ALMOST COMPLETELY DECOMPOSABLE GROUPS

Let X be an almost completely decomposable group. For each type τ there are one or more *Butler decompositions* $X(\tau) = A_\tau \oplus X^\sharp(\tau)$ where the τ-*Butler complement* A_τ turns out to be τ-homogeneous completely decomposable. The *critical typeset* $\mathrm{T}_{\mathrm{cr}}(X)$ of X consists of all those types τ for which $X(\tau) \neq X^\sharp(\tau)$. The subgroups $\sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ have finite index in X and are direct sums: $A = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ and hence completely decomposable groups. Note that $\mathrm{T}_{\mathrm{cr}}(A) = \mathrm{T}_{\mathrm{cr}}(X)$. The subgroups A are the *regulating subgroups* and Lady has shown that they are the completely decomposable subgroups of X of smallest index $\operatorname{rgi} X$ (*regulating index*). The regulating subgroups are special members of the set TSgps(X) consisting of all completely decomposable subgroups of X that are maximal with respect to containment. We follow Faticoni and Vinsonhaler (personal communication) in calling such subgroups *tight* in X. While regulating subgroups are tight, the converse need not be true (Example 2.13). We observe first that tight subgroups have finite index.

LEMMA 2.1. *Every tight subgroup of an almost completely decomposable group has finite index.*

Proof. We use induction on rank. Since almost completely decomposable groups with linearly ordered critical typesets are necessarily completely decomposable, the claim is true for small ranks. Let X be an almost completely decomposable group and assume that the claim holds for almost completely decomposable groups of rank less than rk X. Let A be a tight subgroup of X. Choose a minimal critical type μ and observe that the index $[X : (X(\mu) + X[\mu])]$ is finite. Therefore it suffices to show that $[X(\mu) : A(\mu)]$ and $[X[\mu] : A[\mu]]$ are both finite. This is the case by induction unless $X = X(\mu)$ or $X = X[\mu]$. The latter case cannot occur since $X[\mu] = \left(\oplus_{\rho\neq\mu} A_\rho\right)_*^X$, μ being minimal in $\mathrm{T}_{\mathrm{cr}}(X)$, and $A_\mu \neq 0$. In the former case, $X = X_\mu \oplus X^\sharp(\mu)$ and $A = A_\mu \oplus A^\sharp(\mu)$ since obviously $\mathrm{T}_{\mathrm{cr}}(A) \subset$ typeset(X). Certainly X/A is a torsion group and it follows from

$$\frac{X}{A} = \frac{X_\mu \oplus X^\sharp(\mu)}{A_\mu \oplus A^\sharp(\mu)}$$

by an easy rank calculation that $\operatorname{rk} A_\mu = \operatorname{rk} X_\mu$ and hence $X_\mu \cong A_\mu$. Now consider the short exact sequence

$$\frac{A + X^\sharp(\mu)}{A} \rightarrowtail \frac{X}{A} \twoheadrightarrow \frac{X + X^\sharp(\mu)}{A + X^\sharp(\mu)}.$$

The left end is isomorphic to $X^\sharp(\mu)/A \cap X^\sharp(\mu) = X^\sharp(\mu)/A^\sharp(\mu)$ and finite by induction. To show that the right end of the short exact sequence is also finite, and so finishing the proof, set $Y = A_\mu \oplus X^\sharp(\mu)$. Then

$$\frac{X + X^\sharp(\mu)}{A + X^\sharp(\mu)} = \frac{X_\mu \oplus X^\sharp(\mu)}{A_\mu \oplus X^\sharp(\mu)} = \frac{X_\mu \oplus X^\sharp(\mu)}{(Y \cap X_\mu) \oplus X^\sharp(\mu)} \cong \frac{X_\mu}{X_\mu \cap Y}.$$

Since $Y = A_\mu \oplus X^\sharp(\mu) = (X_\mu \cap Y) \oplus X^\sharp(\mu)$, it follows that $X_\mu \cap Y \cong A_\mu \cong X_\mu$ and therefore ([Mad99, Proposition 2.3.1]) the right-hand end of the short exact sequence is also finite. □

We are mainly interested in tight and regulating subgroups but it is possible to give a description of all completely decomposable subgroups of finite index, and we will do this first.

LEMMA 2.2. *Let X be an almost completely decomposable group. Suppose that for each $\tau \in \mathrm{T}_{\mathrm{cr}}(X)$ a τ-homogeneous completely decomposable group C_τ is chosen such that $X(\tau) \geq C_\tau \oplus X^\sharp(\tau)$ and $k_\tau X(\tau) \subset C_\tau \oplus X^\sharp(\tau)$ for some positive integer k_τ. Put $C = \sum_{\rho\in\mathrm{T}_{\mathrm{cr}}(X)} C_\rho$. Then there is an integer k such that $kX \subset C$ and $C = \bigoplus_{\rho\in\mathrm{T}_{\mathrm{cr}}(X)} C_\rho$,* i.e., *$C$ is the direct sum of the groups C_τ.*

Proof. We can start with a regulating subgroup $A = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(A)} A_\rho$ that is known to have finite index in X and replace one by one the homogeneous components A_τ by C_τ. The following Lemma 2.3 then completes the proof. □

LEMMA 2.3. *Let $C = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(C)} C_\rho$ be a completely decomposable subgroup of the almost completely decomposable group X such that $\gamma X \subset C$ for some positive integer γ. Suppose that C'_τ is a τ-homogeneous completely decomposable subgroup of X disjoint from $X^\sharp(\tau)$ and δ a positive integer such that $\delta(C_\tau \oplus X^\sharp(\tau)) \subset C'_\tau \oplus X^\sharp(\tau)$. Then $C' := C'_\tau \oplus \bigoplus_{\rho \neq \tau} C_\rho$ is completely decomposable and $\delta\gamma^2 X \subset C'$.*

Proof. We have

$$\delta\gamma^2 X \subset \delta\gamma C \subset \gamma(\delta(C_\tau \oplus X^\sharp(\tau)) + \bigoplus_{\rho \neq \tau} C_\rho$$
$$\subset \gamma(C'_\tau \oplus X^\sharp(\tau)) + \bigoplus_{\rho \neq \tau} C_\rho \subset C'_\tau + C^\sharp(\tau) + \bigoplus_{\rho \neq \tau} C_\rho = C'.$$

The intersection $C'_\tau \cap \bigoplus_{\rho \neq \tau} C_\rho$ must be trivial since $\mathrm{rk}\, C' = \mathrm{rk}\, X$, and hence C' is completely decomposable. □

The situation of Lemma 2.2 arises if one selects, for each critical type τ, a subgroup C_τ of $X(\tau)$ that is maximal disjoint from $X^\sharp(\tau)$. Then clearly $\mathrm{rk}\,(C_\tau) = \mathrm{rk}\,(X(\tau)/X^\sharp(\tau))$ and C_τ is pure in $X(\tau)$. This implies that C_τ is τ-homogeneous and $(C_\tau \oplus X^\sharp(\tau))/X^\sharp(\tau) \cong C_\tau$ is also τ-homogeneous of maximal rank in the τ-homogeneous completely decomposable group $X(\tau)/X^\sharp(\tau)$. By a theorem of Baer/Kolettis ([Fuc73, Theorem 86.6]), $(C_\tau \oplus X^\sharp(\tau))/X^\sharp(\tau)$ is completely decomposable. Therefore it is isomorphic with $X(\tau)/X^\sharp(\tau)$ and thus, being a monomorphic image of $X(\tau)/X^\sharp(\tau)$, has finite index in $X(\tau)/X^\sharp(\tau)$. Thus $C_\tau \oplus X^\sharp(\tau)$ has finite index in $X(\tau)$ and the groups C_τ satisfy the hypotheses of Lemma 2.2.

The following proposition describes how all completely decomposable subgroups of finite index can be obtained in terms of some given regulating subgroup.

PROPOSITION 2.4. *Fix a regulating subgroup $A = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ of the almost completely decomposable group X. Suppose that $C = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} C_\rho$ is a completely decomposable subgroup of finite index in X, say $eX \subset C$. Then there exist subgroups A'_τ of A_τ such that $eA_\tau \subset A'_\tau$ and functions $\phi_\tau :\ A'_\tau \to X^\sharp(\tau)$ such that $C_\tau = A'_\tau(1 + \phi_\tau)$. Conversely, given a positive integer e, groups A'_τ such that $eA_\tau \leq A'_\tau \leq A_\tau$, and maps $\phi_\tau \in \mathrm{Hom}(A'_\tau, X^\sharp(\tau))$, then the group $C' = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A'_\rho(1 + \phi_\rho)$ is completely decomposable and has finite index in X.*

Proof. Let $\pi_\tau : X(\tau) \to A_\tau$ be the projection along $X^\sharp(\tau)$. Then $eA_\tau \subset C(\tau) = C_\tau \oplus C^\sharp(\tau)$, hence $eA_\tau = eA_\tau\pi_\tau \subset C_\tau\pi_\tau \subset A_\tau$. Suppose $x \in C_\tau$ and $x\pi_\tau = 0$. Then $x \in C_\tau \cap X^\sharp(\tau) = 0$, so $\pi_\tau : C_\tau \to A_\tau$ is injective. Let $A'_\tau = C_\tau\pi_\tau$. Then $\phi_\tau : A'_\tau \to X^\sharp(\tau) : \phi_\tau = \pi_\tau^{-1}(1-\pi_\tau)$ is well defined and $A'_\tau(1+\phi_\tau) = C_\tau$.

Conversely assume that $eA_\tau \leq A'_\tau \leq A_\tau$ and $\phi_\tau \in \mathrm{Hom}(A'_\tau, X^\sharp(\tau))$. Clearly A'_τ is a τ-homogeneous completely decomposable group and therefore $A'_\tau(1+\phi_\tau)$ is also τ–homogeneous completely decomposable. We will show that $A'_\tau(1+\phi_\tau) \cap X^\sharp(\tau) = 0$ and that $A'_\tau(1+\phi_\tau) \oplus X^\sharp(\tau)$ has finite index in $X(\tau)$. For the first claim suppose that $a(1+\phi_\tau) \in X^\sharp(\tau)$, $a \in A'_\tau$. Then $a \in A_\tau \cap X^\sharp(\tau) = 0$ and $a(1+\phi_\tau) = 0$. For the second claim let $x = a + y \in X(\tau)$, $a \in A_\tau$, $y \in X^\sharp(\tau)$. Then $ex = ea + ea\phi_\tau - ea\phi_\tau + ey = ea(1+\phi_\tau) + (ey - ea\phi_\tau) \in A'_\tau(1+\phi_\tau) \oplus X^\sharp(\tau)$. The rest is a consequence of Lemma 2.2. □

Changing the subject somewhat we record some obvious conditions that a tight subgroup must satisfy.

LEMMA 2.5. *Let A be tight in the almost completely decomposable group X, and let $A = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ be a homogeneous decomposition of A. Then the following hold.*

(1) *If S is a linearly ordered subset of $\mathrm{T}_{\mathrm{cr}}(X)$, then $\bigoplus_{\rho \in S} A_\rho$ is pure in X.*

(2) *If $\sigma, \tau \in \mathrm{T}_{\mathrm{cr}}(X)$ and $\sigma \cup \tau = \mathbb{Q}$, then $A_\sigma \oplus A_\tau$ is pure in X.*

Proof. (1) Since S is linearly ordered, $\left(\bigoplus_{\rho \in S} A_\rho\right)_*^X$ is completely decomposable, so $\left(\bigoplus_{\rho \in S} A_\rho\right)_*^X = \bigoplus_{\rho \in S} A_\rho$ or else A is contained in a strictly larger completely decomposable group.

(2) The hypothesis assures that any finite extension of $A_\sigma \oplus A_\tau$ is completely decomposable, so A, being tight, must coincide with its purification. □

The following proposition characterizes tight subgroups as those completely decomposable subgroups of finite index all of whose rank-one summands are pure. This criterion is useful in establishing tightness. A preliminary result is the following.

LEMMA 2.6. *Let A be a completely decomposable group of finite rank and B a completely decomposable subgroup of A with the property that every rank-one summand of B is pure in A. Then B is a direct summand of A.*

Proof. We first consider the case that A is homogeneous. Let S be a rank-one summand of B. By hypothesis S is pure in A and hence $A = S \oplus Y$

for some Y. It follows that $B = S \oplus B \cap Y$ and every rank-one summand of $B \cap Y$ is pure in Y. By induction on rank, $Y = Z \oplus B \cap Y$ and $Z \oplus B = A$.

Now suppose that A is not homogeneous and induct on rank. Let τ be a maximal critical type of A. Then $A(\tau)$ is τ-homogeneous completely decomposable and every rank-one summand of $B(\tau)$ is a rank-one summand of B and is pure in $A(\tau)$. By the homogeneous case $Z \oplus B(\tau) = A(\tau)$. It follows that $B \subset A = Y \oplus B(\tau)$ for some Y, and hence $B = (B \cap Y) \oplus B(\tau)$. Now every rank-one summand of $B \cap Y$ is pure in Y and by induction hypothesis $W \oplus B \cap Y = Y$, so $W \oplus B = A$. □

[MMR94, Lemma 2.3] shows that the finite rank assumption in Lemma 2.6 cannot be omitted.

PROPOSITION 2.7. *Let X be an almost completely decomposable group and A a completely decomposable subgroup. Then the following hold.*

(1) *If A is tight in X, then every homogeneous summand of A is pure in X.*

(2) *If every rank-one summand of A is pure in X, then A is tight in A_*^X.*

Proof. (1) Assume that A is tight in X and that $A = A_1 \oplus A_2$ where A_1 is homogeneous. Then $(A_1)_*^X \oplus A_2$ is completely decomposable and contains A, hence equals A, consequently $A_1 = (A_1)_*^X$.

(2) If the completely decomposable group A is contained in a completely decomposable subgroup B of A_*^X, then, by Lemma 2.6, $A = B$. □

The next proposition lists some almost completely decomposable groups without proper tight subgroups. Some additional concepts are required. Burkhardt ([Bur84]) defined the *regulator* $\mathrm{R}(X)$ of an almost completely decomposable group to be the intersection of all of its regulating subgroups. He showed that $\mathrm{R}(X)$ is completely decomposable by verifying the formula

$$\mathrm{R}(X) = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} \beta_\rho^X A_\rho,$$

where $A = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ is an arbitrary homogeneous decomposition of any regulating subgroup A of X and the *Burkhardt invariants* are given by

$$\beta_\tau^X = \exp \frac{X^\sharp(\tau)}{\mathrm{R}(X^\sharp(\tau))}.$$

We say that X has a *regulating regulator* if the regulator is a regulating subgroup. If this happens, then there is a unique regulating subgroup, $\beta_\tau^X = 1$ for each $\tau \in \mathrm{T}_{\mathrm{cr}}(X)$, and $\mathrm{R}(X) = \sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} X(\rho)$.

PROPOSITION 2.8. *Let X be an almost completely decomposable group and $A = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ a completely decomposable subgroup.*

(1) *If X has a regulating regulator and A is tight in X, then A is regulating in X $\big($and $A = \mathrm{R}(X)\big)$.*

(2) *If A is tight in X and for each $\tau \in \mathrm{T}_{\mathrm{cr}}(X)$ either the Butler equation $X(\tau) = A_\tau \oplus X^\sharp(\tau)$ holds or $X^\sharp(\tau) = A^\sharp(\tau)$, then A is regulating in X.*

(3) *If the depth of $\mathrm{T}_{\mathrm{cr}}(X)$ is less than or equal to 1, A is tight, and the Butler equation $X(\tau) = A_\tau \oplus X^\sharp(\tau)$ holds for all minimal critical types τ, then A is regulating in X.*

Proof. (1) In this case $\mathrm{R}(X) = \sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} X(\rho)$ ([Mad99, Proposition 4.5.1]), is completely decomposable and clearly $A = \sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(A)} A(\rho) \subset \mathrm{R}(X)$. Since A is tight, $A = \mathrm{R}(X)$.

(2) We need to establish the Butler equations at each critical type. Let $\tau \in \mathrm{T}_{\mathrm{cr}}(X)$, and suppose that the Butler equation $X(\tau) = A_\tau \oplus X^\sharp(\tau)$ is not valid to begin with. Then $X^\sharp(\tau) = A^\sharp(\tau)$. By Proposition 2.4 there is $\phi \in \mathrm{Hom}(A_\tau, X^\sharp(\tau))$, such that $A_\tau(1+\phi)_*^X$ is a Butler complement, and hence completely decomposable. By hypothesis $X^\sharp(\tau) = A^\sharp(\tau)$ and so $\phi \in \mathrm{Hom}(A_\tau, X^\sharp(\tau)) = \mathrm{Hom}(A_\tau, A^\sharp(\tau))$ which means that $A_\tau(1+\phi)$ is a homogeneous summand of A, and since A is tight, it must be pure in X. So $A_\tau(1+\phi)$ is a Butler complement, and so is A_τ.

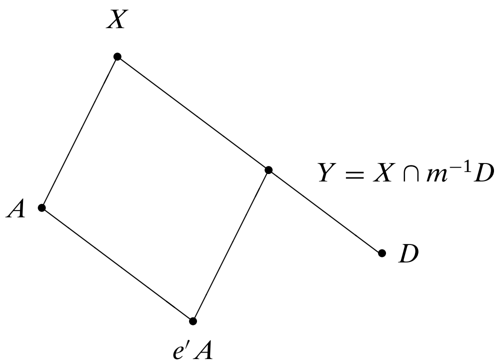
(3) The assumption implies that a non-minimal critical type is maximal, and for a maximal critical τ and a tight subgroup A it is always true that $A^\sharp(\tau) = X^\sharp(\tau) = 0$. $\square$

We will use an unpublished result of Vinsonhaler in order to obtain information about the possible indices of tight subgroups.

PROPOSITION 2.9. (Vinsonhaler). *Let C and D be completely decomposable groups of finite rank such that $\mathbb{Q}C = \mathbb{Q}D$. Suppose that $(C+D)/C$ and $(C+D)/D$ have finite relatively prime orders. Then $C+D$ and $C \cap D$ are completely decomposable.*

COROLLARY 2.10. *Let X be an almost completely decomposable group and D a tight subgroup. Then the prime factors of $[X : D]$ divide $\mathrm{rgi}(X)$ and $\mathrm{rgi}(X)$ divides $[X : D]$.*

Proof. Choose $A \in \mathrm{Regg}(X)$, and set $e = \mathrm{rgi}(X)$. Write $[X : D] = e'm$ such that every prime factor of e' divides e, and $\gcd(e, m) = 1$. Let $Y = X \cap m^{-1}D$. The situation is depicted in the following diagram.



Then

$$D \subset Y \qquad \text{and} \qquad mY \subset D, \tag{2.11}$$

Also $m(e'A) \subset (me')X \subset D$, so $e'A \subset X \cap m^{-1}D = Y$. Further

$$ee'Y \subset ee'X \subset e'A. \tag{2.12}$$

We now have the almost completely decomposable group Y containing the completely decomposable groups $e'A$ and D whose indices in Y are relatively prime because of (2.11) and (2.12), and therefore the indices $[e'A + D : D]$ and $[e'A + D : e'A]$ are relatively prime. By Proposition 2.9 the group $e'A + D$ is completely decomposable and it contains D. But D is tight, so $D = e'A + D$, and therefore $ee'X \subset e'A \subset D$. This implies that the prime factors of $[X : D]$ divide ee' and hence $m = 1$ as desired.

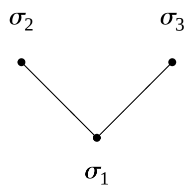
It is an old result of Lady that the regulating index divides the index of any completely decomposable subgroup of finite index. □

A poset T is *∨-free* if for each $\tau \in T$ the set $\{\sigma \in T : \sigma \geq \tau\}$ is totally ordered. We call a group ∨-free if its critical typeset is ∨-free. It is well-known that ∨-free groups have regulating regulators, and thus contain no tight subgroups that are not regulating. An almost completely decomposable group X containing a tight subgroup that is not regulating must have rank at least three since its critical typeset cannot be ∨-free. The following is the simplest example showing that tight subgroups need not be regulating subgroups. The arguments use the methods of [BM98], but the example is simple enough to be checked by ad hoc computations.

EXAMPLE 2.13. Let $A = \sigma_1 a_1 \oplus \sigma_2 a_2 \oplus \sigma_3 a_3$, and $X = A + \mathbb{Z}\frac{1}{p^2}(pa_1 + a_2 + a_3)$. We consider the three distinct types σ_i to be rational groups, $\sigma_1 \subset \sigma_2, \sigma_3$, assume that σ_2 and σ_3 are incomparable as types, and that

$\gcd(p, a_1) = \gcd(p, a_2) = \gcd(p, a_3) = 1$. Then A is tight in X but not regulating. Furthermore, X is the direct sum of two groups in which tight subgroups are regulating.

Proof. The critical typeset of X is the following.



According to the above discussion, purity and purifications are determined by inspection of the augmented matrix

$$[\ p^2 \quad p \quad 1 \quad 1\ ].$$

It is immediate that $\sigma_1 a_1$, $\sigma_2 a_2$, $\sigma_3 a_3$, $\sigma_1 a_1 \oplus \sigma_2 a_2$, and $\sigma_1 a_1 \oplus \sigma_3 a_3$ are all pure in X. Next $[X(\sigma_1) : A(\sigma_1)] = [X : A] = p^2$, while

$$X^{\sharp}(\sigma_1) = (\sigma_2 a_2 \oplus \sigma_3 a_3)_*^X = (\sigma_2 a_2 \oplus \sigma_3 a_3) + \mathbb{Z}\frac{1}{p}(a_2 + a_3),$$

and $[X^{\sharp}(\sigma_1) : A^{\sharp}(\sigma_1)] = p$. This shows that A is not regulating in X since otherwise $[X(\sigma_1) : A(\sigma_1)] = [X^{\sharp}(\sigma_1) : A^{\sharp}(\sigma_1)]$. It remains to show that A is tight in X. We do this by showing that every rank-one summand of A is pure in X. Hence assume that $A = B \oplus C$ with $\operatorname{rk} B = 1$.

Case $\operatorname{tp} B = \sigma$ where $\sigma \in \{\sigma_2, \sigma_3\}$. Here $A(\sigma) = X(\sigma)$ and so $B \lhd X$.

Case $\operatorname{tp} B = \sigma_1$. We have $A^{\sharp}(\sigma_1) = C^{\sharp}(\sigma_1) = \sigma_2 a_2 \oplus \sigma_3 a_3$. Hence $A = A(\sigma_1) = B \oplus A^{\sharp}(\sigma_1)$. Write $pa_1 + a_2 + a_3 = b + a^{\sharp}$, where $b \in B$ and $a^{\sharp} \in A^{\sharp}(\sigma_1)$. Then $X = (B \oplus A^{\sharp}(\sigma_1)) + \mathbb{Z}(1/p^2)(b + a^{\sharp})$ and $\gcd^A(p^2, b + a^{\sharp}) = 1$. If $\gcd^A(p^2, a^{\sharp}) \neq 1$, i.e., if B were not pure in X, then $\gcd^A(p^2, b) = 1$. We have shown that $A^{\sharp}(\sigma_1)$ is pure in X contrary to fact. This shows that B is pure in X and hence A is tight in X.

For the second claim consider a Butler decomposition $X = X(\sigma_1) = Y \oplus X^{\sharp}(\sigma_1)$. Both summands have regulating regulators and hence their tight subgroups are regulating. □

3. INTERSECTION OF TIGHT SUBGROUPS

The examples above indicate that proper tight subgroups are plentiful. Conversely, having no proper tight subgroups must be rare. This is confirmed by the following theorems. Let $\operatorname{Core}(X)$ denote the intersection of all tight subgroups of X. Since regulating subgroups are tight, certainly $\operatorname{Core}(X) \subset \operatorname{R}(X)$. The following general result places a lower bound on $\operatorname{Core}(X)$.

LEMMA 3.1. *Let X be an almost completely decomposable group. Then every tight subgroup of X contains the completely decomposable group*

$$\sum\{X(\rho) : \rho \in \mathrm{T}_{\mathrm{cr}}(X), \beta^X_\rho = 1\}.$$

Proof. Let T be a tight subgroup of X. Suppose that $\tau \in \mathrm{T}_{\mathrm{cr}}(X)$ and $\beta^X_\tau = 1$. Then $X(\tau) = \mathrm{R}(X(\tau))$ is completely decomposable and $T(\tau)$ is tight in $X(\tau)$. Thus $T(\tau) = X(\tau)$. The group $\sum\{X(\rho) : \rho \in \mathrm{T}_{\mathrm{cr}}(X), \beta^X_\rho = 1\}$ is completely decomposable by [Mad99, Proposition 4.5.1]. $\square$

We will show next that for any almost completely decomposable group X the above lower bound is assumed, i.e., $\mathrm{Core}(X) = \sum\{X(\rho) : \rho \in \mathrm{T}_{\mathrm{cr}}(X), \beta^X_\rho = 1\}$.

LEMMA 3.2. *Let X be an almost completely decomposable group, and let $A = \bigoplus_{\rho\in\mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ a regulating subgroup of X. Suppose that $\tau \in \mathrm{T}_{\mathrm{cr}}(X)$ and $\beta^X_\tau \neq 1$. Let $0 \neq a \in A_\tau$. Then for any decomposition $A_\tau = \langle a\rangle^{A_\tau}_* \oplus A'_\tau$, there is a tight subgroup T such that $a \notin T$ but $T \supset A'_\tau \oplus \bigoplus_{\rho\neq\tau} A_\rho$.*

Proof. As $\beta^X_\tau = \exp(X^\sharp(\tau)/\mathrm{R}(X^\sharp(\tau))$, the type τ cannot be divisible by any prime divisor of β^X_τ. Therefore there is a positive integer m such that $a = mb$ with $b \in A_\tau$, $\gcd^A(\beta^X_\tau, b) = 1$, and every prime divisor of m is a prime divisor of β^X_τ. In other words, we divide a in such a way that the quotient $a/m = b$ has p-height 0 for every prime divisor of β^X_τ. Set $A''_\tau = \langle a\rangle^{A_\tau}_* = \langle b\rangle^{A_\tau}_*$. Since $\beta^X_\tau > 1$, certainly $A^\sharp(\tau) \neq X^\sharp(\tau)$ and every prime divisor of the index $[X^\sharp(\tau) : A^\sharp(\tau)]$, which is the regulating index of $X^\sharp(\tau)$, is a prime divisor of β^X_τ. Choose any $x' \in X^\sharp(\tau) - A^\sharp(\tau)$ and observe that $\mathrm{tp}^{X^\sharp(\tau)}(x') \geq \tau = \mathrm{tp}^{A_\tau}(b)$. By the choice of b, there is an integer n relatively prime to β^X_τ such that the characteristic $\chi^X(nx')$ is greater than or equal to the characteristic $\chi^{A_\tau}(b)$. So $x := nx' \in X^\sharp(\tau) - A^\sharp(\tau)$ and $\chi^{X^\sharp(\tau)}(x) \geq \chi^{A_\tau}(b) = \chi^{A''_\tau}(b) = \chi^{mA''_\tau}(mb) = \chi^{mA''_\tau}(a)$. Hence there is a well-defined homomorphism

$$\phi : mA''_\tau \oplus A'_\tau \to X^\sharp(\tau) : a\phi = x,\ \ A'_\tau\phi = 0.$$

By Proposition 2.4 the subgroup

$$D = mA''_\tau(1+\phi) \oplus A'_\tau \oplus \bigoplus_{\rho\neq\tau} A_\rho$$

is a completely decomposable subgroup of X that has finite index in X. There is a tight subgroup T of X containing D. Clearly, T contains $A'_\tau \oplus \bigoplus_{\rho\neq\tau} A_\rho$. Assume, by way of contradiction, that $a \in T$. Then $x = a(1+\phi) - a \in T \cap X^\sharp(\tau) = T^\sharp(\tau) \supset A^\sharp(\tau)$. But $A^\sharp(\tau)$ is regulating (hence tight) in $X^\sharp(\tau)$ and $T^\sharp(\tau)$ is completely decomposable, so actually $T^\sharp(\tau) = A^\sharp(\tau)$ which places x in $A^\sharp(\tau)$ contrary to the choice of x. $\square$

COROLLARY 3.3. *Let X be an almost completely decomposable group. Then the intersection of all of the tight subgroups of X is*

$$\mathrm{Core}\,(X) = \sum\{X(\rho) : \rho \in \mathrm{T_{cr}}\,(X), \beta_\rho^X = 1\}.$$

Proof. The core contains $C := \sum\{X(\rho) : \rho \in \mathrm{T_{cr}}\,(X), \beta_\rho^X = 1\}$ by Lemma 3.1. To show that $\mathrm{Core}\,(X) \subset C$ choose some regulating subgroup $A = \bigoplus_{\rho\in \mathrm{T_{cr}}\,(X)} A_\rho$. Of course, $\mathrm{Core}\,(X) \subset A$ so that we only need to consider elements of A. Suppose that $a = \sum_{\rho\in\mathrm{T_{cr}}\,(X)} a_\rho$, $a_\rho \in A_\rho$. If $a_\tau \neq 0$ for some $\tau \in \mathrm{T_{cr}}\,(X)$ such that $\beta_\tau^X \neq 1$, then (Lemma 3.2) there is a tight subgroup of X not containing a_τ but containing $\bigoplus_{\rho\neq\tau} A_\rho$ and hence $\sum_{\rho\neq\tau} a_\rho$. This means that $a \notin T$. Therefore, if $a \in \mathrm{Core}\,(X)$, then $a_\tau = 0$ whenever $\beta_\tau^X \neq 1$, and $a \in \sum\{X(\rho) : \rho \in \mathrm{T_{cr}}\,(X), \beta_\rho^X = 1\}$. □

COROLLARY 3.4. *Every tight subgroup of an almost completely decomposable group X is regulating if and only if X has a regulating regulator. The class of all almost completely decomposable groups without proper tight subgroups is closed under direct summands.*

The proof of the crucial Lemma 3.2 worked with a tight subgroup T that was not precisely known but contained a completely decomposable subgroup of finite index D that was explicitly stated. It would be nice to know whether D itself is already tight. This is our next topic. For any torsion-free group G and an element $g \in G$, let $\mathbb{Q}_g^G = \{r \in \mathbb{Q} : rg \in G\}$, the *coefficient group* of g in G. For example, if $A = \tau a \oplus A'$ where τ is a rational group, i.e., an additive subgroup of $\mathbb{Q}$ that contains $\mathbb{Z}$, then $\mathbb{Q}_a^A = \tau$.

PROPOSITION 3.5. *Let X be an almost completely decomposable group such that $\beta = \beta_\tau^X > 1$ for some critical type τ of X. Let $A = \bigoplus_{\rho\in\mathrm{T_{cr}}\,(X)} A_\rho$ be a regulating subgroup of X and write $A_\tau = \tau a \oplus A'_\tau$ where the element a is so chosen that* $\mathrm{gcd}^A(\beta_\tau^X, a) = 1$ *and τ doubles as a rational group (containing $\mathbb{Z}$) that represents the type τ. There exists an element $x \in X^\sharp(\tau)$ such that* $\mathrm{ord}(x + A^\sharp(\tau)) = m$ *is divisible by every prime factor of β and $\mathbb{Q}_x^X \supset \tau = \mathbb{Q}_a^A$. For each $k \geq 1$ let*

$$D_k = \tau(\beta^k a + x) \oplus A'_\tau \oplus \bigoplus_{\rho\neq\tau} A_\rho.$$

Then D_k is tight in X and β^k divides $\exp(X/D_k)$.

Proof. The choice of a is possible since the type τ cannot be divisible by any prime divisor p of β. Note that the assumption that $\mathrm{gcd}^A(\beta_\tau^X, a) = 1$ is equivalent to $p^{-1} \notin \tau$ for every prime divisor p of β. The assumption that $\beta > 1$ implies that $X^\sharp(\tau) \neq A^\sharp(\tau)$ and since $A^\sharp(\tau)$ is regulating in $X^\sharp(\tau)$ the prime divisors of β are exactly the prime divisors of the index $[X^\sharp(\tau) : A^\sharp(\tau)]$.

We show next that an element x exists having the stated properties. First choose an element $x' \in X^\sharp(\tau)$ such that $\mathrm{ord}(x' + A^\sharp(\tau))$ is divisible by every prime factor of β. Note that x' may be multiplied by any integer relatively prime to β without changing its order modulo $A^\sharp(\tau)$. Now $\mathrm{tp}^X(x') \geq \tau = \mathrm{tp}^A(a)$ and $p^{-1} \notin \tau$ for prime divisors p of β, therefore there exists an integer n relatively prime to β such that $\mathbb{Q}^X_{nx'} \supset \tau$. Let $x = nx'$.

The choice of x assures that there is a well-defined homomorphism $\phi_k : \tau\beta^k a \oplus A'_\tau \to X^\sharp(\tau)$ such that $(\beta^k a)\phi_k = x$ and $A'_\tau \phi_k = 0$. Hence D_k is well-defined, completely decomposable and has finite index in X (Proposition 2.4). By Corollary 2.10 the prime divisors of $[X : D_k]$ are all prime divisors of β and therefore a summand of D_k is pure in X if and only if it is p-pure in X for the prime divisors p of β.

To verify that D_k is tight, we will show that every rank-one summand of D_k is pure in X. So suppose that $D_k = B \oplus C$ where $\mathrm{rk}\, B = 1$ and $\mathrm{tp}\, B = \sigma$.

First consider the case $\sigma \not\leq \tau$. Then $D_k(\sigma) = \bigoplus_{\rho \geq \sigma} A_\rho = B \oplus C(\sigma)$ and therefore B is a summand of A and pure in X.

Secondly, consider the case $\sigma < \tau$. Write $C(\sigma) = C_\sigma \oplus C^\sharp(\sigma)$ and note that $C^\sharp(\sigma) = D_k^\sharp(\sigma)$. Then $D_k(\sigma) = B \oplus C_\sigma \oplus D_k^\sharp(\sigma) = A_\sigma \oplus D_k^\sharp(\sigma)$. Hence $B \oplus C_\sigma = A_\sigma(1 + \phi)$ for some $\phi \in \mathrm{Hom}(A_\sigma, D_k^\sharp(\sigma))$. But $D_k^\sharp(\sigma) \subset X^\sharp(\sigma)$, so $\phi \in \mathrm{Hom}(A_\sigma, X^\sharp(\sigma))$ and therefore $B \oplus C_\sigma = A_\sigma(1 + \phi)$ is actually another σ-Butler complement of X consequently B is pure in X.

Finally, assume that $\sigma = \tau$ as types. Choose a decomposition $C(\tau) = C_\tau \oplus C^\sharp(\tau)$ and note that $C^\sharp(\tau) = A^\sharp(\tau) = D_k^\sharp(\tau)$. It follows that $D_k(\tau) = B \oplus C_\tau \oplus A^\sharp(\tau) = \tau(\beta^k a + x) \oplus A'_\tau \oplus A^\sharp(\tau)$. Hence there exists a homomorphism $\phi : \tau(\beta^k a + x) \oplus A'_\tau \to A^\sharp(\tau)$ such that

$$B \oplus C_\tau = \tau(\beta^k a + x)(1 + \phi) \oplus A'_\tau(1 + \phi). \tag{3.6}$$

There is $b \in B \cong \tau$ such that $B = \tau b$. By (3.6) we obtain

$$\begin{aligned} b &= t_b(\beta^k a + x)(1 + \phi) \oplus a'(1 + \phi) \\ &= t_b(\beta^k a + x) + a' + t_b(\beta^k a + x)\phi + a'\phi, \\ &\qquad\qquad \text{where } t_b \in \tau,\ a' \in A'_\tau. \end{aligned} \tag{3.7}$$

Let p be a prime divisor of β and suppose that $0 \neq z \in X$ and $pz = tb \in B$ where $t \in \tau$. Then $\mathrm{tp}^X(z) = \mathrm{tp}^X(pz) = \mathrm{tp}^{D_k}(pz) = \mathrm{tp}^B(pz) = \tau$ and $z \in X(\tau) = \tau a \oplus A'_\tau \oplus X^\sharp(\tau)$. Hence

$$z = t_z a + a_z + y', \qquad \text{where} \qquad t_z \in \tau,\ a_z \in A'_\tau,\ y' \in X^\sharp(\tau).$$

It follows that, for some $t \in \tau$,

$$pz = pt_z a + pa_z + py' = tt_b(\beta^k a + x) + ta' + tt_b(\beta^k a + x)\phi + ta'\phi.$$

Comparing terms we obtain that

$$(1) \quad pt_z = tt_b\beta^k, \qquad (2) \quad pa_z = ta',$$

$$(3) \quad py' = tt_b x + tt_b(\beta^k a + x)\phi + ta'\phi.$$

The goal is to show that $t = pt'$ for some $t' \in \tau$. If a' is not divisible by p in D_k, this follows from (2). On the other hand, if a' is divisible by p in D_k, then it follows from (3.6) and $p^{-1} \notin \tau$ that t_b is not divisible by p in τ. Using (3) we obtain that

$$tt_b x = py' - tt_b(\beta^k a + x)\phi - ta'\phi \in X^\sharp(\tau) \cap A(\tau) = A^\sharp(\tau).$$

Hence m must divide tt_b in τ and since p does not divide t_b, it follows that $t = pt'$ for some $t' \in \tau$ as desired.

To obtain information about X/D_k consider $a + D_k$. Suppose that $na \in D_k$ for some positive integer n. Then $na = t\beta^k a$ for some $t \in \tau$ and it follows that β^k divides n. This confirms the last claim. □

This result shows once more that the absence of proper tight subgroups implies that $\beta_\tau^X = 1$ for every critical type τ and so that X has a regulating regulator.

Example 2.13 shows that the direct sum of two almost completely decomposable groups without proper tight subgroups may have proper tight subgroups.

4. REGULATING QUOTIENTS

Let X be an almost completely decomposable group and A a tight subgroup of X. The quotient X/A has certain properties. The question arises whether there exists a regulating subgroup with a quotient having the same properties. In particular, the following two questions arise. Recall the *width* of a finite abelian group G:

$$\mathrm{width}\,(G) = \max\{\dim G[p] : p \in \mathbb{P}\}.$$

QUESTION 4.1. *Let X be an almost completely decomposable group.*

(1) *If X contains a tight subgroup A such that $p^e(X/A) = 0$, then does X contain a regulating subgroup B such that $p^e(X/B) = 0$?*

(2) *If X contains a tight subgroup A such that* $\mathrm{width}\,(X/A) \leq w$, *then does X contain a regulating subgroup B such that* $\mathrm{width}\,(X/B) \leq w$?

For $e = 1$ we establish the stronger result that the tight subgroup itself must be regulating. For $w = 1$ the second question was answered affirmatively by Campagna ([Cam95]).

LEMMA 4.2. *Let X be an almost completely decomposable group with a least critical type τ, and suppose that X contains a tight subgroup A such that $p(X/A) = 0$. Then $X = X(\tau) = A_\tau \oplus X^\sharp(\tau)$ for some maximal τ-homogeneous summand A_τ of A.*

Proof. Write $A = \tau v \oplus B$ where τ doubles as a rational group containing $\mathbb{Z}$ and where $\gcd^A(p, v) = 1$. If $k = \dim(X/A)$, then X can be presented in the form

$$X = A + \sum_{i=1}^{k} \mathbb{Z}\frac{1}{p}(m_i v + b_i), \qquad m_i \in \mathbb{Z}, b_i \in B.$$

If p divides each of the integers m_i, then clearly

$$X = (\tau v \oplus B) + \sum_{i=1}^{k} \mathbb{Z}\frac{1}{p} b_i = \tau v \oplus B_*^X. \tag{4.3}$$

On the other hand, if p does not divide all of the m_i, then we assume without loss of generality that $\gcd(p, m_1) = 1$. Set $m = m_1$ and $b = b_1$. Write $1 = sm + tp$ with integers s, t. Now consider the generator

$$x = \frac{1}{p}(mv + b) \in X$$

and note that

$$p(sx + tv) = psx + ptv = smv + sb + ptv = v + sb. \tag{4.4}$$

We may assume that v was chosen so that $\chi^{\tau v}(v) \leq \chi^B(sb)$. Then there is a well-defined homomorphism

$$\phi : \tau v \to B : \ v\phi = sb$$

that can be used to produce a new decomposition $A = \tau v(1 + \phi) \oplus B = \tau(v + sb) \oplus B$. However, by (4.4), the summand $\tau(v + sb)$ of A is not pure and this contradicts the assumption that A is tight (Proposition 2.7). Hence only the case (4.3) can occur. Now B is obviously tight in $Y = B_*^X$, $p(Y/B) = 0$ and $Y(\tau) = Y$. By induction on rank, $Y = B_\tau \oplus Y^\sharp(\tau)$ where $B_\tau = 0$ is a possibility. It follows that

$$X = X(\tau) = \tau v \oplus B_\tau \oplus Y^\sharp(\tau) = \tau v \oplus B_\tau \oplus X^\sharp(\tau)$$

as claimed. □

THEOREM 4.5. *Let X be an almost completely decomposable group containing a tight subgroup A such that $p(X/A) = 0$ for some prime p. Then A is regulating in X.*

Proof. As A is tight in X, it follows that $A(\tau)$ is tight in $X(\tau)$ for every critical type τ, and by Lemma 4.2 we have $X(\tau) = A_\tau \oplus X^\sharp(\tau)$. Hence $A = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ is regulating in X. □

COROLLARY 4.6. *Let X be an almost completely decomposable group containing a completely decomposable subgroup A such that $p(X/A) = 0$ for some prime p. Then A is contained in a regulating subgroup B of X such that $p(X/B) = 0$.*

Proof. The completely decomposable group A is contained in some tight subgroup B of X. Then $p(X/B) = 0$ and B is regulating by Theorem 4.5. □

Corollary 4.6 can be combined with results of [MV95] to give a quick proof of the following striking result of Burkhardt ([Bur84]). For unexplained notation and terminology we refer the reader to [MV95].

THEOREM 4.7. (Burkhardt). *Let X be an almost completely decomposable group whose regulating index is p^n where p is some prime. Assume that X possesses a cyclic regulating quotient as well as a regulating quotient that is elementary*, i.e., *a direct sum of n cyclic groups of order p. Then the following hold.*

(1) *The depth of $\mathrm{T}_{\mathrm{cr}}(X)$ is $n-1$, X has $n-1$ equivalence classes of sharp types $S_1 < S_2 < \cdots < S_{n-1}$ with corresponding Burkhardt invariants $\beta_i = p^{n-i}$. In any indecomposable decomposition $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ the critical typesets $\mathrm{T}_{\mathrm{cr}}(X_i)$ are anti-chains and $\operatorname{rgi}(X_i) = p$.*

(2) *Every group of order p^n is isomorphic to some regulating quotient of X.*

Proof. Suppose first that X is indecomposable. By [MV95, 4.7 Indecomposability Criterion] every regulating quotient of X is cyclic, and by hypothesis one of them is p-elementary also, so that $\operatorname{rgi} X = p$. Being indecomposable, X is clipped and this implies that $\mathrm{T}_{\mathrm{cr}}(X)$ is an anti-chain by the [MV95, 3.2 Structure Theorem]. It is now clear that Burkhardt's theorem holds for indecomposable X. Suppose then that X is decomposable. Select a minimal sharp type μ. Then, according to [MV95, Proposition 4.4],

$$X = X^\flat(\mu) \oplus X^\sharp(\mu), \qquad \text{where} \qquad X^\flat(\mu) = B^\flat(\mu) + \mathbb{Z}(\beta_\mu^X p^{-n})b,$$

for some regulating subgroup $B = B^\flat(\mu) \oplus B^\sharp(\mu)$ of X, and some $b \in B^\sharp(\mu)$. Since for any regulating subgroup A of X, the type subgroup $A^\sharp(\mu)$ is a regulating subgroup of $X^\sharp(\mu)$, it follows that $X^\sharp(\mu)$ satisfies the hypothesis of the theorem. On the other hand, $X^\flat(\mu)$ is indecomposable and has a cyclic regulating quotient. It is not clear off-hand whether it has a p-elementary regulating quotient also. However, $X^\flat(\mu) \cong X/X^\sharp(\mu)$ and, if $C = C^\flat(\mu) \oplus C^\sharp(\mu)$ is a regulating subgroup of X such that X/C is p-elementary, then $X/X^\sharp(\mu)$ contains the completely decomposable group $(C + X^\sharp(\mu))/X^\sharp(\mu) \cong C^\flat(\mu)$ with p-elementary factor group. By Corollary 4.6 $X^\flat(\mu) \cong X/X^\sharp(\mu)$ has a p-elementary regulating quotient. So both summands $X^\flat(\mu)$ and $X^\sharp(\mu)$ satisfy the hypotheses of the theorem and by induction on rank, have all claimed properties. In particular, $\operatorname{rgi}(X^\flat(\mu)) = p$, and hence $\operatorname{rgi}(X^\sharp(\mu)) = p^{n-1}$. The theorem follows. □

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