Problems on \((f^2)^{(k)}\)

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Abstract

Let \(f\) be a transcendental meromorphic function in complex plane and have only zeros of multiplicity at least \(\left\lfloor \frac{k}{2} \right\rfloor + 1\), and \(k (\geq 1)\) be an integer. Then \((f^2)^{(k)}\) assumes every finite non-zero value infinitely often.

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1. Introduction and main results

In 1992, Yuefei Wang [3] proved the following result.

**Theorem A.** Let \(f\) be a transcendental meromorphic function in complex plane and let \(n \geq 3\), \(k \geq 0\) be two integers. Then \((f^n)^{(k)}\) assumes every finite non-zero value infinitely often.

In 1995, Huaihui Chen and Mingliang Fang [1] proved one of Hayman’s conjecture.

**Theorem B.** Let \(f\) be a transcendental meromorphic function in complex plane. Then \(f'f\) assumes every finite non-zero value infinitely often.

It is obvious that we can replace \(f'f\) by \((f^2)’\). In this note, we consider the value distribution of \((f^n)^{(k)}\) with \(n = 2\), \(k \geq 1\), and generalize Theorem B. Our main results are stated as follows.
Theorem 1. Let \( f \) be a transcendental meromorphic function in complex plane and have only zeros of multiplicity at least \( \left[ \frac{k}{2} \right] + 1 \), \( k \geq 1 \) be an integer. Then \( (f^{\frac{k}{2}})^{(k)} \) assumes every finite non-zero value infinitely often.

Theorem 2. Let \( n \geq 2 \), \( k \geq 0 \) be two integers, and \( k + n \neq 2 \). Let \( F \) be a family of functions meromorphic in a domain \( D \subset \mathbb{C} \) and \( b \) be a finite non-zero value in \( \mathbb{C} \), if each function \( f \in F \) satisfies \( (f^{n})^{(k)} \neq b \) and has only zeros of multiplicity \( \geq 1 + \left[ \frac{k}{n} \right] \), then \( F \) is normal.

Remark 1. The condition that all zeros of \( f \) have multiplicity \( \geq 1 + \left[ \frac{k}{n} \right] \) cannot be dropped in Theorem 2, as showed by the following.

Example 1. \( D = \{ z : |z| < 1 \} \), \( F = \{ f_{j}(z) : f_{j} = \left( \frac{j + 1}{(n \cdot k)!} \right)^{\frac{1}{n}} \cdot z^{k}, \ z \in D, \ n \in \mathbb{N}, \ k \in \mathbb{N}, \ j = 1, 2, \ldots \} \).

clearly, all zeros of \( f_{j}(z) \) have multiplicity \( k \), \( (f_{j}^{n}(z))^{n} \neq 1 \), and \( k < 1 + \left[ \frac{n}{k} \right] = l + 1 \), but \( F \) fails to be normal in \( z = 0 \).

Remark 2. Let \( n = 2, k = 0 \) in Theorem 2, however, \( F \) may fail to be normal in \( D \).

Example 2. \( D = \{ z : |z| < 1 \} \), \( F = \{ f_{n} : f_{n}(z) = \frac{2}{(e^{z})^{n} - 1} + 1, \ z \in D, \ n = 1, 2, 3, \ldots \} \).

Obviously, for each \( n \in \mathbb{N} \), \( f_{n}^{2} \neq 1 \), but \( F \) fails to be normal in \( D \).

Remark 3. Let \( n = 1, k \geq 0 \) in Theorem 2. Obviously, \( F \) may be not normal and we do not give examples.

2. Lemmas

Lemma 1. (See [5].) Let \( k \) be a positive integer and \( f(z) \) be a transcendental meromorphic function in the complex plane. Then

\[
T(r, f) < \left( 2 + \frac{1}{k} \right) N \left( r, \frac{1}{f} \right) + \left( 2 + \frac{2}{k} \right) N \left( r, \frac{1}{f^{(k)} - 1} \right) + S(r, f).
\]

Lemma 2. (See [4].) Let \( f \) be a meromorphic function of finite order. If \( f \) has infinitely many multiple zeros, then \( f' \) assumes every finite non-zero value infinitely often.

Lemma 3. (See [6].) Let \( Q(z) = a_{n}z_{n} + a_{n-1}z_{n-1} + \cdots + a_{0} + \frac{q(z)}{p(z)} \), where \( a_{0}, a_{1}, \ldots, a_{n} \) are constants with \( a_{n} \neq 0 \), \( q(z) \) and \( p(z) \) be two co-prime polynomials with \( \deg(q(z)) < \deg(p(z)) \), \( m \) be a positive integer. If \( Q^{(m)}(z) \neq 1 \) for each \( z \in C \), then

\[
Q(z) = \frac{z^{m}}{m!} + \cdots + a_{0} + \frac{1}{(az + b)^{n}}.
\]
If all zeros of $Q(z)$ have multiplicity at least $m + 1$, then

$$Q(z) = \frac{(cz + d)^{m+1}}{az + b}$$

where $a (\neq 0)$, $c (\neq 0)$, $b, d$ are constants.

**Lemma 4.** (See [2].) Let $k \in \mathbb{N}$ and $F$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$ with the property that each function in $F$ has only zeros of order at least $k$. If $F$ is not normal at $z_0$, $z_0 \in D$, then for any real number $0 \leq \alpha < k$, there exist a sequence $\{f_j\} \subset F$, a sequence $z_j \to 0$ ($z_j \in D$) and a positive sequence $\rho_j \to 0$ such that $\rho_j^{-\alpha} f_j(z_j + \rho_j \xi)$ uniformly spherically convergent to a nonconstant meromorphic function $g(\xi)$ on compact subsets of $\mathbb{C}$. Moreover, the order of $g(\xi)$ is at most 2 and the zeros of $g(\xi)$ are of multiplicity $\geq k$.

**Lemma 5.** Let $f$ be a transcendental meromorphic functions of finite order, and the zeros of $f$ be of multiplicity $\geq 1 + \lfloor \frac{k}{2} \rfloor$, $k \in \mathbb{N}$. Then $(f^2)^{(k)}$ assumes every finite non-zero value infinitely often.

**Proof.** If $f$ has finitely many zeros, then $f^2$ has finitely many zeros, and the conclusion follows from Lemma 1. If $f$ has infinitely many zeros and $z_0$ is a zero of $f$, $z_0$ is a zero of $(f^2)^{(k-1)}$ with multiplicity $\geq 2 \times (1 + \lfloor \frac{k}{2} \rfloor) - (k - 1) \geq 2$ and $(f^2)^{(k-1)}$ has infinitely many multiple zeros. Applying Lemma 2 to $(f^2)^{(k-1)}$, the conclusion follows.

**3. Proof of Theorem 1**

If $f$ is a transcendental meromorphic function of finite order, then the conclusion follows from Lemma 5. Hence we assume $f$ is a transcendental meromorphic function of infinite order.

**Step 1.** Suppose that the equation $(f^2)^{(k)} = a$ has a finite set of solutions for some $a \neq 0$, we may assume without loss of generality that $a = 1$. Let

$$D = \left\{ z : \frac{1}{3} - \varepsilon < |z| < 3 + \varepsilon \right\}, \quad 0 < \varepsilon < \frac{1}{10}, \quad D_0 = \left\{ z : \frac{1}{3} < |z| < 3 \right\},$$

$$f^\# = \frac{|f'|}{1 + |f|^2}.$$ 

Define a family $F$ consisting of all functions

$$f_n(z) = n^{-\frac{k}{2}} f(nz), \quad z \in D, \; n = 1, 2, \ldots.$$ 

This family cannot be normal in $D$. For otherwise we would have some $M > 0$,

$$M > f_n^\#(z) = \frac{n^{-\frac{k}{2} + 1} |f'(nz)|}{1 + n^{-k} |f(nz)|^2} \geq \frac{n^{-\frac{k}{2} + 1} |f'(nz)|}{1 + |f(nz)|^2} = n^{-\frac{k}{2} + 1} f^\#(nz), \quad z \in D_0. \quad (1)$$

From (1) we can obtain $f^\#(nz) \leq n^{\frac{k}{2} - 1} M$. Clearly, $\forall z \in D_0, \; \frac{1}{3} < |z|$, hence $n < 3|nz|$. Thus for each $n$, we have $f^\#(nz) \leq (3|nz|)^{\frac{k}{2} - 1} M$, $(z \in D_0)$, that is, for each $z \in \{ z : |\frac{1}{3} < |z| < \infty \}$, we have $f^\#(z) \leq (3|z|)^{\frac{k}{2} - 1} M$. 

Without effect on the result, we can assume that for each $z \in C$, $f^\#(z) \leq (3|z|)^{\frac{1}{2}-1}M$. Let

$$A(r) = \frac{1}{\pi} \int_0^r \int_0^{2\pi} (f^\#(e^{\rho i}))^2 \rho d\theta d\rho \leq \frac{1}{\pi} \times 2\pi \int_0^r ((3\rho)^{\frac{1}{2}-1}M)^2 \rho d\rho = \frac{3^{k-2} \times 2\pi^k M^2}{k}.$$ 

Hence $T_0(r) = \int_0^r A(t) dt \leq \int_0^r \frac{3^{k-2} \times 2\pi^{k-1}M^2}{k} dt = O(r^k)$. So the order of $f$ is finite and this contradicts the assumption. Hence $F$ is not normal in $D$.

Now, notice that

$$\left(f_n^2\right)^{(k)}(z) = (f^2)^{(k)}(nz). \quad (2)$$

Let $t = \frac{1}{z} - \xi$. Obviously, $\forall z \in D$, $|z| > t > 0$. Since the equation $(f^2)^k = 1$ has a finite set of solutions, let $\{z_1, z_2, \ldots, z_j\}$ be all zeros of $(f^2)^k - 1$, $(j \in N)$.

Let

$$M_0 = \max\{|z_1|, |z_2|, \ldots, |z_j|\}. \quad (3)$$

Let $n_0 = \left\lceil \frac{M_0}{16} + 1 \right\rceil$. We claim that $\forall n > n_0$, $\forall z \in D$, $(f_n^2)^{(k)}(z) \neq 1$. For otherwise there exists $n_1 > n_0$, and exists $z_0 \in D$ satisfying $(f_n^2)^{(k)}(z_0) = 1$, by (2), we have $(f^2)^{(k)}(n_1z_0) = 1$, that is, $n_0z$ is also a zero of $(f^2)^{(k)}(z) - 1$. However, $|nz_0| > \left\lceil \frac{M_0}{16} + 1 \right\rceil \times t > M_0$, this contradicts (3).

Without loss of generality, we assume $(f_n^2)^{(k)}(z) \neq 1$ in $D$ for each $n \in N$. Since $f_n(z) = n^{-\frac{k}{2}}f(nz)$, the zeros of $f_n$ are of multiplicity $\geq \left\lceil \frac{k}{2} \right\rceil + 1$.

**Step 2.** Assume $F$ is not normal at $z_0$. Applying Lemma 4 to $\alpha = -\frac{k}{2}$, there exist a sequence $\{f_j\} \subset F$, a sequence $z_j \to z_0$ and a positive sequence $\rho_j \to 0$ such that $g_j(\xi) = \rho_j^{\frac{k}{2}}f_j(z_j + \rho_j \xi)$ uniformly spherically convergent to a nonconstant meromorphic function $g(\xi)$ on compact subsets of $C$. Moreover, the order of $g(\xi)$ is at most 2 and the zeros of $g(\xi)$ are of multiplicity $\geq 1 + \left\lceil \frac{k}{2} \right\rceil$.

By the assumption $g_j^2(\xi)^{(k)} - 1 = (f^2)(z_j + \rho_j \xi)^{(k)} - 1 \neq 0$. Thus, by Hurwitz’s theorem, we have either $(g_2^2(\xi))^{(k)} - 1 \equiv 0$ or $(g_2^2(\xi))^{(k)} - 1 \neq 0$. If $(g_2^2(\xi))^{(k)} - 1 \equiv 0$, there exists a polynomial $p(z)$ (deg$(p(z)) \leq k$) satisfying $g_2^2(\xi) = p(z)$. However, the zeros of $g_2^2(\xi)$ are of multiplicity $\geq (1 + \left\lceil \frac{k}{2} \right\rceil) \times 2 \geq k + 1$. This is impossible.

If

$$\left(g_2^2(\xi)\right)^{(k)} - 1 \neq 0. \quad (4)$$

Lemma 5 implies that $g(\xi)$ is not transcendental. If $g(\xi)$ is a nonconstant polynomial, then by (4), there exists a polynomial $q(z)$ (deg$(q(z)) \leq k - 1$) satisfying $g_2^2(\xi) = q(z)$. This also contradicts that the zeros of $g_2^2(\xi)$ are of multiplicity $\geq k + 1$. The remaining case is that $g(\xi)$ is a nonconstant rational function. By (4), there exists a polynomial $h(\xi)$ such that

$$\left(g_2^2(\xi)\right)^{(k)} = \frac{h(\xi) + 1}{h(\xi)}. \quad (5)$$

Let $p_0$ and $q_0$ be the degree of the numerator and the denominator of $g(\xi)$, respectively. It is easy to verify that the difference between the degree of the numerator of $(g_2^2(\xi))^{(k)}$ and the degree of the denominator of $(g_2^2(\xi))^{(k)}$ is $2p_0 - 2q_0 - k$. It follows form (5) that $2p_0 - 2q_0 - k = 0$, that is, $k = 2(p_0 - q_0)$ $(k \geq 1)$, the zeros of $g_2^2(\xi)$ are of multiplicity $\geq 2 \times (1 + \left\lceil \frac{2(p_0 - q_0)}{2} \right\rceil) \times 2 = 2(p_0 - q_0) + 2 = k + 2$. It follows form Lemma 3 that $Q(z) = \frac{(cz + d)^{k+1}}{az + b}$ where $a \neq 0$, $c \neq 0$, $b$
and $d$ are constants, this contradicts that $g^2(\zeta)$ is of multiplicity $\geq k + 2$. Hence the equation $(f^2)^{(k)} = 1$ has a infinite set of solutions.

This completes the proof of Theorem 1.

4. Proof of Theorem 2

By using Theorem A, Theorem 2 can be proved like the argument of Step 2 of Theorem 1 without much difficult. We here omit the detail.

References