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Subnormality for arbitrary powers of 2-variable weighted shifts whose restrictions to a large invariant subspace are tensor products ☆

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Abstract

The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions for a pair of subnormal operators on Hilbert space to admit commuting normal extensions. We study LPCS within the class of commuting 2-variable weighted shifts $\mathbf{T} \equiv (T_1, T_2)$ with subnormal components T_1 and T_2, acting on the Hilbert space $\ell^2(\mathbb{Z}_+^2)$ with canonical orthonormal basis $\{e_{(k_1,k_2)}\}_{k_1,k_2 \geqslant 0}$. The *core* of a commuting 2-variable weighted shift $\mathbf{T}$, $c(\mathbf{T})$, is the restriction of $\mathbf{T}$ to the invariant subspace generated by all vectors $e_{(k_1,k_2)}$ with $k_1, k_2 \geqslant 1$; we say that $c(\mathbf{T})$ is of *tensor form* if it is unitarily equivalent to a shift of the form $(I \otimes W_\alpha, W_\beta \otimes I)$, where W_α and W_β are subnormal unilateral weighted shifts. Given a 2-variable weighted shift $\mathbf{T}$ whose core is of tensor form, we prove that LPCS is solvable for $\mathbf{T}$ if and only if LPCS is solvable for any power $\mathbf{T}^{(m,n)} := (T_1^m, T_2^n)$ $(m, n \geqslant 1)$.

Keywords: Jointly hyponormal pairs; Subnormal pairs; 2-Variable weighted shifts; Tensor form; Core

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1. Introduction

The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions for a pair of subnormal operators on Hilbert space to admit commuting normal extensions. In previous work [6–9,14–16,27] we have studied LPCS from a number of different approaches. One such approach is to consider commuting pairs $\mathbf{T} \equiv (T_1, T_2)$ of subnormal operators and to ask to what extent the existence of liftings for the powers $\mathbf{T}^{(m,n)} := (T_1^m, T_2^n)$ $(m, n \geqslant 1)$ can guarantee a lifting for $\mathbf{T}$. For the class of 2-variable weighted shifts $\mathbf{T}$, it is often the case that the powers of $\mathbf{T}$ are less complex than the initial pair; thus it becomes especially significant to unravel how subnormality behaves under the action $(m, n) \mapsto \mathbf{T}^{(m,n)}$ $(h, \ell \geqslant 1)$.

Within the class of 2-variable weighted shifts, we consider the subclass $\mathcal{TC}$ consisting of pairs whose cores are of tensor form; that is

$$\mathcal{TC} := \{\mathbf{T} \in \mathfrak{H}_0\colon\ c(\mathbf{T}) \text{ is of tensor form}\}.$$

This subclass has proved to be particularly attractive, since it is possible to separate, within it, subnormality from k-hyponormality; thus, results about LPCS for pairs in $\mathcal{TC}$ are especially useful. The class $\mathcal{TC}$ is small enough to allow for a simple description of its pairs, yet large enough to be used as test ground for many significant hypotheses.

Before we proceed, we briefly pause to establish our terminology. For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geqslant 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. Similarly, consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, and let $\{e_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$ be the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+^2)$. We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ acting on $\ell^2(\mathbb{Z}_+^2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \quad \text{and} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. The *core* of a commuting 2-variable weighted shift $\mathbf{T}$ (in symbols, $c(\mathbf{T})$) is the restriction of $\mathbf{T}$ to the invariant subspace generated by all vectors $e_{(k_1,k_2)}$ with $k_1, k_2 \geqslant 1$; we say that $c(\mathbf{T})$ is of *tensor form* if it is unitarily equivalent to a shift of the form $(I \otimes W_\alpha, W_\beta \otimes I)$, where W_α and W_β are subnormal unilateral weighted shifts. Fig. 1 shows both the weight and Berger measure diagrams of a typical pair in $\mathcal{TC}$. As shown in [8], each $\mathbf{T} \in \mathcal{TC}$ is completely determined by five parameters, i.e., the 1-variable measures σ, τ, ξ and η, and the positive number $a \equiv \alpha_{(0,1)}$. As we mentioned before, $\mathcal{TC}$ is of substantial interest to us, since it provides a fertile ground to test results on subnormality and k-hyponormality, and in particular about the solubility of LPCS.

Let us now denote the class of commuting pairs of subnormal operators on Hilbert space by $\mathfrak{H}_0$, the class of subnormal pairs by $\mathfrak{H}_\infty$, and for an integer $k \geqslant 1$, the class of k-hyponormal pairs in $\mathfrak{H}_0$ by $\mathfrak{H}_k$. Clearly, $\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0$; the main results in [14] and [6] show that these inclusions are all proper; moreover, examples illustrating these proper inclusions can be found in $\mathcal{TC}$.

In this paper we show that for $\mathbf{T} \in \mathcal{TC}$, the subnormality of *any* power $\mathbf{T}^{(m,n)}$ implies the subnormality of $\mathbf{T}$. To accomplish this, we first show that every power of $\mathbf{T} \in \mathcal{TC}$ is the orthogonal direct sum of 2-variable weighted shifts in $\mathcal{TC}$. Since each 2-variable weighted shift in $\mathcal{TC}$

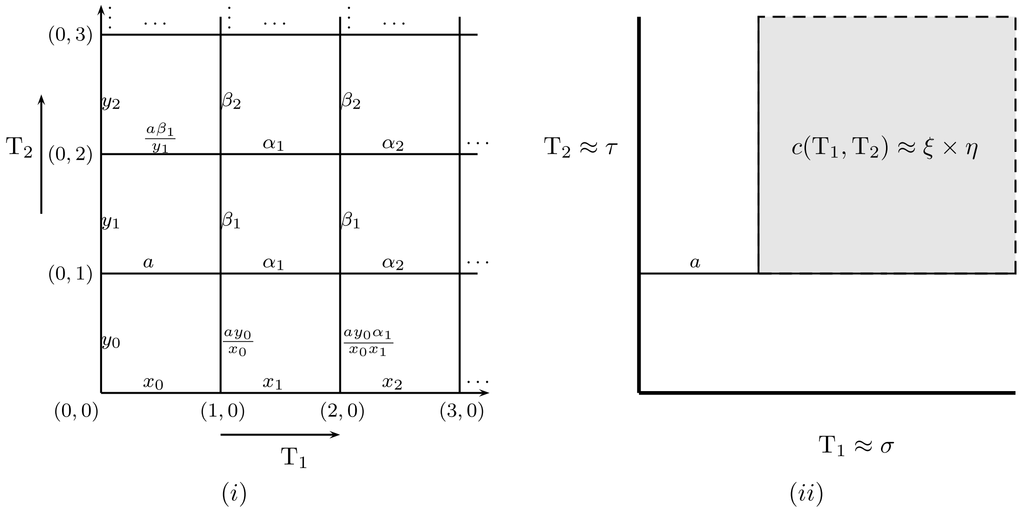

Fig. 1. Weight and Berger measure diagrams of a typical 2-variable weighted shift in $\mathcal{TC}$.

is completely determined by five parameters, we then study how the five parameters of each direct summand in a power are related to the five parameters in the initial 2-variable weighted shift. Next, we recall from [8] that each $\mathbf{T} \in \mathcal{TC}$ is associated with a pair of linear functionals $\varphi \equiv \varphi(\mathbf{T})$ and $\psi \equiv \psi(\mathbf{T})$ (each depending on the five parameters), and that $\mathbf{T}$ is subnormal if and only if $\varphi \geqslant 0$ and $\psi \geqslant 0$. With all of this at our disposal, we proceed to establish a connection between the pair (φ, ψ) associated with $\mathbf{T}$ and those associated with the summands in the orthogonal direct sum decomposition of $\mathbf{T}^{(m,n)}$. This eventually leads to the proof of our main result (Theorem 3.1).

This result provides a complete generalization of Theorem 3.9 in [7]. At the time we wrote [7], the techniques available to us allowed us to deal only with the quadratic powers $\mathbf{T}^{(1,2)}$ and $\mathbf{T}^{(2,1)}$; with the aid of a number of additional examples, together with the main result in [8], we have now been able to handle the case of arbitrary powers.

As an application of Theorem 3.1, we can exhibit a hyponormal 2-variable weighted shift such that none of its powers is subnormal. We describe the shift in Example 8.1. This provides a striking and concrete example of the big gap that exists between hyponormality and subnormality for 2-variable weighted shifts, even within a relatively simple class like $\mathcal{TC}$.

2. Notation and preliminaries

To describe our results in detail we need some notation; we further expand on our terminology later in this section. For $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}_+^2$, we shall let $\mathcal{M}_i$ (resp. $\mathcal{N}_j$) be the subspace of $\ell^2(\mathbb{Z}_+^2)$ which is spanned by the canonical orthonormal basis vectors associated to indices $\mathbf{k}$ with $k_1 \geqslant 0$ and $k_2 \geqslant i$ (resp. $k_1 \geqslant j$ and $k_2 \geqslant 0$). The *core* of a 2-variable weighted shift $\mathbf{T}$ is the restriction of $\mathbf{T}$ to $\mathcal{M}_1 \cap \mathcal{N}_1$; in symbols, $c(\mathbf{T}) := \mathbf{T}|_{\mathcal{M}_1 \cap \mathcal{N}_1}$. A 2-variable weighted shift $\mathbf{T}$ is said to be of *tensor form* if $\mathbf{T} \cong (I \otimes W_\xi, W_\eta \otimes I)$, where W_ξ and W_η are unilateral weighted shifts. The class of all 2-variable weighted shifts $\mathbf{T} \in \mathfrak{H}_0$ whose cores are of tensor form is denoted by $\mathcal{TC}$; that is,

$$\mathcal{TC} := \{\mathbf{T} \in \mathfrak{H}_0\colon\ c(\mathbf{T}) \text{ is of tensor form}\}$$

(see Fig. 1(i)).

It is well known that the commutativity of a pair of subnormals is necessary but not sufficient for the existence of a lifting [1,20–22], and it has recently been shown that the joint hyponormality of the pair is necessary but not sufficient [14]. Our previous work [14–17,6–9,26,27] has revealed that the nontrivial aspects of the LPCS are best detected within the class $\mathfrak{H}_0$, especially within $\mathcal{TC}$; we thus focus our attention on this class.

For a single operator T, the subnormality of all powers T^n ($n \geqslant 2$) does not imply the subnormality of T, even if T is a unilateral weighted shift [24, pp. 378–379]. Thus one might guess that if we were to impose a further condition such as the subnormality of a restriction of T to an invariant subspace, e.g., the subnormality of $T|_{\bigvee\{e_k \in \ell^2(\mathbb{Z}_+)\colon k \geqslant i\}}$ (for some $i \geqslant 1$), that T would then be subnormal. However, even if we assume this for $i = 1$, the subnormality of T is not guaranteed. For example, let $T := \text{shift}(\frac{1}{3}, \frac{1}{2}, 1, 1, \ldots)$. Then $T|_{\bigvee\{e_k \in \ell^2(\mathbb{Z}_+)\colon k \geqslant 1\}} \equiv \text{shift}(\frac{1}{2}, 1, 1, \ldots)$ is subnormal, and also all powers T^n ($n \geqslant 2$) are subnormal, but T is not subnormal. As a mater of fact, no backward extension $\text{shift}(\alpha_0, \frac{1}{2}, 1, 1, \ldots)$ can be subnormal [5, Corollary 6]. More generally, the necessary and sufficient conditions for a unilateral weighted shift W_α to be subnormal when we assume that $W_\alpha|_{\bigvee\{e_k \in \ell^2(\mathbb{Z}_+)\colon k \geqslant 1\}}$ is subnormal (with Berger measure μ) were obtained in [5, Proposition 8]: W_α is subnormal if and only if $\frac{1}{t} \in L^1(\mu)$ and $\alpha_0^2 \|\frac{1}{t}\|_{L^1(\mu)} \leqslant 1$.

In the multivariable case, the analogous results are highly nontrivial, if one further assumes that each component is subnormal. In 1-variable, the subspace $\bigvee\{e_k \in \ell^2(\mathbb{Z}_+)\colon k \geqslant 1\}$ can be regarded as “the core of T”; as we move into two variables it is therefore natural to consider the condition $\mathbf{T} \in \mathcal{TC}$.

To prove our results, we require a number of tools and techniques introduced in previous work, e.g., the Six-point Test (Lemma 4.1), the Backward Extension Theorem for 2-variable weighted shifts (Lemma 4.2) and the formula to reconstruct the Berger measure of a unilateral weighted shift (4.1), together with a new direct sum decomposition for powers of 2-variable weighted shifts which parallels the decomposition used in [11] to analyze k-hyponormality for powers of (1-variable) weighted shifts. Concretely, to analyze the power $\mathbf{T}^{(m,n)}$ of a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$, we split the ambient space $\ell^2(\mathbb{Z}_+^2)$ as an orthogonal direct sum $\bigoplus_{p=0}^{m-1} \bigoplus_{q=0}^{n-1} \mathcal{H}_{(p,q)}^{(m,n)}$, where for $p = 0, 1, \ldots, m-1$, and $q = 0, 1, \ldots, n-1$,

$$\mathcal{H}_{(p,q)}^{(m,n)} := \bigvee\{e_{(m\ell+p,nk+q)}\colon\ k, \ell \geqslant 0\}. \tag{2.1}$$

Each of the subspaces $\mathcal{H}_{(p,q)}^{(m,n)}$ reduces T_1^m and T_2^n, and $\mathbf{T}^{(m,n)}$ is subnormal if and only if each summand $\mathbf{T}^{(m,n)}|_{\mathcal{H}_{(p,q)}^{(m,n)}}$ is subnormal. For a set of pairs $\mathcal{X}$, let $\bigoplus \mathcal{X}$ denote the set of pairs that can be written as orthogonal sums of pairs in $\mathcal{X}$. We will show in Section 5 that $\bigoplus \mathcal{TC}$ is invariant under the action $(m, n) \mapsto \mathbf{T}^{(m,n)}$ ($m, n \geqslant 1$).

Briefly stated, our strategy to prove our main result (Theorem 3.1) is as follows: (i) if $\mathbf{T} \in \mathcal{TC}$ then each power $\mathbf{T}^{(m,n)} \in \bigoplus \mathcal{TC}$; (ii) without loss of generality, we can always assume $m = 1$; (iii) the pair (φ, ψ) associated with $\mathbf{T}$ is directly related to the pairs $(\varphi^{(p,q)}, \psi^{(p,q)})$ associated to the direct summands in the orthogonal decomposition of $\mathbf{T}^{(1,n)}$; (iv) if a power $\mathbf{T}^{(1,n)}$ is subnormal, the functionals $\varphi^{(0,0)}$ and $\psi^{(0,1)}$ are both positive; and (v) it then follows that φ and ψ are both positive, and therefore $\mathbf{T}$ is subnormal.

We devote the rest of this section to establishing some additional terminology and notation. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. Recall that a bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$. An operator T is said to be *hyponormal* if $T^*T \geqslant TT^*$. For $S, T \in \mathcal{B}(\mathcal{H})$, let $[S,T] := ST - TS$. An n-tuple $\mathbf{T} := (T_1, \dots, T_n)$ of operators on $\mathcal{H}$ is said to be (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive semidefinite on the direct sum of n copies of $\mathcal{H}$ (cf. [2,10,12,13,23]). For instance, if $n = 2$,

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix}.$$

The n-tuple $\mathbf{T} \equiv (T_1, T_2, \dots, T_n)$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. In particular, a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is said to be *k-hyponormal* ($k \geqslant 1$) [6] if

$$\mathbf{T}(k) := \left(T_1, T_2, T_1^2, T_2T_1, T_2^2, \dots, T_1^k, T_2T_1^{k-1}, \dots, T_2^k\right)$$

is hyponormal, or equivalently

$$\left[\mathbf{T}(k)^*, \mathbf{T}(k)\right] = \left(\left[\left(T_2^q T_1^p\right)^*, T_2^m T_1^n\right]\right)_{\substack{1 \leqslant p+q \leqslant k \\ 1 \leqslant n+m \leqslant k}} \geqslant 0.$$

Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ k-hyponormal. For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geqslant 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. For a weighted shift W_α, *the moments of* α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0, \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0. \end{cases} \tag{2.2}$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leqslant \alpha_1 \leqslant \cdots$. Similarly, consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$. We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \quad \text{and} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1,0)$ and $\varepsilon_2 := (0,1)$. Clearly,

$$T_1T_2 = T_2T_1 \quad \Leftrightarrow \quad \beta_{\mathbf{k}+\varepsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}} \quad \left(\text{all } \mathbf{k} \in \mathbb{Z}_+^2\right). \tag{2.3}$$

In an entirely similar way one can define multivariable weighted shifts. Given $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, the *moment* of (α, β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha,\beta) := \begin{cases} 1 & \text{if } k_1 = 0 \text{ and } k_2 = 0, \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geqslant 1 \text{ and } k_2 = 0, \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geqslant 1, \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geqslant 1 \text{ and } k_2 \geqslant 1. \end{cases}$$

We remark that, due to the commutativity condition (2.3), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0,0)$ to (k_1,k_2).

We now recall a well-known characterization of subnormality for multivariable weighted shifts [19] (due to C. Berger [3, II.6.10] and independently established by R. Gellar and L.J. Wallen [18] in the single variable case): $\mathbf{T}$ admits a commuting normal extension if and only if there is a probability measure μ (which we call the Berger measure of $\mathbf{T}$) defined on the 2-dimensional rectangle $R = [0,a_1] \times [0,a_2]$ (where $a_i := \|T_i\|^2$) such that $\gamma_{\mathbf{k}} = \int_R s^{k_1} t^{k_2}\, d\mu(s,t)$, for all $\mathbf{k} \equiv (k_1,k_2) \in \mathbb{Z}_+^2$.

The following well-known result, which links the Berger measure of a subnormal unilateral weighted shift with the Berger measure of its restriction to a suitable invariant subspace, will be needed in Section 5.

Lemma 2.1. *(See [14, p. 5140].) Let W_α be a subnormal unilateral weighted shift and let ξ denote its Berger measure. For $n \geqslant 1$ let $\mathcal{L}_n := \bigvee\{e_h\colon h \geqslant n\}$ denote the invariant subspace obtained by removing the first n vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$. Then the Berger measure of $W_\alpha|_{\mathcal{L}_n}$ is*

$$d\xi_n(s) := \frac{s^n}{\gamma_n}\, d\xi(s), \tag{2.4}$$

where γ_n is the n-th moment of α, given by (2.2)*.*

We will occasionally write $\mathrm{shift}(\alpha_0,\alpha_1,\ldots)$ to denote the weighted shift with weight sequence $\{\alpha_k\}_{k=0}^\infty$. We also denote by $U_+ := \mathrm{shift}(1,1,\ldots)$ the (unweighted) unilateral shift, and for $0 < a < 1$ we let $S_a := \mathrm{shift}(a,1,1,\ldots)$. Observe that U_+ and S_a are subnormal, with Berger measures δ_1 and $(1-a^2)\delta_0 + a^2\delta_1$, respectively, where δ_p denotes the point-mass probability measure with support the singleton $\{p\}$.

Given integers i and m ($m \geqslant 1$, $0 \leqslant i \leqslant m-1$), consider $\mathcal{H} \equiv \ell^2(\mathbb{Z}_+) = \bigvee\{e_n\colon n \geqslant 0\}$ and define $\mathcal{H}_i := \bigvee\{e_{mj+i}\colon j \geqslant 0\}$, so $\mathcal{H} = \bigoplus_{i=0}^{m-1}\mathcal{H}_i$. For a sequence $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$, let $\alpha(m:i) := \{\Pi_{k=0}^{m-1}\alpha_{mj+i+k}\}_{j=0}^\infty$, that is, $\alpha(m:i)$ denotes the sequence of products of numbers in adjacent packets of size m, beginning with the product $\alpha_i \cdots \alpha_{i+m-1}$. For example, $\alpha(2:0): \alpha_0\alpha_1, \alpha_2\alpha_3, \alpha_4\alpha_5, \ldots,$ and $\alpha(3:2): \alpha_2\alpha_3\alpha_4, \alpha_5\alpha_6\alpha_7, \ldots.$ Then for $m \geqslant 1$ and $0 \leqslant i \leqslant m-1$, $W_{\alpha(m:i)}$ is unitarily equivalent to $W_\alpha^m|_{\mathcal{H}_i}$. Therefore, W_α^m is unitarily equivalent to $\bigoplus_{i=0}^{m-1} W_{\alpha(m:i)}$. This analysis naturally leads to the following result, which will be needed in Section 6.

Lemma 2.2. *(See [11, Theorem 2.9].) Let W_α be a subnormal unilateral weighted shift with Berger measure μ. Then $W_{\alpha(m,i)}$ is subnormal with Berger measure $\mu_{(m,i)}$, where*

$$d\mu_{(m,0)}(s) = d\mu\big(s^{\frac{1}{m}}\big) \quad \textit{and} \quad d\mu_{(m,i)}(s) = \frac{s^{\frac{i}{m}}}{\gamma_i}\, d\mu\big(s^{\frac{1}{m}}\big) \quad \textit{for } 1 \leqslant i \leqslant m-1.$$

3. Statement of the main result

In [7] we showed that if $\mathbf{T} \in \mathcal{TC}$ (see Fig. 1(i)), then

$$\mathbf{T}^{(1,2)} \in \mathfrak{H}_\infty \quad \Leftrightarrow \quad \mathbf{T}^{(2,1)} \in \mathfrak{H}_\infty \quad \Leftrightarrow \quad \mathbf{T} \in \mathfrak{H}_\infty. \tag{3.1}$$

The main result in this paper, which follows, is based on (3.1) and a myriad of examples that have arisen in our previous research.

Theorem 3.1. *Let $\mathbf{T} \in \mathcal{TC}$. The following statements are equivalent.*

(i) $\mathbf{T} \in \mathfrak{H}_\infty$;
(ii) $\mathbf{T}^{(m,n)} \in \bigoplus \mathfrak{H}_\infty$ *for all* $m, n \geqslant 1$;
(iii) $\mathbf{T}^{(m,n)} \in \bigoplus \mathfrak{H}_\infty$ *for some* $m, n \geqslant 1$.

4. Some basic facts

For the reader's convenience, in this section we list several well-known auxiliary results and definitions which are needed for the proof of the main result. First, to detect hyponormality for 2-variable weighted shifts we use a simple criterion involving a base point $\mathbf{k}$ in $\mathbb{Z}_+^2$ and its five neighboring points in $\mathbf{k} + \mathbb{Z}_+^2$ at path distance at most 2.

Lemma 4.1 *(Six-point Test). (See [4].) Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift, with weight sequences α and β. Then*

$$[\mathbf{T}^*, \mathbf{T}] \geqslant 0$$
$$\Leftrightarrow \quad H_{(k_1,k_2)}(1) := \begin{pmatrix} \alpha_{\mathbf{k}+\varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta_{\mathbf{k}+\varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geqslant 0 \quad \big(\textit{all } \mathbf{k} \in \mathbb{Z}_+^2\big).$$

Next, we present a criterion to detect the subnormality of 2-variable weighted shifts. First, we need some definitions.

(i) Let μ and ν be two positive measures on $\mathbb{R}_+$. We say that $\mu \leqslant \nu$ on $X := \mathbb{R}_+$, if $\mu(E) \leqslant \nu(E)$ for all Borel subset $E \subseteq \mathbb{R}_+$; equivalently, $\mu \leqslant \nu$ if and only if $\int f\,d\mu \leqslant \int f\,d\nu$ for all $f \in C(X)$ such that $f \geqslant 0$ on $\mathbb{R}_+$.
(ii) Let μ be a probability measure on $X \times Y$, and assume that $\frac{1}{t} \in L^1(\mu)$. The *extremal measure* μ_{ext} (which is also a probability measure) on $X \times Y$ is given by $d\mu_{ext}(s,t) := (1 - \delta_0(t)) \frac{1}{t\|\frac{1}{t}\|_{L^1(\mu)}}\,d\mu(s,t)$.
(iii) Given a measure μ on $X \times Y$, the *marginal measure* μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$.

To state the following result, recall the notation in (2.1), and let $\mathcal{M} := \mathcal{M}_1 \equiv \mathcal{H}_{(0,1)}^{(1,1)}$.

Lemma 4.2 *(Subnormal backward extension). (See [14, Proposition 3.10].) Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift, and assume that $\mathbf{T}|_{\mathcal{M}}$ is subnormal with associated measure $\mu_{\mathcal{M}}$*

and that $W_0 := \mathrm{shift}(\alpha_{00}, \alpha_{10}, \ldots)$ *is subnormal with associated measure* σ. *Then* $\mathbf{T}$ *is subnormal if and only if*

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \leqslant (\|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;
(iii) $\beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leqslant \sigma$.

Moreover, if $\beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})} = 1$, *then* $(\mu_{\mathcal{M}})_{ext}^X = \sigma$. *In the case when* $\mathbf{T}$ *is subnormal, its Berger measure* μ *is given by*

$$d\mu(s,t) = \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) + \left(d\sigma(s) - \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s) \right) d\delta_0(t). \tag{4.1}$$

5. The structure of powers of 2-variable weighted shifts in $\mathcal{TC}$

Consider a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{TC}$ (see Fig. 1(i)). Since T_1 (resp. T_2) is subnormal, we know that $\mathrm{shift}(\alpha_1, \alpha_2, \alpha_3 \cdots)$ (resp. $\mathrm{shift}(\beta_1, \beta_2, \beta_3 \cdots)$) is subnormal; let ξ (resp. η) be its Berger measure. Similarly, let σ (resp. τ) denote the Berger measure of $\mathrm{shift}(x_0, x_1, x_2, \ldots)$ (resp. $\mathrm{shift}(y_0, y_1, y_2, \ldots)$). Finally, let τ_1 be the Berger measure of $\mathrm{shift}(y_1, y_2, y_3, \ldots) \equiv \mathrm{shift}(y_0, y_1, y_2, \ldots)|_{\bigvee\{e_k:\ k \geqslant 1\}}$. Fig. 1 shows the general form of a pair in $\mathcal{TC}$, and that it is uniquely determined by the five parameters σ, τ, a, ξ and η. Thus, in what follows we will identify a pair $\mathbf{T} \in \mathcal{TC}$ with the 5-tuple $\langle \sigma, \tau, a, \xi, \eta \rangle$. We shall also let $[a, \xi]$ denote the Berger measure of the subnormal unilateral weighted shift W whose 0-th weight is a and with ξ as the Berger measure of $W|_{\mathcal{L}_1}$, where $\mathcal{L}_1 := \bigvee\{e_k:\ k \geqslant 1\}$. For instance, in Fig. 1(i) the Berger measure for the first row is $[a, \xi]$, and for the second row is $[\frac{a\beta_1}{y_1}, \xi]$. Finally, we shall let $z_j \equiv z_j(\eta)$ denote the j-th weight of the unilateral weighted shift whose Berger measure is η; that is, $\mathrm{shift}(z_0, z_1, \ldots)$ has Berger measure η.

Lemma 5.1. *Let* $\mathbf{T} \in \mathcal{TC}$, *and let* $m, n \geqslant 1$. *Then* $\mathbf{T}^{(m,n)} \in \bigoplus \mathcal{TC}$.

Proof. First recall from (2.1) that $\ell^2(\mathbb{Z}_+^2)$ can be written as an orthogonal direct sum $\bigoplus_{p=0}^{m-1} \bigoplus_{q=0}^{n-1} \mathcal{H}_{(p,q)}^{(m,n)}$, where for $p = 0, 1, \ldots, m-1$ and $q = 0, 1, \ldots, n-1$ we have $\mathcal{H}_{(p,q)}^{(m,n)} := \bigvee\{e_{(m\ell+p, nk+q)}:\ k, \ell \geqslant 0\}$. Now write $\mathbf{T} \equiv \langle \sigma, \tau, a, \xi, \eta \rangle$. We shall establish that

$$\mathbf{T}^{(m,n)} = \bigoplus_{p=0}^{m-1} \bigoplus_{q=0}^{n-1} \langle \sigma^{(p,q)}, \tau^{(p,q)}, a^{(p,q)}, \xi^{(p,q)}, \eta^{(p,q)} \rangle, \tag{5.1}$$

where $\langle \sigma^{(p,q)}, \tau^{(p,q)}, a^{(p,q)}, \xi^{(p,q)}, \eta^{(p,q)} \rangle$ is the 5-tuple associated to the restriction of $\mathbf{T}^{(m,n)}$ to the reducing subspace $\mathcal{H}_{(p,q)}^{(m,n)}$. Since $\mathbf{T}^{(m,n)} = (\mathbf{T}^{(m,1)})^{(1,n)}$, and since $\mathbf{T}^{(m,1)} \in \bigoplus \mathcal{TC}$ if and only if $\mathbf{T}^{(1,m)} \in \bigoplus \mathcal{TC}$, it suffices to prove (5.1) in the case $m = 1$. The proof is simple but a bit tedious, and it entails careful diagram chasing in Fig. 3. Visual inspection of that figure reveals that when $m = 1$ we have $\sigma^{(0,0)} = \sigma$, $\sigma^{(0,1)} = [a, \xi], \ldots, \sigma^{(0,n-1)} = [\frac{az_0 \cdots z_{n-3}}{y_1 \cdots y_{n-2}}, \xi]$ $(n \geqslant 3)$; moreover,

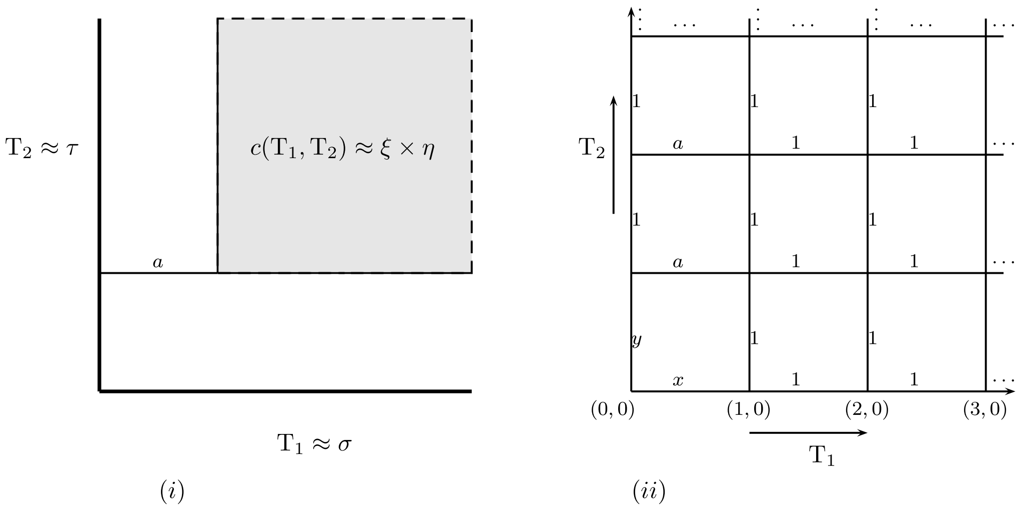

Fig. 2. Berger measure diagram and weight diagram of the 2-variable weighted shifts in Lemma 5.3 and Example 5.4, respectively.

$\tau^{(0,q)} = \tau_{(n,q)}$, where the latter notation was introduced in Lemma 2.2. Still looking at Fig. 1, we observe that $a^{(0,q)} = \frac{az_0 \cdots z_{n-2+q}}{y_1 \cdots y_{n-1+q}}$ and that $\xi^{(0,q)} = \xi$, $\eta^{(0,0)} = \eta_{(n,n-1)}$ and $\eta^{(0,q)} = (\eta_{(n,q-1)})_1$ ($q \geqslant 1$). This completes the proof. $\square$

Corollary 5.2. *For $m, n \geqslant 1$ we have $(\bigoplus \mathcal{TC})^{(m,n)} \subseteq \bigoplus \mathcal{TC}$.*

We now restate the main result in [8]. First, recall that if τ is the Berger measure of shift$(y_0, y_1, \ldots)$, we denote by τ_1 the Berger measure of shift$(y_1, y_2, \ldots)$. As described in Lemma 2.1, we know that $d\tau_1(t) \equiv \frac{t}{y_0^2}\, d\tau(t)$.

Lemma 5.3. *(See [8, Theorem 2.3].) Let $\mathbf{T} \equiv \langle \sigma, \tau, a, \xi, \eta \rangle \in \mathcal{TC}$ be as in Fig.* 2(i) *and let*

$$\psi := \tau_1 - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \eta,$$

$$\varphi := \sigma - y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 - a^2 y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\eta)} \frac{\xi}{s}, \tag{5.2}$$

where $y_0 \equiv \beta_{00} := \sqrt{\int t\, d\tau(t)}$. Then $\mathbf{T}$ is subnormal if and only if $\psi \geqslant 0$ and $\varphi \geqslant 0$.

Example 5.4. Assume the very simple case of $\mathbf{T} \equiv \langle \sigma, \tau, a, \xi, \eta \rangle$, where $\sigma := [x, \delta_1]$, $\tau := [y, \delta_1]$, $0 < a < 1$, $\xi := \delta_1$ and $\eta := \delta_1$ (cf. Fig. 2(ii)). Then $\psi = \delta_1 - a^2\delta_1 = (1 - a^2)\delta_1$ and $\varphi = (1 - x^2)\delta_0 + x^2\delta_1 - y^2(1 - a^2)\delta_0 - a^2y^2\delta_1 = \{(1 - x^2) - y^2(1 - a^2)\}\delta_0 + (x^2 - a^2y^2)\delta_1$. Thus, $\mathbf{T}$ is subnormal if and only if $(1 - x^2) - y^2(1 - a^2) \geqslant 0$, a condition identical to that in [14, Proposition 2.11].

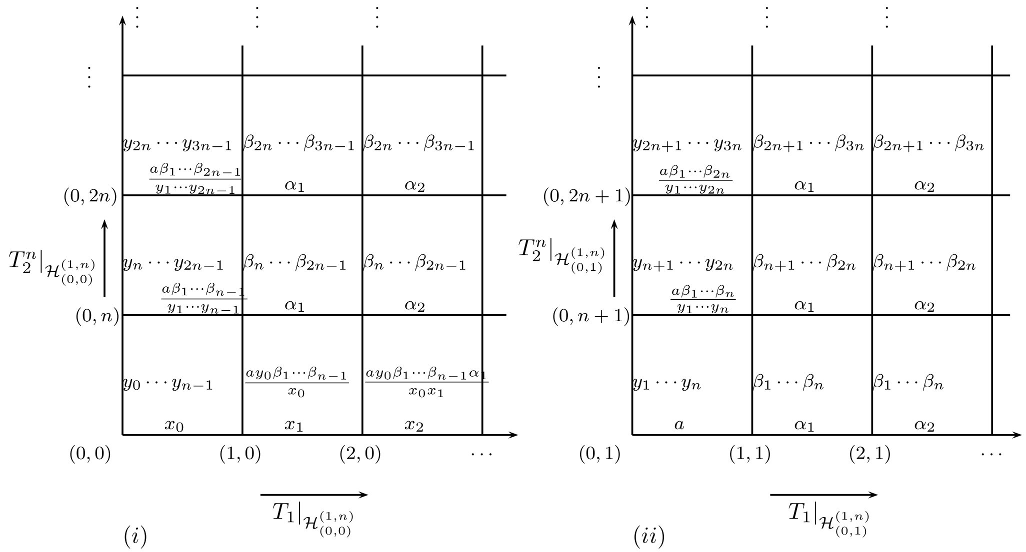

Fig. 3. Weight diagrams of $\mathbf{T}^{(1,n)}|_{\mathcal{H}_{(0,0)}^{(1,n)}}$ and $\mathbf{T}^{(1,n)}|_{\mathcal{H}_{(0,1)}^{(1,n)}}$.

6. The pairs (ψ, φ) for $\langle \sigma, \tau, a, \xi, \eta \rangle$ and $\langle \sigma^{(p,q)}, \tau^{(p,q)}, a^{(p,q)}, \xi^{(p,q)}, \eta^{(p,q)} \rangle$

In this section we establish a direct relationship between the pair (ψ, φ) associated to $\langle \sigma, \tau, a, \xi, \eta \rangle \in \mathcal{TC}$ and some of the pairs $(\psi^{(p,q)}, \varphi^{(p,q)})$ associated to the direct summands $\langle \sigma^{(p,q)}, \tau^{(p,q)}, a^{(p,q)}, \xi^{(p,q)}, \eta^{(p,q)} \rangle$ in $\langle \sigma, \tau, a, \xi, \eta \rangle^{(m,n)}$.

Proposition 6.1. *Let $\langle \sigma, \tau, a, \xi, \eta \rangle \in \mathcal{TC}$, and let $n \geqslant 2$. Consider the decomposition $\langle \sigma, \tau, a, \xi, \eta \rangle^{(1,n)} = \bigoplus_{q=0}^{n-1} \langle \sigma^{(0,q)}, \tau^{(0,q)}, a^{(0,q)}, \xi^{(0,q)}, \eta^{(0,q)} \rangle$, and let (ψ, φ) (resp. $(\psi^{(0,q)}, \varphi^{(0,q)})$) be the associated pair in Lemma* 5.3. *Then*

(i) $\psi^{(0,1)} \geqslant 0 \Leftrightarrow \psi \geqslant 0$; *and*
(ii) $\varphi^{(0,0)} \geqslant 0 \Leftrightarrow \varphi \geqslant 0$.

Proof. We refer the reader to Fig. 3. By Lemma 5.1, we have $\sigma^{(0,0)} = \sigma$, $\sigma^{(0,1)} = [a, \xi]$, $\tau^{(0,0)} = \tau_{(n,0)}$, $\tau^{(0,1)} = \tau_{(n,1)}$, $a^{(0,0)} = \frac{a z_0 \cdots z_{n-2}}{y_1 \cdots y_{n-1}}$, $a^{(0,1)} = \frac{a z_0 \cdots z_{n-1}}{y_1 \cdots y_n}$, $\xi^{(0,0)} = \xi^{(0,1)} = \xi$, $\eta^{(0,0)} = \eta_{(n,n-1)}$ and $\eta^{(0,1)} = (\eta_{(n,0)})_1$. Then

$$\psi^{(0,1)} = \left(\tau^{(0,1)}\right)_1 - \left(a^{(0,1)}\right)^2 \left\| \frac{1}{s} \right\|_{L^1(\xi^{(0,1)})} \eta^{(0,1)}$$

$$= (\tau_{(n,1)})_1 - \frac{a^2 z_0^2 \cdots z_{n-1}^2}{y_1^2 \cdots y_n^2} \left\| \frac{1}{s} \right\|_{L^1(\xi)} (\eta_{(n,0)})_1. \tag{6.1}$$

We now calculate $(\tau_{(n,1)})_1$ and $(\eta_{(n,0)})_1$. From Lemma 2.2 we know that

$$d\tau_{(n,1)}(t) = \frac{t^{\frac{1}{n}}}{y_0^2}\, d\tau\big(t^{\frac{1}{n}}\big),$$

so that

$$d(\tau_{(n,1)})_1(t) = \frac{t}{y_1^2 \cdots y_n^2}\, d\tau_{(n,1)}(t) = \frac{t^{1+\frac{1}{n}}}{y_0^2 y_1^2 \cdots y_n^2}\, d\tau\big(t^{\frac{1}{n}}\big) \quad \text{(by Lemma 2.1).}$$

On the other hand, and again using Lemma 2.2, we have

$$d\eta_{(n,0)}(t) = d\eta\big(t^{\frac{1}{n}}\big),$$

so that

$$d(\eta_{(n,0)})_1(t) = \frac{t}{z_0^2 \cdots z_{n-1}^2}\, d\eta_{(n,0)}(t) = \frac{t}{z_0^2 \cdots z_{n-1}^2}\, d\eta\big(t^{\frac{1}{n}}\big) \quad \text{(again by Lemma 2.1).}$$

It follows from (6.1) that

$$\begin{aligned} d\psi^{(0,1)}(t) &= \frac{t^{1+\frac{1}{n}}}{y_0^2 y_1^2 \cdots y_n^2}\, d\tau\big(t^{\frac{1}{n}}\big) - \frac{a^2 z_0^2 \cdots z_{n-1}^2}{y_1^2 \cdots y_n^2} \left\| \frac{1}{s} \right\|_{L^1(\xi)} \cdot \frac{t}{z_0^2 \cdots z_{n-1}^2}\, d\eta\big(t^{\frac{1}{n}}\big) \\ &= \frac{t}{y_1^2 \cdots y_n^2} \left\{ \frac{t^{\frac{1}{n}}}{y_0^2}\, d\tau\big(t^{\frac{1}{n}}\big) - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} d\eta\big(t^{\frac{1}{n}}\big) \right\} \\ &= \frac{t}{y_1^2 \cdots y_n^2} \left\{ d\tau_1\big(t^{\frac{1}{n}}\big) - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} d\eta\big(t^{\frac{1}{n}}\big) \right\} \\ &= \frac{t}{y_1^2 \cdots y_n^2} d\psi\big(t^{\frac{1}{n}}\big). \end{aligned}$$

It now readily follows that $\psi^{(0,1)} \geqslant 0$ if and only if $\psi \geqslant 0$, which establishes (i).

To prove (ii), we begin by calculating $\psi^{(0,0)}$. As in (6.1), we have

$$\begin{aligned} d\psi^{(0,0)}(t) &= d\big(\tau^{(0,0)}\big)_1(t) - \big(a^{(0,0)}\big)^2 \left\| \frac{1}{s} \right\|_{L^1(\xi^{(0,0)})} d\eta^{(0,0)}(t) \\ &= d(\tau_{(n,0)})_1(t) - \frac{a^2 z_0^2 \cdots z_{n-2}^2}{y_1^2 \cdots y_{n-1}^2} \left\| \frac{1}{s} \right\|_{L^1(\xi)} d\eta_{(n,n-1)}(t) \\ &= \frac{t}{y_0^2 \cdots y_{n-1}^2}\, d\tau\big(t^{\frac{1}{n}}\big) - \frac{a^2 z_0^2 \cdots z_{n-2}^2}{y_1^2 \cdots y_{n-1}^2} \left\| \frac{1}{s} \right\|_{L^1(\xi)} \frac{t^{\frac{n-1}{n}}}{z_0^2 \cdots z_{n-2}^2}\, d\eta\big(t^{\frac{1}{n}}\big) \\ &\quad \text{(by Lemmas 2.1 and 2.2)} \\ &= \frac{1}{y_0^2 y_1^2 \cdots y_{n-1}^2} \left\{ t\, d\tau\big(t^{\frac{1}{n}}\big) - a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} t^{\frac{n-1}{n}}\, d\eta\big(t^{\frac{1}{n}}\big) \right\}. \end{aligned}$$

It follows that

$$y_0^2y_1^2\cdots y_{n-1}^2\int\frac{1}{t}d\psi^{(0,0)}(t)=\int d\tau\bigl(t^{\frac{1}{n}}\bigr)-a^2y_0^2\left\|\frac{1}{s}\right\|_{L^1(\xi)}\int t^{-\frac{1}{n}}\,d\eta\bigl(t^{\frac{1}{n}}\bigr)$$
$$=1-a^2y_0^2\left\|\frac{1}{s}\right\|_{L^1(\xi)}\left\|\frac{1}{t}\right\|_{L^1(\eta)}.$$

On the other hand,

$$y_0^2\int\frac{1}{t}d\psi(t)=y_0^2\left\{\int\frac{1}{t}\,d\tau_1(t)-a^2\left\|\frac{1}{s}\right\|_{L^1(\xi)}\int\frac{1}{t}\,d\eta(t)\right\}$$
$$=y_0^2\int\frac{1}{t}\cdot\frac{t}{y_0^2}\,d\tau(t)-a^2y_0^2\left\|\frac{1}{s}\right\|_{L^1(\xi)}\left\|\frac{1}{t}\right\|_{L^1(\eta)}$$
$$=1-a^2y_0^2\left\|\frac{1}{s}\right\|_{L^1(\xi)}\left\|\frac{1}{t}\right\|_{L^1(\eta)}.$$

Thus,

$$y_0^2y_1^2\cdots y_{n-1}^2\left\|\frac{1}{t}\right\|_{L^1(\psi^{(0,0)})}=y_0^2\left\|\frac{1}{t}\right\|_{L^1(\psi)}. \tag{6.2}$$

Consider now

$$\varphi^{(0,0)}=\sigma^{(0,0)}-y_0^2\cdots y_{n-1}^2\left\|\frac{1}{t}\right\|_{L^1(\psi^{(0,0)})}\delta_0-\bigl(a^{(0,0)}\bigr)^2y_0^2\cdots y_{n-1}^2\left\|\frac{1}{t}\right\|_{L^1(\eta^{(0,0)})}\frac{\xi^{(0,0)}}{s}.$$

We know that $\sigma^{(0,0)}=\sigma$, that $a^{(0,0)}=\frac{az_0\cdots z_{n-2}}{y_1\cdots y_{n-1}}$ and that $\xi^{(0,0)}=\xi$, so using (6.2) we obtain

$$\varphi^{(0,0)}=\sigma-y_0^2\left\|\frac{1}{t}\right\|_{L^1(\psi)}\delta_0-a^2y_0^2z_0^2\cdots z_{n-2}^2\left\|\frac{1}{t}\right\|_{L^1(\eta^{(0,0)})}\frac{\xi}{s}.$$

Since $\varphi=\sigma-y_0^2\|\frac{1}{t}\|_{L^1(\psi)}\delta_0-a^2y_0^2\|\frac{1}{t}\|_{L^1(\eta)}\frac{\xi}{s}$, it is easy to see that it suffices to prove that $z_0^2\cdots z_{n-2}^2\|\frac{1}{t}\|_{L^1(\eta^{(0,0)})}=\|\frac{1}{t}\|_{L^1(\eta)}$. We know that $\eta^{(0,0)}=\eta_{(n,n-1)}$, so

$$z_0^2\cdots z_{n-2}^2\left\|\frac{1}{t}\right\|_{L^1(\eta^{(0,0)})}=z_0^2\cdots z_{n-2}^2\int\frac{1}{t}\,d\eta_{(n,n-1)}(t)$$
$$=z_0^2\cdots z_{n-2}^2\int\frac{1}{t}\frac{t^{\frac{n-1}{n}}}{z_0^2\cdots z_{n-2}^2}\,d\eta\bigl(t^{\frac{1}{n}}\bigr)$$
$$=\int t^{-\frac{1}{n}}\,d\eta\bigl(t^{\frac{1}{n}}\bigr)=\left\|\frac{1}{t}\right\|_{L^1(\eta)},$$

as desired. $\square$

Corollary 6.2. *Let $\langle\sigma,\tau,a,\xi,\eta\rangle\in\mathcal{TC}$, and let $n\geqslant 2$. Assume that $\langle\sigma,\tau,a,\xi,\eta\rangle^{(1,n)}$ is subnormal. Then $\langle\sigma,\tau,a,\xi,\eta\rangle$ is subnormal.*

Proof. Assume that $\langle\sigma,\tau,a,\xi,\eta\rangle^{(1,n)}$ is subnormal, and recall that the power of a 2-variable weighted shift splits as an orthogonal direct sum of 2-variable weighted shifts. Moreover, each summand is in $\mathcal{TC}$ (because $\langle\sigma,\tau,a,\xi,\eta\rangle\in\mathcal{TC}$). The fact that $\langle\sigma,\tau,a,\xi,\eta\rangle^{(1,n)}$ is subnormal readily implies that each direct summand is subnormal, and then Lemma 5.3 says that $\psi^{(0,q)}\geqslant 0$ and $\varphi^{(0,q)}\geqslant 0$ (all $q\geqslant 0$). In particular, $\psi^{(0,1)}\geqslant 0$ and $\varphi^{(0,0)}\geqslant 0$. It follows from Proposition 6.1 that $\psi\geqslant 0$ and $\varphi\geqslant 0$. Applying Lemma 5.3 once again, we see that $\langle\sigma,\tau,a,\xi,\eta\rangle$ is subnormal. $\square$

Corollary 6.3. *Let $\langle\sigma,\tau,a,\xi,\eta\rangle\in\mathcal{TC}$, and let $m\geqslant 2$. Assume that $\langle\sigma,\tau,a,\xi,\eta\rangle^{(m,1)}$ is subnormal. Then $\langle\sigma,\tau,a,\xi,\eta\rangle$ is subnormal.*

7. Proof of the main theorem

We are now ready to prove our main result, which we restate for the reader's convenience.

Theorem 7.1. *Let $\mathbf{T}\in\mathcal{TC}$. The following statements are equivalent.*

(i) $\mathbf{T}\in\mathfrak{H}_\infty$;
(ii) $\mathbf{T}^{(m,n)}\in\bigoplus\mathfrak{H}_\infty$ *for all $m,n\geqslant 1$;*
(iii) $\mathbf{T}^{(m,n)}\in\bigoplus\mathfrak{H}_\infty$ *for some $m,n\geqslant 1$.*

Proof. It is clear that (i) $\Rightarrow$ (ii) and that (ii) $\Rightarrow$ (iii). Assume that (iii) holds, with $n\geqslant 2$. Since $\mathbf{T}^{(m,n)}=(\mathbf{T}^{(m,1)})^{(1,n)}$, we can use Corollary 6.2 to conclude that $\mathbf{T}^{(m,1)}$ is subnormal. If we now apply Corollary 6.3, we obtain that $\mathbf{T}$ is subnormal, as desired. $\square$

8. An application

In our previous work [14–17,6–9,26,27], we have shown that there are many different families of commuting pairs of subnormal operators, jointly hyponormal but not admitting commuting normal extensions, that is, $\mathbf{T}\in\mathfrak{H}_1$ but $\mathbf{T}\notin\mathfrak{H}_\infty$ (all $m,n\geqslant 1$). As a simple application of Theorem 7.1, we now show that $\mathfrak{H}_1\cap\mathcal{TC}\neq\mathfrak{H}_\infty\cap\mathcal{TC}$; moreover, there exists $\mathbf{T}\in\mathcal{TC}$, such that $\mathbf{T}\in\mathfrak{H}_1$ but $\mathbf{T}^{(m,n)}\notin\bigoplus\mathfrak{H}_\infty$ (all $m,n\geqslant 1$). We recall that $\mathrm{shift}(x_0,x_1,\ldots)$ and $\mathrm{shift}(y_0,y_1,\ldots)$ are subnormal unilateral weighted shifts with Berger measures σ and τ, respectively. Consider a contractive 2-variable weighted shift $\mathbf{T}\in\mathfrak{H}_0$ whose weight diagram is given by Fig. 4(i); that is, in the 5-tuple $\langle\sigma,\tau,a,\xi,\eta\rangle$, we have

* $d\sigma(t):=(1-\kappa^2)d\delta_0(t)+\frac{\kappa^2}{2}dt+\frac{\kappa^2}{2}d\delta_1(t)$,
* τ is the Berger measure of shift $(y_0,y_1,\ldots)$, with τ_1 the 2-atomic Berger measure $\rho_0\delta_{t_0}+\rho_1\delta_{t_1}$ of the Stampfli subnormal completion of $\sqrt{\omega_0}<\sqrt{\omega_1}<\sqrt{\omega_2}$,
* a is a positive number,
* $\xi:=\delta_1$, and
* $\eta:=\delta_1$.

Example 8.1. Let $\mathbf{T}\equiv\langle\sigma,\tau,a,\xi,\eta\rangle$ be the 2-variable weighted shift given by Fig. 4(i), with σ, τ_1, a, ξ and η as above. Then $\mathbf{T}\in\mathfrak{H}_1$ and $\mathbf{T}^{(m,n)}\notin\mathfrak{H}_\infty$ (all $m,n\geqslant 1$) if and only if $s(\kappa)<y_0<h(\kappa)$, where

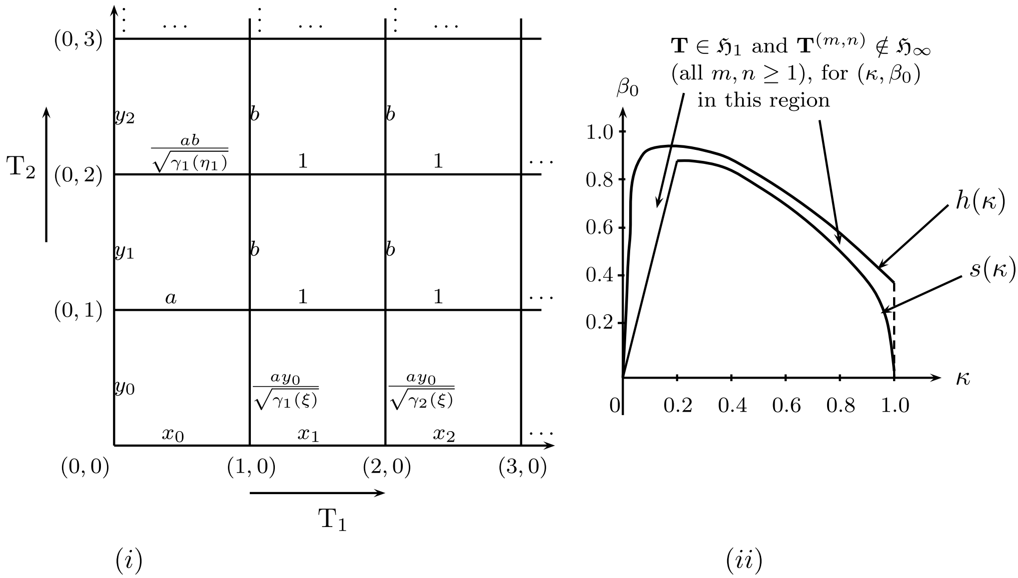

Fig. 4. Weight diagram of the 2-variable weighted shift in Example 8.1 and graphs of $s(\kappa)$ and $h(\kappa)$ on the interval [0, 1], respectively; here $\omega_0 := \frac{5}{6}$, $\omega_1 := \frac{6}{7}$, $\omega_2 := \frac{7}{8}$ and $y_0 := \beta_0$.

$$s(\kappa) := \min\left\{ \frac{\sqrt{t_1}}{a}\sqrt{\rho_1}, \sqrt{\frac{(1-\kappa^2)}{\|\frac{1}{t}\|_{L^1(\tau_1)} - \frac{a^2}{t_1}}}, \frac{\sqrt{t_1}}{a}\sqrt{\frac{\kappa^2}{2}}, \sqrt{\frac{1}{\|\frac{1}{t}\|_{L^1(\tau_1)}}} \right\}$$

and

$$h(\kappa) := \sqrt{\frac{x_0^2 y_1^2 (x_1^2 - x_0^2)}{x_0^2(x_1^2 - x_0^2) + (a^2 - x_0^2)^2}}.$$

Fig. 4(ii) specifies a region in the (κ, β_0) plane where $\mathbf{T}$ is hyponormal but none of its powers is subnormal. A detailed analysis of Example 8.1 and of other applications of Theorem 3.1 will be discussed elsewhere.

Acknowledgment

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