ELSEVIER
Topology and its Applications 108 (2000) 53–65
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

Continuous images of H^* and its subcontinua

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Received 17 September 1998; received in revised form 27 April 1999

Abstract

We give new sufficient conditions for a continuum to be a remainder of H. We also show that any non-degenerate subcontinuum of H^* maps onto any continuum of weight $\leqslant \omega_1$, thus generalizing a result of D.P. Bellamy.

Keywords: Continuum; Remainder; H^*

AMS classification: 54D05; 54D40

1. Introduction

Let X be a completely regular space. A compact Hausdorff space αX is a compactification of X if αX contains a dense copy of X. A space of the form $\alpha X \setminus X$ where αX is a compactification of X is called a remainder of X. As usual we denote by βX the Stone–Čech compactification of the space X and by X^* its Stone–Čech remainder $X^* = \beta X \setminus X$. By a theorem of Magill [8] we know that the remainders of a locally compact space X are exactly the continuous images of X^*. For this reason, given a locally compact space X and a compact space K, it is equivalent to say that "K is a remainder of X" or that "K is a continuous image of X^*". If the space X is also locally compact we denote by ωX the Alexandroff one-point compactification of X.

One of the general problems in the study of the compactifications of a space X is to characterize internally the class of its remainders. This is generally a very hard task and we are usually not able to give a definitive answer. However it is sometimes possible to find classes of spaces that are remainders of X, thus giving a partial answer to the problem.

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[1] This paper represents part of the Ph.D. Thesis of the author at the Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, during which time the author was partially supported by a grant of the Consiglio Nazionale delle Ricerche (Italy).

0166-8641/00/$ – see front matter
PII: S0166-8641(99)00116-9

The most studied space X from this point of view is probably ω, the discrete set of natural numbers. Some of the most important results on the remainders of ω are the following:

Fact 1. *Let X be a compact space of weight $\leqslant \omega_1$. Then X is a remainder of ω.*

The original proof is due to Parovičenko [9]. An interesting different proof can be found in [3].

Fact 2. *Let X be a compact perfectly normal space. Then X is a remainder of ω.*

The proof is due to Przymusiński and can be found in [10].

Fact 3. *Let X be a compact separable space. Then X is a remainder of ω.*

This fact is well known and can be seen for instance as follows. Let $f:\omega \to X$ be an infinite to one function that maps ω onto a countable dense subset of X. Define on the set $Z = \omega \cup X$ the following topology: ω is discrete and a basic neighborhood of a point $x \in X$ is given by $U \cup (f^{-1}(U) \setminus F)$ where U is a neighborhood of x in the original topology of X and $F \subset \omega$ is finite. It is easily checked that Z is a compactification of ω with remainder X. (This is an example of a singular compactification, for more information see, for example, [6].)

Another space that deserves a special attention is the half open interval $H = [0, \infty)$. The reason why we do not usually consider the real line $\mathbb{R}$ is that the Stone–Čech remainder of $\mathbb{R}$ is the topological sum of two copies of H^*. Some of the most important results on the remainders of H are the following:

Fact 4. *Let X be a remainder of H, then X is a continuum* (*i.e., compact and connected*).

This is well known, see, for example, [12].

Fact 5. *Let X be a continuum of weight $\leqslant \omega_1$. Then X is a remainder of H.*

The proof, involving elementarity consideration, is due to Dow and Hart [5].

We recall that a weak Peano space is a compact Hausdorff space that contains a dense continuous image of the real line $\mathbb{R}$.

Fact 6. *Let X be a weak Peano space. Then X is a remainder of H.*

This is a consequence of Fact 7, or we can use a procedure similar to that one we used to show Fact 3. Let $g:\mathbb{R} \to X$ be a continuous dense function. Define $f:H \to X$ as $f(x) = g(x\sin(x))$. Define on the set $Z = H \cup X$ the following topology: the topology on H is the same as before and a basic open neighborhood of a point $x \in X$ is given by a set of the form $U \cup (f^{-1}(U) \setminus F)$ where U is a neighborhood of x in the original topology of X and $F \subset H$ is compact. It is easily checked that Z is a compactification of H with remainder X.

We recall that a continuum X is irreducible about a subset $S \subset X$ if there is no proper subcontinuum of X containing S. In particular we say that a continuum X is irreducible between two points a and b if X is irreducible about $\{a, b\}$.

Fact 7. *Let X be a continuum irreducible about some separable subset $S \subset X$. Assume that X can be embedded as a G_δ-set into a connected, locally pathwise connected, locally compact space Y. Then X is a remainder of H.*

This was proved by Bellamy in his Ph.D. Thesis [2].

By a "Tychonoff cube of weight κ" we mean a product of κ intervals $\prod_{\alpha<\kappa} I_\alpha$. As an immediate consequence of Fact 7 we get that any separable continuum that can be embedded as a G_δ-set into a Tychonoff cube of weight $\leqslant c$ is a remainder of H.

In this paper we will find more classes of spaces that are remainders of H. In particular we will show the following

(1) Let X be a continuum that can be embedded as a G_δ-set into a Tychonoff cube of weight $\leqslant c$. Then X is a remainder of H (Corollary 3).
(2) Let X_1 be any remainder of H and let X_2 be a separable pathwise connected continuum. Then $X_1 \times X_2$ is a remainder of H (Theorem 5).
(3) Let X be a continuum that can be embedded as a G_{ω_1}-set into a Tychonoff cube of weight $\leqslant c$. Then X is a remainder of H (Corollary 6).
(4) Let X be a continuum irreducible about some separable subset $S \subset X$. Assume that X can be embedded into a connected locally compact space Y as a countable intersection of a family of pathwise connected sets $\{A_n\colon n < \omega\}$ such that $\overline{A_{n+1}} \subset A_n$ for all $n < \omega$. Then X is a remainder of H (Theorem 7).
(5) Let X be a product of no more than c separable continua X_α, and assume that each X_α can be embedded into a connected locally compact space Y_α as a countable intersection of a family of pathwise connected sets $\{A_n^\alpha\colon n < \omega\}$ such that $\overline{A_{n+1}^\alpha} \subset A_n^\alpha$ for all $n < \omega$ and for all α. Then X is a remainder of H (Theorem 9).

Observe that (4) is a generalization of Bellamy's Theorem (Fact 7). In fact, suppose that the space X is a G_δ-set in a locally pathwise connected locally compact space Y. Then using the fact that Y is locally pathwise connected and locally compact we can write X as a countable intersection of a family of pathwise connected (open) sets $\{A_n\}$ such that $\overline{A_{n+1}} \subset A_n$ for all $n < \omega$. An example of a space that satisfies the conditions of (4) but is not a G_δ-set is given in Theorem 9. Both Theorems 7 and 9 have been suggested by some unpublished notes of Faulkner and Vipera.

In parallel with the study of remainders, there is an entire theory of mappings of the remainders, embeddings into the remainders, and mappings from and onto special subsets of the remainders. Many of these theorems have remnants of the universality of the mappings from the Stone–Čech remainder. An example is the following theorem from Bellamy's Thesis [2].

Theorem 1. *Any non-degenerate subcontinuum of H^* maps onto any metric continuum.*

One of the main steps in the proof of Theorem 1 is to show that H^* maps onto any metric continuum. This was proved in [2] and also in [1]. Since we know now that something more is actually true, i.e., that H^* maps onto any continuum of weight $\leqslant \omega_1$, we naturally ask if any non-degenerate subcontinuum of H^* maps onto any continuum of weight $\leqslant \omega_1$. The proof that this is actually true is the main result of Section 3 of this paper (Theorem 14).

2. Remainders of H

Let $X = \prod_{\alpha<\kappa} X_\alpha$ be a product of κ spaces. Let $B = \bigcap_{\delta\in\Delta} \pi_\delta^{-1}(U_\delta)$ be a basic open set of X, where $U_\delta \subset X_\delta$ is open, $U_\delta \neq X_\delta$ and $\Delta \subset \kappa$ is a finite set of indices. We say that Δ is the support of B ($\Delta = \mathrm{supp}(B)$). If $A = \bigcup_{i=1}^n B_i$ is a finite union of basic open sets we say that $S = \bigcup_{i=1}^n \mathrm{supp}(B_i)$ is the support of A ($S = \mathrm{supp}(A)$).

Lemma 2. *A compact space X can be embedded into a Tychonoff cube of weight κ as the intersection of $\mu < \kappa$ open sets if and only if X is homeomorphic to the product $M \times I^\kappa$ of a compact space M of weight $\leqslant \mu$ and the Tychonoff cube I^κ.*

Proof. Assume $X = M \times I^\kappa$ where $w(M) \leqslant \mu < \kappa$. Since M is compact and $w(M) \leqslant \mu$, M can be embedded into the Tychonoff cube I^μ as the intersection of μ open sets. Say $M = \bigcap_{\alpha<\mu} B_\alpha$ where $B_\alpha \subset I^\mu$ are open sets. Then $X = \bigcap_{\alpha<\mu}(B_\alpha \times I^\kappa) \subset I^\mu \times I^\kappa$ is the intersection of μ open sets in $I^\mu \times I^\kappa$.

Assume now that $X = \bigcap_{\alpha<\mu} A_\alpha \subset I^\kappa$ where A_α are open sets in the Tychonoff cube I^κ. Without loss of generality we can assume that A_α is a finite union of basic open sets, so that A_α is supported by a set of finitely many indeces S_n. Put $S = \bigcup_{n<\omega} S_n$. Observe that $|S| \leqslant \mu$.

Let $\pi_S : I^\kappa \to \prod_{\delta\in S} I_\delta$ be the projection. Then $X = \pi_S(X) \times I^{\kappa\setminus S} \simeq \pi_S(X) \times I^\kappa$ and $w(\pi_S(X)) \leqslant \mu$. □

In particular the lemma above says that a compact space X can be embedded as a G_δ-set in a Tychonoff cube $I^\kappa = [0,1]^\kappa$ if and only if X is either metrizable or it is homeomorphic to the product $M \times I^\kappa$ of a compact metric space M and the Tychonoff cube I^κ.

Corollary 3. *Let X be a continuum that can be embedded as a G_δ-set into a Tychonoff cube of weight $\leqslant c$. Then X is a remainder of H.*

Proof. By Lemma 2, $X = M \times I^\kappa$ where M is metrizable. Since $\kappa \leqslant c$, X is separable. Therefore X is a separable continuum that can be embedded as a G_δ-set into the connected locally pathwise connected locally compact space I^κ and we can apply Fact 7. □

In particular, Corollary 3 together with Lemma 2 tell us that the product of a metric continuum and a Tychonoff cube of weight $\leqslant c$ is always a remainder of H.

We will show that something more is actually true.

Lemma 4. *A compact space X is a remainder of H if and only if there exists a compact space Z in which X is embedded and a continuous function $h : H \to Z$ such that*

(1) *for any neighborhood U of X in Z, there is an $n \in \omega$ such that $h([n, \infty)) \subset U$,*
(2) *for any open set $V \subset Z$ such that $X \cap V \neq \emptyset$ the set $\{t \in H\colon\ h(t) \in V\}$ is unbounded.*

Proof. Assume X is a remainder of H. Let αH be a compactification of H such that $\alpha H \setminus H = X$. Let $e : \alpha H \to I^{w(\alpha H)}$ be the Tychonoff embedding of the completely regular space αH into the cube $I^{w(\alpha H)} = Z$. Let $\alpha : H \to \alpha H$ be the inclusion mapping. We claim that the function $h = e \circ \alpha$ is as required.

To check (1) observe that any unbounded sequence of $h(H)$ must cluster to some points of X. To check (2) observe that any open set of $\alpha H \setminus H$ meets H in an unbounded set.

Assume now that X is embedded into a compact space Z and that there is a function $h : H \to Z$ satisfying (1) and (2). We can embed Z into a Tychonoff cube of weight κ and consider the map $h : H \to I^\kappa$. This new map still satisfies (1) and (2). Define $f : H \to I^\kappa \times I$ as $f(t) = \langle h(t), 2^{-t}\rangle$. Then the closure of $f(H)$ in $Z \times I$ is the compactification needed. □

Observe that the conditions of the above lemma are equivalent to X being the singular set of the mapping h [4].

Theorem 5. *Let X_1 be any remainder of H and let X_2 be a separable pathwise connected continuum. Then $X_1 \times X_2$ is a remainder of H.*

Proof. Since X_1 is a remainder of H by Lemma 4 there is a compact space Z in which X_1 is embedded and there is a continuous function $h : H \to Z$ such that

(1) for any neighborhood U of X_1 in Z, there is an $n \in \omega$ such that $h([n, \infty)) \subset U$,
(2) for any open set $V \subset Z$ such that $X_1 \cap V \neq \emptyset$ the set $\{t \in H\colon\ h(t) \in V\}$ is unbounded.

Let $\{d_n\colon\ n \in \omega\}$ be a dense subset of X_2.

Claim. *Let $f : H \to Z \times X_2$ be a continuous function and assume that*

(a) *for every $n \geqslant 0$ and $t \geqslant n$, $f(t) \in h([n, \infty)) \times X_2$,*
(b) *for $n - 1 \leqslant t < n$ and $i \leqslant n$ the point $\langle h(t), d_i\rangle$ is in the range of $f|_{[n-1,n)}$.*

Then the function f satisfies also the properties:

(c) *for any neighborhood $U \times X_2$ of $X_1 \times X_2$ in $Z \times X_2$, there is an $n \in \omega$ such that $f([n, \infty)) \subset U \times X_2$,*
(d) *for any basic open set $(V \times W) \subset (Z \times X_2)$ such that $(X_1 \times X_2) \cap (V \times W) \neq \emptyset$ the set $\{t \in H\colon\ f(t) \in V \times W\}$ is unbounded.*

Proof. Suppose that $f : H \to Z \times X_2$ is a continuous function that satisfies (a) and (b). To show that f satisfies (c) let $U \times X_2$ be a neighborhood of $X_1 \times X_2$ in $Z \times X_2$. By

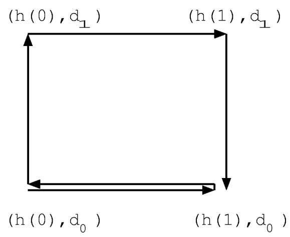

Fig. 1. $f^0([0,1])$.

property (1) of the function h there is an n such that $h([n,\infty)) \subset X_1$; by (a) we have $f([n,\infty)) \subset U \times X_2$.

To show that f satisfies (d) let $V \times W$ be an open set of $Z \times X_2$ such that $(U \times W) \cap (X_1 \times X_2) = (V \cap X_1) \times (W \cap X_2) \neq \emptyset$. There are infinitely many $d_n \in W \cap X_2$. Pick one, say d_{n_1}. By property (2) of the function h there is a $t \in [k_1, k_1+1)$ with $k_1 \geqslant n_1$ such that $h(t) \in V$. By property (b) the point $\langle h(t), d_{n_i}\rangle$ is in the range of $f|_{[k_i,k_i+1)}$ and therefore there is an $s \in [k_1, k_1+1)$ such that $f(s) = \langle h(t), d_{n_i}\rangle \in V \times W$. □

The claim shows that a function f satisfying properties (a) and (b) satisfies the hypothesis of Lemma 4 and therefore, once we have such a function, we can conclude that the space $X_1 \times X_2$ is a remainder of H.

It remains to show that we can construct a function satisfying properties (a) and (b). We will define it by induction on each interval of the form $[n, n+1]$ as follows.

Define $f^0 : [0,1] \to Z \times X_2$ as follows. Subdivide the interval $[0,1]$ into 5 parts I_i with $0 \leqslant i \leqslant 4$; let

$$f_0^0 : I_0 \to \bigl\{\langle h(r), d_0\rangle\colon\ r \in [0,1]\bigr\} \subset Z \times X_2$$

be defined as

$$f_0^0(s) = \bigl\langle h(5s), d_0\bigr\rangle.$$

Let f_1^0 be the path backwards $f_1^0(s) = \langle h(1-5s), d_0\rangle$. Then we move the second variable along a path connecting d_0 to d_1; let $g_2 : [\frac{2}{5}, \frac{3}{5}] \to \{h(0)\} \times X_2$ be such a path, then put $f_2^0(s) = \langle h(0), g_2(s)\rangle$. Next let us move the first variable again; define f_3^0 to be a path parallel to f_0^0 at the level d_1, connecting $\langle h(0), d_1\rangle$ with $\langle h(1), d_1\rangle$. Finally we go down to level d_0 with a path connecting d_1 to d_0 and fixing the second variable $h(1)$. See Fig. 1.

Suppose we have defined $f^k : [k, k+1] \to Z \times X_2$ for all $k < n$. Define $f^n : [n, n+1] \to Z \times X_2$ as follows. Subdivide the interval $[n, n+1]$ into $5+3n$ parts I_i with $0 \leqslant i \leqslant 4+3n$. For $0 \leqslant i \leqslant 3n$ define $f_i^n : I_i \to Z \times X_2$ as f_i^{n-1} with the appropriate parametrization and replacing $h(n-1)$ with $h(n)$ and $h(n)$ with $h(n+1)$. f_{3n+1}^n will be as f_{3n}^n backwards, connecting $\langle h(n+1), d_n\rangle$ with $\langle h(n), d_n\rangle$; f_{3n+2}^n is a path connecting $\langle h(n), d_n\rangle$ to $\langle h(n), d_{n+1}\rangle$ and fixing the first variable; f_{3n+3}^n connects $\langle h(n), d_{n+1}\rangle$ to $\langle h(n+1), d_{n+1}\rangle$. Finally f_{3n+4}^n connects $\langle h(n+1), d_{n+1}\rangle$ back to $\langle h(n+1), d_0\rangle$. See Fig. 2.

It is now easily seen that the map $f : H \to Z \times X_2$ given by the union of the f_n satisfies (a) and (b). □

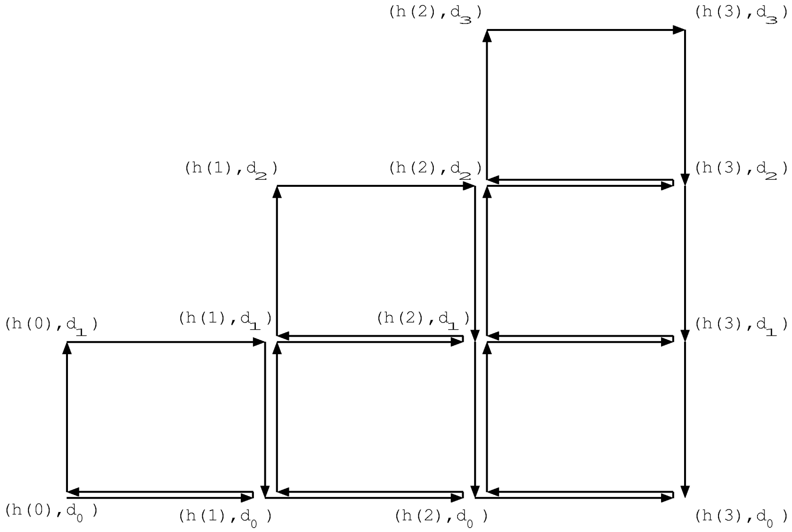

Fig. 2. $f^0([0,1]) \cup f^1([1,2]) \cup f^2([2,3])$.

Corollary 6. *Let X be a continuum that can be embedded as a G_{ω_1}-set (i.e., as the intersection of ω_1 open sets) into a Tychonoff cube of weight $\leqslant c$. Then X is a remainder of H.*

Proof. By Lemma 2 we know that such a space can be written as the product of a Tychonoff cube T and a space of weight $\leqslant \omega_1$. Since by Fact 5 a continuum of weight $\leqslant \omega_1$ is a remainder of H we can apply Theorem 5. □

We can generalize both Fact 7 and Corollary 3 by weakening the requirement that the space is a G_δ as follows:

Theorem 7. *Let X be a continuum irreducible about some separable subset $S \subset X$. Assume that X can be embedded into a connected locally compact space Y as a countable intersection of a family of pathwise connected sets $\{A_n\colon n < \omega\}$ such that $\overline{A_{n+1}} \subset A_n$ for all $n < \omega$. Then X is a remainder of H.*

Proof. Let $D = \{d_n\colon n < \omega\}$ be a dense set in S. For any $n < \omega$ let $h_n : I \to A_n$ be a path connecting d_n with d_{n+1}. Let ωY be the one-point compactification of the locally compact space Y.

Define a continuous function $h : H \to \omega Y$ by $h(x) = h_n(x - n)$ if $x \in [n, n+1]$.

The function $h : H \to \omega Y$ is such that

(1) for any neighborhood U of X in ωY, there is an $n \in \omega$ such that $h([n,\infty)) \subset U$,

(2) for any open set $V \subset \omega Y$ such that $S \cap V \neq \emptyset$ the set $\{t \in H\colon h(t) \in V\}$ is unbounded.

To check (1) let U be a neighborhood of X in ωY. Since $\overline{A_{n+1}} \subset A_n$ for all n there is an $n < \omega$ with $A_n \subset U$ (suppose not, then $\bigcap_{n<\omega} \overline{A_n} \setminus U \neq \emptyset$ since $\{\overline{A_n} \setminus U\colon n < \omega\}$ has the finite intersection property). Therefore $f(x) \in U$ for all $x \geqslant n$.

To check (2) let $V \subset \omega Y$ be an open set and suppose that $S \cap V \neq \emptyset$. Since D is dense in S there are infinitely elements $d_i \in D \cap V$. Since $d_i = h(i)$ we conclude that teh set $\{t \in H\colon h(t) \in V\}$ is unbounded.

We just showed that the function h satisfies properties (1) and (2) of Lemma 4 except that in property (2) above the set S plays the role of X. So we cannot immediately conclude that X is a remainder of H. As in the proof of Lemma 4 we can assume without loss of generality that the function h is one–one and that $h(H) \cap X = \emptyset$ (otherwise we can replace ωY with $\omega Y \times I$ and h with $f: H \to \omega Y \times I$ given by $f(t) = \langle h(t), 2^{-t}\rangle$). By property (1) we get that $\mathrm{cl}_{\omega Y}(h(H)) \subseteq X$. By property (2) and by the fact that $h(H) \cap X = \emptyset$ we get that $S \subset \mathrm{cl}_{\omega Y}(h(H)) \setminus h(H)$. Since the space $\mathrm{cl}_{\omega Y}(h(H)) \setminus h(H)$ is a remainder of H, by Fact 4 it is a continuum. Since X is irreducible about S we conclude that $\mathrm{cl}_{\omega Y}(h(H)) \setminus h(H) = X$. □

Corollary 8. *Let X be a separable continuum that is a countable intersection in a Tychonoff cube of weight $\leqslant c$ of a family of pathwise connected sets $\{A_n\colon n < \omega\}$ such that $\overline{A_{n+1}} \subset A_n$ for all $n < \omega$. Then X is a remainder of H.*

As a consequence of Theorem 7 we have the following interesting result:

Theorem 9. *Let X be a product of no more than c separable continua X_α, and assume that each X_α can be embedded into a connected locally compact space Y_α as a countable intersection of a family of pathwise connected sets $\{A_n^\alpha\colon n < \omega\}$ such that $\overline{A_{n+1}^\alpha} \subset A_n^\alpha$ for all $n < \omega$ and for all α. Then X is a remainder of H.*

Proof. We have $X \subset \prod_{\alpha \leqslant \kappa} Y_\alpha = Y$ for some $\kappa \leqslant c$. Y is connected and locally compact. We will apply Theorem 7 by showing that X can be written as a countable intersection of pathwise connected sets nested in the right way. We have

$$X = \prod_{\alpha \leqslant \kappa} X_\alpha = \prod_{\alpha \leqslant \kappa} \left(\bigcap_{n<\omega} A_n^\alpha \right) = \bigcap_{n<\omega} \left(\prod_{\alpha \leqslant \kappa} A_n^\alpha \right).$$

Put $B_n = \prod_{\alpha \leqslant \kappa} A_n^\alpha$. B_n is pathwise connected and

$$\overline{B_{n+1}} = \prod_{\alpha \leqslant \kappa} \overline{A_{n+1}^\alpha} \subset \prod_{\alpha \leqslant \kappa} A_n^\alpha = B_n \quad \text{for all } n < \omega. \qquad \square$$

As a consequence we get the following corollary, proved by Faulkner and Vipera.

Corollary 10 (Faulkner and Vipera). *Let X be a product of no more than c metric continua. Then X is a remainder of H.*

Proof. We observe that any metric continuum X_α is a G_δ-set in the metric Tychonoff cube I^ω. Moreover, since a Tychonoff cube is locally pathwise connected we can assume without loss of generality that $X_\alpha = \bigcap_{n<\omega} U_n^\alpha$ where U_n^α is pathwise connected and $\overline{U_{n+1}^\alpha} \subset U_n^\alpha$ for all $n < \omega$. Therefore we can apply Theorem 9. □

Theorems 7, 9 and its corollaries give conditions for a continuum to be a remainder of H. All of them consider practically separable spaces that can be embedded in a cube in a special way. The next step would be to drop at least one of these conditions. In this sense we can ask the following:

Question 11. Is any separable continuum a remainder of H?

Question 12. Is any continuum that is a remainder of ω and that can be embedded in a Tychonoff cube as a countable intersection of strongly nested pathwise connected sets a remainder of H?

3. Do non-degenerate subcontinua of H^* map onto H^*?

In his Ph.D. Thesis Bellamy proved the following results:

(a) Any non-degenerate subcontinuum of H^* maps onto βH (and hence onto any weak Peano space).

(b) Any non-degenerate subcontinuum of H^* maps onto any metric continuum.

We ask if it is possible to generalize the above mentioned results. In particular the final goal is to see if it is true that any non-degenerate subcontinuum of H^* can be mapped onto H^* itself and hence onto any continuum that is a remainder of H. Bellamy proves (b) by using the following three lemmas:

(B1) Let K be a metric continuum, then there is a compactification αH of H whose remainder is homeomorphic to K.

(B2) Any metric continuum is a retract of a metric continuum which is irreducible between two of its points.

(B3) Any non-degenerate subcontinuum of H^* maps onto any metric continuum which is irreducible between two of its points.

The reader may notice that (B1) follows by (B2) and (B3); however Bellamy first proved (B1) and then he used that result to prove (B3).

It is known [5] that (B1) can be generalized to any continuum of weight $\leqslant \omega_1$. It is also possible to see [5] that any standard subcontinuum I_u of H^* can be mapped onto any continuum of weight $\leqslant \omega_1$. We want to show that actually the same is true for all non-degenerate subcontinua of H^*.

We first prove a lemma that generalizes Bellamy's (B2).

Lemma 13. *Let K be a continuum of weight κ, then there is a continuum T of weight κ such that*

(1) *There is a subspace $J \subset T$ homeomorphic to the unit interval I.*

(2) *There is a subspace $K' \subset T$ homeomorphic to K.*

(3) *For any* $t \in [0,1) \subset J$ *there exists a continuous function* $f : T \setminus [0,t) \to K$ *onto.*
(4) $T \setminus [0,t)$ *is irreducible between* $t \in [0,1) \subset J$ *and any point of* K'.

Proof. Let $\{a_\alpha\colon \alpha < \kappa\}$ be a dense set of points of K and assume that $a_{2\alpha} = a_0$ for any $\alpha < \kappa$, in particular for any limit ordinal γ we have $a_\gamma = a_0$. We also assume that for any $\gamma < \kappa$ the set $\{a_\alpha\colon \alpha \geqslant \gamma\}$ is still dense in K.

We denote by $W(\kappa)$ the long segment (= the compactified long line) of length κ. By transfinite induction we will construct for all $\alpha \leqslant \kappa$ a continuum $L_\alpha \subset K$ and a continuum $T_\alpha \subset K \times W(\kappa)$ with the following properties:

- $T_\alpha \subset K \times [0,\alpha]$ and $T_\alpha \cap K \times \{\alpha\} = L_\alpha \times \{\alpha\}$;
- $T_\alpha \subset T_\beta$ for $\alpha < \beta$;
- $(a_\alpha, \alpha) \in T_\alpha$;
- $T_\gamma \cap K \times (\alpha, \alpha+1) = \{a_\alpha\} \times (\alpha, \alpha+1)$ for any $\alpha < \gamma$.

Let $T_0 = \{(a_0, 0)\}$ and $L_0 = \{a_0\}$. Assume we have defined L_α and T_α for all $\alpha < \gamma$.

Assume first $\gamma = \mu + 1$. Let L_γ be a continuum irreducible between a_μ and a_γ. Put $T_\gamma = T_\mu \cup (\{a_\mu\} \times [\mu, \gamma]) \cup (L_\gamma \times \{\gamma\})$. T_γ is clearly closed and it is connected because the point (a_μ, μ) belongs to both T_μ and $\{a_\mu\} \times [\mu, \gamma]$ and the point (a_μ, γ) belongs to both $\{a_\mu\} \times [\mu, \gamma]$ and $L_\gamma \times \{\gamma\}$. All the properties required are easily checked.

Assume next that γ is a limit ordinal. Define

$$T_\gamma = \mathrm{cl}_{K \times W(\kappa)}\Bigl(\bigcup_{\alpha<\gamma} T_\alpha\Bigr) \quad \text{and} \quad L_\gamma = \pi_K\Bigl(T_\gamma \Big\backslash \Bigl(\bigcup_{\alpha<\gamma} T_\alpha\Bigr)\Bigr),$$

where π_K is the projection of $K \times W(\kappa)$ onto its first factor. The space T_γ is connected being the continuous image of the closure of a nested union of connected sets. We clearly have $L_\gamma \times \{\gamma\} \subset T_\gamma \cap K \times \{\gamma\}$; if $(x, \gamma) \in T_\gamma \cap K \times \{\gamma\}$ we have $(x, \gamma) \in T_\gamma \setminus \bigcup_{\alpha<\gamma} T_\alpha$ hence $x \in L_\gamma$. Therefore $L_\gamma \times \{\gamma\} = T_\gamma \cap K \times \{\gamma\}$ and L_γ is connected. Finally observe that $(a_\gamma, \gamma) = (a_0, \gamma)$ and this point is clearly in T_γ being the limit of the sequence $\{(a_{2\alpha}, 2\alpha)\} \subset \bigcup_{\alpha<\gamma} T_\alpha$. See Fig. 3.

We claim that the space $T = T_\kappa$ is as required.

(1) holds taking $J = \{a_0\} \times [0,1]$, where $[0,1]$ is the initial interval in $W(\kappa)$;
(2) holds taking $K' = L_\kappa \times \{\kappa\} = K \times \{\kappa\}$;
(3) holds taking as f the projection $\pi : K \times W(\kappa) \to K$ restricted to $T \setminus [0,t)$.

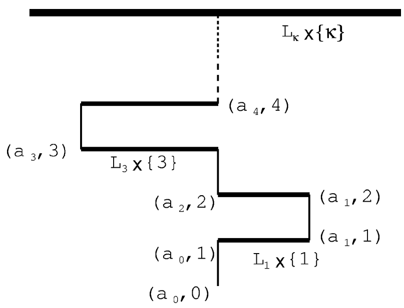

Fig. 3. The space T.

To prove (4) pick a point $z \in K$ and assume that $P \subset T$ is a subcontinuum that contains both (z, κ) and $(a_0, 0)$ (for sake of simplicity we pick the point $(a_0, 0) \in J$, clearly the same argument would work for any other point (a_0, t) with $t \in [0, 1)$).

Since the projection of P onto $W(\kappa)$ contains $\{0, \kappa\}$ it is all of $W(\kappa)$. Since $T_\gamma \cap K \times (\alpha, \alpha+1) = \{a_\alpha\} \times (\alpha, \alpha+1)$, P must contain all intervals $\{a_\alpha\} \times (\alpha, \alpha+1)$, hence ($P$ is closed) all closed intervals $\{a_\alpha\} \times [\alpha, \alpha+1]$. Let $P_\alpha = P \cap (K \times \{\alpha\})$.

We observe that for any non limit ordinal α we have $P_\alpha = L_\alpha \times \{\alpha\}$. To show this it is sufficient to prove that P_α is connected, since in this case P_α is a subcontinuum of $L_\alpha \times \{\alpha\}$ containing both points $(a_{\alpha-1}, \alpha)$ and (a_α, α) and the continuum $L_\alpha \times \{\alpha\}$ is irreducible between these points. So let us assume that P_α is not connected. Then we can write $P_\alpha = U \cup V$ where U and V are both open in P_α, $U \cap V = \emptyset$, $(a_{\alpha-1}, \alpha) \in U$ and $(a_\alpha, \alpha) \in V$ (if the two points are in the same component, this must be all $L_\alpha \times \{\alpha\}$ since L_α is irreducible between $a_{\alpha-1}$ and a_α). Then the sets

$$U \cup \bigl\{(x, t)\colon t \in W(\kappa), t < \alpha\bigr\} \quad \text{and} \quad V \cup \bigl\{(x, t)\colon t \in W(\kappa),\ t > \alpha\bigr\}$$

disconnect P and we get a contradiction.

We have shown that $P \cap (K \times \{\alpha\}) = T \cap (K \times \{\alpha\})$ for all non limit ordinals α. Since P is closed we must have $P = T$. $\square$

Theorem 14. *Any non-degenerate subcontinuum of H^* maps onto any continuum of weight $\leqslant \omega_1$.*

Proof. Let $F \subset H^*$ be a non-degenerate subcontinuum of H^* and let K be any continuum of weight ω_1. Let $T \subset K \times W(\omega_1)$ be as in Lemma 1 and let S be the following "longer" version of T: $S = T \cup [0, 1]$ where we identify the point $0 \in [0, 1]$ with the point $\langle a_0, \omega_1 \rangle \in K \times \{\omega_1\} \subset T$. We will denote with $0, 1, s, t, \ldots$ the points of $J \subset T$ and with $0', 1', s', t', \ldots$ the points of the new interval we added. It is clear that for any choice of $t \in [0, 1)$ and $t' \in (0', 1']$ the continuum $S \setminus ([0, t) \cup (t', 1'])$ is irreducible between the points t and t'. S is a continuum of weight ω_1 and therefore by Fact 5 it is a remainder of H. Let αH be a compactification of H such that $\alpha H \setminus H = S$. Let A and B be open sets of αH such that $\overline{A} \cap S \subset \{a_0\} \times [0, \frac{1}{2})$ and $\overline{B} \cap S \subset ((\frac{1}{2})', 1']$.

Let $\{a_n\colon n < \omega\}$ and $\{b_n\colon n < \omega\}$ be two sequences in H such that $a_n \in A$, $b_n \in B$ and $a_n < b_n < a_{n+1}$ for all $n < \omega$.

Fix two points x and y of F and pick two standard neighborhoods with disjoint closures U and V of x and y, respectively (we recall that a standard open set in βH is an open set of the form $\beta H \setminus \mathrm{cl}_{\beta H}(H \setminus \bigcup_{n<\omega}(x_n, y_n))$ where $\{(x_n, y_n)\colon n < \omega\}$ is an unbounded sequence of disjoint open intervals in H; see [7]). Let (p_n, q_n) and (r_n, s_n) be the intervals in H that generate U and V, respectively. Without loss of generality we can assume that $p_n < q_n < r_n < s_n < p_{n+1}$ for all $n < \omega$. Define a function $g : H \to H$ as follows:

$$g(t) = \begin{cases} a_n & \text{if } t \in [p_n, q_n], \\ b_n & \text{if } t \in [r_n, s_n], \\ \text{connected linearly} & \text{elsewhere.} \end{cases}$$

Consider g as a map from H into αH and let $f:\beta H\to\alpha H$ be its Stone–Čech extension.

Observe that $f(x)\in\overline{A}$ and $f(y)\in\overline{B}$, so that $S\setminus([0,f(x)\cup(f(y),1'])$ is irreducible between $f(x)$ and $f(y)$. Since $f(F)$ is connected we have $T\subset f(F)$.

Finally define $h:F\to K$ as $h(p)=\pi(f(p))$ where $\pi:S\to K$ is the projection. □

Corollary 15. (CH) *Any non-degenerate subcontinuum of H^* maps onto H^*, and hence onto any remainder of H.*

In the proof of Theorem 14 we used the fact that $w(K)\leqslant\omega_1$ only when we claimed that the space S is a remainder of H. The same argument of the proof can be used to show that any non-degenerate subcontinuum of H^* maps onto K provided that the corresponding space S is a remainder of H. This observation is of no use to try to generalize Theorem 14 in ZFC; in fact if S is a remainder of H then the projection $pr_{W(\kappa)}S$ is also a remainder of H and we know that if we add ω_2 Cohen reals to a model of GCH the long segment $W(\kappa)$ is not a remainder of H for $\kappa>\omega_1$ (see [5]). However we have the following:

Proposition 16. *Let κ be a cardinal number and assume that any continuum of weight $<\kappa$ is a remainder of H. Then any non-degenerate subcontinuum of H^* can be mapped onto any continuum of weight $<\kappa$.*

Corollary 17. (MA) *Any non-degenerate subcontinuum of H^* can be mapped onto any continuum of weight $<c$.*

Do we have a similar situation for ω? In this case of course we should replace continua with compact spaces and to avoid trivial cases non-degenerate subcontinua with infinite closed subsets of ω^*. The theorem above is easily shown to be false for ω. In fact there are non-trivial (= infinite) separable closed subsets of ω^* (e.g., copies of $\beta\omega$) and clearly such spaces cannot be mapped onto a non-separable compact space. Let us remark that no separable continua can be found in H^* (in particular βH cannot be embedded into H^*). Therefore the only question about ω is about compact separable spaces. We can show the following:

Theorem 18. *Any non-trivial closed subspace of ω^* can be mapped onto any separable compact space.*

Proof. Let K be a separable compact space, and let F be any non-trivial closed subset of ω^*. It is well known that any infinite closed subset of ω^* contains a copy of $\beta\omega$; let S be one of these. Since S has countable π-weight, S is a retract of $\beta\omega$ [11], hence it is a retract of K with retraction r. Let f be a function mapping S onto K, then $f\circ r$ is the required surjection. □

Question 19. Can we map any non-degenerate subcontinuum of H^* onto any separable continuum ?

In particular,

Question 20. Can we map any non-degenerate subcontinuum of H^* onto any separable continuum that is a G_δ-subset (or better, a countable intersection of strongly nested pathwise connected sets) of a Tychonoff cube of weight $\leqslant c$?

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