JOURNAL OF ALGEBRA **183,** 396–419 (1996)
ARTICLE NO. 0227

On the Block Structure of Supersolvable Restricted Lie Algebras

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1. INTRODUCTION

Block theory is an important tool in the modular representation theory of finite groups (cf. [18]). Apart from a few papers (e.g. [17, 13, 16]) dealing with restricted simple Lie algebras there apparently has been no effort to do the same for other classes of restricted Lie algebras despite a good knowledge of the simple modules (cf. e.g. [21] for the solvable case).

The aim of this paper is to develop the block theory for reduced universal enveloping algebras of a finite dimensional supersolvable restricted Lie algebra as far as possible in close analogy to modular group algebras. In order to organize the paper in a concise way, we include some open questions on tensor products of simple modules. Under certain conditions either on the Lie algebra or on the character they have affirmative answers which are decisively used in the proofs of our main results. But unfortunately this is not true in general as we show by an example. In the following, we are going to describe the contents of the paper in more detail.

The second section provides the necessary background from the block theory of associative algebras which perhaps is not as well-known to Lie theorists. In particular, we stress that the block decomposition induces an equivalence relation (“linkage relation”) on the (finite) set of isomorphism classes of simple modules and indicate the proof of a cohomological characterization of the linkage relation which is fundamental for some of the following results. The latter can be used to give a combinatorial description of the linkage relation via the Gabriel quiver. We include some results on the number of blocks, resp. the principal block, that hold for any restricted Lie algebra.

0021-8693/96 $18.00

In Section 3, we use the cohomological characterizations of finite dimensional nilpotent and supersolvable restricted Lie algebras obtained by the author in [7] in order to give a short proof of a block-theoretic characterization which originally was due to Voigt [23] in the context of infinitesimal algebraic group schemes over an algebraically closed ground field. This also gives good information on the structure of arbitrary blocks for the corresponding reduced universal enveloping algebras. All the remaining results are obtained by considering the principal block as an (elementary) abelian p-group which acts (via the tensor product) on the isomorphism classes of simple modules (with a fixed character) (cf. [22, Section 5.8]). In the case of a strongly solvable Lie algebra, we have a complete description of this group in terms of the roots of the p-nilpotent radical. This enables us to give a proof of an unpublished result of Farnsteiner on the number of blocks of a finite dimensional strongly solvable Lie algebra without using the (well-known) projective covers of the simple modules. A consequence of this is the characterization of the finite dimensional restricted supersolvable Lie algebras with one block, and slightly more general, a (necessary and) sufficient condition for a simple restricted module to belong to the principal block. Finally, from this we are able to determine the number of isomorphism classes of simple modules in the principal block, and thereby establish an upper bound for the number of isomorphism classes of simple modules in an arbitrary block.

2. PRELIMINARIES

Let L denote a finite dimensional restricted Lie algebra over a commutative field $\mathbb{F}$ of characteristic p. In the following we are interested in the category $u(L, \chi)$-mod of finite dimensional (unitary, left) $u(L, \chi)$-modules for an arbitrary character $\chi \in L^* := \mathrm{Hom}_{\mathbb{F}}(L, \mathbb{F})$ (cf. [22, Chap. 5, Sections 2 and 3]). A complete classification of simple $u(L, \chi)$-modules (over an algebraically closed field) exists only in a few cases (i.e., up to the knowledge of the author only for nilpotent restricted Lie algebras, the three-dimensional simple Lie algebra, and the restricted Lie algebras of Cartan type of ranks one and two). Moreover, this supplies enough information only if $u(L, \chi)$ is semisimple, which is a very strong restriction (cf. [15]). So in the other cases one should try to classify the finite dimensional indecomposable $u(L, \chi)$-modules (up to isomorphism). Since this is in general a very hard problem (cf. [10, 11]), it is quite natural to consider a decomposition of $u(L, \chi)$ into smaller subalgebras such that the category $u(L, \chi)$-mod decomposes into the corresponding smaller module categories for these subalgebras (*block decomposition*)

$$u(L,\chi) = \bigoplus_{j=1}^{b} B_j,$$

where each B_j is an indecomposable two-sided ideal of $u(L,\chi)$. The B_j are called *block ideals* of $u(L,\chi)$. This decomposition is in one-to-one correspondence with a primitive central idempotent decomposition of the identity element 1 of $u(L,\chi)$

$$1 = \sum_{j=1}^{b} c_j,$$

where $B_j = u(L,\chi)c_j$ is a finite dimensional associative $\mathbb{F}$-algebra with identity element c_j. The c_j are called *block idempotents* of $u(L,\chi)$ (cf. [18, Theorem VII.12.1]). Every indecomposable $u(L,\chi)$-module is a (unitary, left) B_j-module for some uniquely determined j, i.e.,

$$u(L,\chi)\text{-mod} = \bigoplus_{j=1}^{b} B_j\text{-mod}.$$

In particular, this induces an equivalence relation "*belonging to the same block*" or "*linked*" on the finite set $\mathrm{Irr}(L,\chi)$ of all isomorphism classes of (irreducible or) simple $u(L,\chi)$-modules

$$\mathrm{Irr}(L,\chi) = \bigcup_{j=1}^{b} \mathbb{B}_j,$$

such that each equivalence class $\mathbb{B}_j = \{[S] \in \mathrm{Irr}(L,\chi) | c_j \cdot S = S\}$, a so-called *block* or *linkage class* of $u(L,\chi)$, is in one-to-one correspondence with the set of isomorphism classes of simple B_j-modules. If the ground field $\mathbb{F}$ is algebraically closed, Schur's Lemma yields for every simple $u(L,\chi)$-module S a unique (unitary) $\mathbb{F}$-algebra homomorphism ζ_S from the center $C(u(L,\chi))$ of $u(L,\chi)$ onto $\mathbb{F}$ such that

$$(c)_S = \zeta_S(c) \cdot \mathrm{id}_S \qquad \forall c \in C(u(L,\chi)),$$

where $(\cdot)_S$ denotes the action of $u(L,\chi)$ on S. ζ_S is called the *central character* of S. One can show that two simple $u(L,\chi)$-modules belong to the same block if and only if their corresponding central characters coincide (cf. [3, Section 55, Exercise 3]).

For the convenience of the reader we state some cohomological features of the linkage relation which will be quite useful in the following.

LEMMA 1. *Let L denote a finite dimensional restricted Lie algebra and $\chi \in L^*$. Then the following statements hold*:

(a) *If two $u(L, \chi)$-modules M and N belong to different blocks, then $\operatorname{Ext}^n_{u(L,\chi)}(M, N)$ vanishes for every integer $n \geq 0$.*

(b) *Two simple $u(L, \chi)$-modules M and N belong to the same block if and only if there exists a finite sequence $S_1, \ldots, S_r$ of simple $u(L, \chi)$-modules such that $M = S_1$, $S_r = N$, and*

$$\operatorname{Ext}^1_{u(L,\chi)}(S_j, S_{j+1}) \neq 0 \quad or \quad \operatorname{Ext}^1_{u(L,\chi)}(S_{j+1}, S_j) \neq 0$$

$$\text{for every } 1 \leq j \leq r-1.$$

Proof. (a) is an immediate consequence of [5, Corollary 4.10] applied to the block idempotents corresponding to M (resp. N) and (b) is well-known (see e.g. [20, Corollary 1]). ∎

There exists a nice combinatorial description of the linkage relation, the so-called *Gabriel quiver* $\mathbf{Q}(L, \chi)$ of $u(L, \chi)$, i.e., the finite directed graph with the set $\operatorname{Irr}(L, \chi)$ as vertices and $\dim_{\mathbb{F}} \operatorname{Ext}^1_{u(L,\chi)}(M, N)$ arrows from $[M]$ to $[N]$. By virtue of Lemma 1(b), it is obvious that the $\mathbb{B}_j$ are in one-to-one correspondence with the *connected components* of the underlying (undirected) graph $Q(L, \chi)$ of $\mathbf{Q}(L, \chi)$.

The aim of the paper is to attack the following problems for *supersolvable* restricted Lie algebras.

PROBLEMS. I. Describe the Gabriel quiver $\mathbf{Q}(L, \chi)$ of $u(L, \chi)$, in particular,

(1) determine the number $b(L, \chi)$ of blocks of $u(L, \chi)$ (i.e., determine the number of connected components of $Q(L, \chi)$),

(2) determine the number $|\mathbb{B}_j|$ of (isomorphism classes of) simple modules in every block $\mathbb{B}_j$ (i.e., determine the number of vertices of the connected component of $Q(L, \chi)$ corresponding to $|\mathbb{B}_j|$).

II. Determine the algebra structure of the block ideals B_j.

Problem II was (at least partially) motivated by the desire to determine all characters χ of L for which the finite dimensional indecomposable $u(L, \chi)$-modules can be classified (up to isomorphism), i.e., to decide for which characters $u(L, \chi)$ is tame. (A finite dimensional associative $\mathbb{F}$-algebra A is called *tame* if for any positive integer d almost all indecomposable A-modules of dimension d belong to a finite number of one-parameter families [1, Definition 4.4.1].) According to a result of Gabriel (see [1, Proposition 4.1.7]), Problem I can be considered as a first step in solving Problem II. It is also of independent interest because the solution of Problem I.2 for all blocks with a fixed character χ would give the number of isomorphism classes of simple $u(L,\chi)$-modules which is still unknown in the non-nilpotent case. In this paper, we will give some first steps in order to solve both problems for *solvable* restricted Lie algebras.

Since a closed formula for the *block invariants* mentioned in Problems I.1 and I.2 seems to be difficult to obtain, we will begin in the next section by considering the simplest possible cases, i.e.,

- $b(L,\chi) = 1$ (i.e., $|\mathbb{B}| = |\mathrm{Irr}(L,\chi)|$ for the *unique* block $\mathbb{B}$ of $u(L,\chi)$) (*"block degeneracy"*),
- $|\mathbb{B}| = 1$ for every block $\mathbb{B}$ of $u(L,\chi)$ (i.e., $b(L,\chi) = |\mathrm{Irr}(L,\chi)|$),

and attempt to discover classes of restricted Lie algebras (resp. characters) for which these conditions hold. It is well-known from classical ring theory that the second condition is equivalent to the statement that every block ideal B is *primary*, i.e., $B/\mathrm{Jac}(B)$ is a simple algebra (cf. [19, Proposition 6.5a]). This property can be generalized to

$(*)$ $\dim_{\mathbb{F}} S = \text{const}$ for every simple module S belonging to $\mathbb{B}$

or

$(**)$ $|\mathbb{B}| = \text{const.}$ for every block $\mathbb{B}$ of $u(L,\chi)$.

It turns out that $(*)$ holds for every block of $u(L,\chi)$ if L is a *supersolvable* restricted Lie algebra (cf. Corollary 2), and even in this case it does not seem to be obvious under which conditions $(**)$ will be satisfied (cf. Question 2 before Proposition 6). Recall that according to Lie's Theorem (which fails in the modular situation) every non-modular solvable Lie algebra (over an algebraically closed field) is supersolvable and thus Theorem 1 (resp. Corollary 2) can be considered as a modular analogue of Lie's Theorem.

In order to state the next result, we need some more notation. The block of $u(L,0)$ containing the one-dimensional trivial L-module is called the *principal block* of L and will be denoted by $\mathbb{B}_0$. The principal block turns out to be the most complicated block of L (see e.g. Corollary 1 and Examples 1 and 2 below). Following Hochschild [14] we define the *restricted cohomology* of L with coefficients in a restricted L-module X by means of

$$H^n_*(L,X) := \mathrm{Ext}^n_{u(L,0)}(\mathbb{F},X) \qquad \forall n \geq 0.$$

In the following we will need a stronger version of the well-known fact that every simple restricted module not belonging to the principal block has vanishing restricted cohomology (cf. Lemma 1(a)):

LEMMA 2. *Let L be a finite dimensional restricted Lie algebra. If the finite dimensional restricted L-module X does not contain a non-zero submodule* *belonging to the principal block of* L, *then the restricted cohomology* $H_*^n(L, X)$ *vanishes for every integer* $n \geq 0$.

Proof. Since restricted cohomology commutes with direct sums, we can assume without loss of generality that X is indecomposable. Then it is clear that every composition factor of X belongs to a unique block, namely the block to which X itself belongs. By hypothesis, the composition factors in the socle of X and thus *all* composition factors of X do not belong to the principal block of L. Hence the assertion is an immediate consequence of Lemma 1(a) and the long exact sequence for restricted cohomology. ∎

We continue by studying the behavior of the number of blocks of reduced universal enveloping algebras under field extensions and restriction to central subalgebras:

LEMMA 3. *Let* L *be a finite dimensional restricted Lie algebra over* $\mathbb{F}$ *and* $\chi \in L^*$. *Then the following statements hold*:

(a) *If* $\mathbb{E}$ *is a field extension of* $\mathbb{F}$, *then* $b(L, \chi) \leq b(L \otimes_{\mathbb{F}} \mathbb{E}, \chi \otimes \mathrm{id}_{\mathbb{E}})$.

(b) *If* K *is a central* p-*subalgebra of* L, *then* $b(K, \chi|_K) \leq b(L, \chi)$.

Proof. Look at the primitive central idempotent decomposition of the identity element in $u(L, \chi)$ for (a) resp. in $u(K, \chi|_K) \hookrightarrow C(u(L, \chi))$ for (b)! ∎

Let $C(L)$ denote the center of L and set $T_p(L) := \{x \in C(L) | x \text{ is semisimple}\}$. Since every toral ideal is central, $T_p(L)$ is the largest toral ideal of L. By the main result of [15] together with [7, Theorem 2.4], every reduced universal enveloping algebra of a torus is semisimple and therefore Lemma 3(b) in conjunction with [22, Theorem 2.3.6(1)] yields:

PROPOSITION 1. *Let* L *be a finite dimensional restricted Lie algebra over an algebraically closed field* $\mathbb{F}$ *and* $\chi \in L^*$. *Then*

$$p^{\dim_{\mathbb{F}} T_p(L)} \leq b(L, \chi).$$

We conclude this section by investigating some elementary properties of the principal block and begin with the following result which follows immediately from Lemma 1(b) and the five-term exact sequence for restricted cohomology:

LEMMA 4. *Let* L *be a finite dimensional restricted Lie algebra*, I *be a* p-*ideal in* L, *and* M *be a restricted* L-*module such that* $IM = 0$. *If* M *belongs to the principal block of* L/I, *then* M *belongs to the principal block of* L.

Remark. Example 1 below (with $I := \mathbb{F}x \oplus \mathbb{F}y \oplus \mathbb{F}z$ and $M := F_\tau$ for $\tau \neq 0$) shows that the converse of Lemma 4 is far from being true!

Using the interplay between restricted and ordinary cohomology provided by a six-term exact sequence due to Hochschild [14, p. 575] we obtain along the same lines as in the proof of [20, Proposition 1]:

PROPOSITION 2. *Let L be a finite dimensional solvable restricted Lie algebra. Then every composition factor of the adjoint module belongs to the principal block of L.*

Proof. Let $S = I/J$ be a composition factor of L. Since I/J is by definition a minimal ideal of L/J and L is assumed to be solvable, S is abelian. Hence S is a trivial I-module and thus the five-term exact sequence for ordinary cohomology specializes to

$$0 \to H^1(L/I, S) \to H^1(L/J, S) \to \mathrm{Hom}_L(S, S) \to H^2(L/I, S)$$
$$\to H^2(L/J, S).$$

Since the third term is non-zero, the second and fourth term must also be non-zero. In the first case, we obtain by the first three terms of Hochschild's exact sequence for L/J that either $0 \neq H^1_*(L/J, S) \hookrightarrow H^1_*(L, S)$ or $S \cong \mathbb{F}$. Therefore S belongs to the principal block of L. In the second case, a similar argument for L/I yields that either S belongs to the principal block of L or $H^2_*(L/I, S) \neq 0$. But then S belongs to the principal block of L/I and thus by Lemma 4 also to the principal block of L. ∎

Remark. The same argument as that in the proof of Proposition 2 shows that every *abelian* composition factor of an *arbitrary* finite dimensional restricted Lie algebra belongs to the principal block of L, but contrary to the case of finite groups (cf. [18, Theorem VII.13.9]), Proposition 2 is not true for non-solvable restricted Lie algebras—consider e.g. the three-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 2$, where the adjoint module has only one composition factor which does *not* belong to the principal block (cf. e.g. [13, Proposition 2.4]). Moreover, we note that the converse of Proposition 2 is far from being true (cf. Example 1 for $p > 3$)!

Let $\mathrm{Ann}_L(M) := \{x \in L | x \cdot M = 0\}$ denote the *annihilator* of an arbitrary L-module M. Then we have the following necessary condition for a simple restricted L-module to belong to the principal block:

PROPOSITION 3. *Let L be a finite dimensional restricted Lie algebra and S be a simple restricted L-module. If S belongs to the principal block of L, then $T_p(L) \subseteq \mathrm{Ann}_L(S)$.*

Proof. Let us assume first that the ground field $\mathbb{F}$ is algebraically closed. If ε denotes the *augmentation mapping* of $u(L,0)$ (i.e., the unique (unitary) $\mathbb{F}$-algebra homomorphism from $u(L,0)$ onto $\mathbb{F}$ extending the zero mapping from L into $\mathbb{F}$) and if ζ_0 denotes the central character corresponding to the principal block of L, then

$$\zeta_0(u)\cdot 1_{\mathbb{F}} = u\cdot 1_{\mathbb{F}} = \varepsilon(u)\cdot 1_{\mathbb{F}} \qquad \forall u \in C(u(L,0))$$

shows that ζ_0 is just the restriction of ε to the center $C(u(L,0))$ of $u(L,0)$. Hence $T_p(L) \subseteq C(L) \hookrightarrow C(u(L,0))$ implies

$$T_p(L) \subseteq \operatorname{Ker}(\varepsilon_{|C(u(L,0))}) \cap L = \operatorname{Ker}(\zeta_0) \cap L \subseteq \operatorname{Ann}_L(S)$$

since by our hypothesis S belongs to the principal block of L.

If $\mathbb{F}$ is arbitrary, let $\overline{\mathbb{F}}$ denote an algebraic closure of $\mathbb{F}$. Then $S \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is a simple $(L \otimes_{\mathbb{F}} \overline{\mathbb{F}})$-module, and in view of Lemma 1(b) it is clear that $S \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ also belongs to the principal block of $L \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. Therefore the already established case of an algebraically closed ground field yields

$$T_p(L) \otimes_{\mathbb{F}} \overline{\mathbb{F}} \subseteq T_p\big(L \otimes_{\mathbb{F}} \overline{\mathbb{F}}\big) \subseteq \operatorname{Ann}_{L\otimes_{\mathbb{F}}\overline{\mathbb{F}}}\big(S \otimes_{\mathbb{F}} \overline{\mathbb{F}}\big) = \operatorname{Ann}_L(S) \otimes_{\mathbb{F}} \overline{\mathbb{F}},$$

i.e., $T_p(L) \subseteq \operatorname{Ann}_L(S)$. ∎

3. MAIN RESULTS

A Lie algebra L is called *supersolvable* if there is a (descending) chain

$$L = L_0 \supset L_1 \supset \cdots \supset L_n = 0$$

of ideals L_j in L such that the factor algebras L_j/L_{j+1} are one-dimensional for every $0 \le j \le n-1$. It is well-known that subalgebras and factor algebras of supersolvable Lie algebras are again supersolvable.

Since it is fundamental for the rest of the paper, we begin with a short cohomological proof of a block-theoretic characterization of supersolvable (resp. nilpotent) restricted Lie algebras using the results of [7] (cf. also [8, Theorem 2, resp. Theorem 3] for the equivalences (a) ⇔ (b) in Theorem 1, resp. Theorem 2).

A finite dimensional associative $\mathbb{F}$-algebra A is called *basic* if $A/\operatorname{Jac}(A)$ is a direct product of $\mathbb{F}$-division algebras. Then the following result should be considered as a modular analogue of Lie's Theorem.

THEOREM 1. *For any finite dimensional restricted Lie algebra L over $\mathbb{F}$ there are the implications*

$$(a) \Leftrightarrow (b) \Rightarrow (c)$$

among the following statements:

(a) *L is supersolvable*.

(b) *Every simple module in the principal block of L is one-dimensional*.

(c) *The principal block ideal of $u(L, 0)$ is a basic algebra*.

If $\mathbb{F}$ *is algebraically closed, then* (b) *and* (c) *are also equivalent*.

Proof. (a) ⇒ (b): Let X and Y be simple restricted L-modules such that $\dim_{\mathbb{F}} X = 1$ and $\dim_{\mathbb{F}} Y \neq 1$. Then $\operatorname{Hom}_{\mathbb{F}}(X, Y) \cong X^* \otimes_{\mathbb{F}} Y$ and $\operatorname{Hom}_{\mathbb{F}}(Y, X) \cong Y^* \otimes_{\mathbb{F}} X$ are also simple restricted L-modules such that $\dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}}(X, Y) \neq 1 \neq \dim_{\mathbb{F}} \operatorname{Hom}_{\mathbb{F}}(Y, X)$. From [7, Proposition 5.9] we derive

$$\begin{aligned}\operatorname{Ext}^1_{u(L,0)}(X, Y) &\cong H^1_*(L, \operatorname{Hom}_{\mathbb{F}}(X, Y)) = 0 = H^1_*(L, \operatorname{Hom}_{\mathbb{F}}(Y, X)) \\ &\cong \operatorname{Ext}^1_{u(L,0)}(Y, X).\end{aligned}$$

If we apply this successively to a finite sequence $\mathbb{F} =: S_1, \ldots, S_r =: S$ of simple $u(L, 0)$-modules satisfying the condition in Lemma 1(b), we obtain $\dim_{\mathbb{F}} S = 1$ for any simple module S belonging to the principal block of L.

The implication (b) ⇒ (a) follows from Lemma 1(a) and [7, Theorem 5.10] and (b) ⇒ (c) is an immediate consequence of the definition (cf. also the proof of Corollary 2 below). Finally, in the case of an algebraically closed ground field, the principal block ideal B_0 is basic if and only if every simple B_0-module is one-dimensional, i.e., (b) and (c) are equivalent. ∎

A finite dimensional associative $\mathbb{F}$-algebra A is called *local* if $A/\operatorname{Jac}(A)$ is a division algebra. Then using [7, Proposition 5.5, resp. Theorem 5.6] instead of [7, Proposition 5.9, resp. Theorem 5.10] we obtain in the nilpotent case:

THEOREM 2. *Let L be a finite dimensional restricted Lie algebra. Then the following statements are equivalent*:

(a) *L is nilpotent*.

(b) *Every simple module in the principal block of L is trivial*.

(c) *The principal block ideal of $u(L, 0)$ is a local algebra*.

Remark. In fact, Lemma 1(a) shows that [7, Theorem 5.10] can also be derived from Theorem 1 and similarly [7, Theorem 5.6] is equivalent to Theorem 2. This was implicitly used by Voigt [23] to obtain the above results in the more general context of infinitesimal algebraic group schemes, but also assuming the ground field as *algebraically closed*. Our approach to proving Theorems 1 and 2 was motivated by the analogous results [20, Corollary 2, resp. Corollary 4] for finite modular group algebras.

Lemma 2 and the implication (a) ⇒ (b) of Theorem 1 in conjunction with Theorem 5 below can be used to give a more precise statement of [22, Theorem 5.8.7(1)]:

THEOREM 3. *Let L be a finite dimensional supersolvable restricted Lie algebra over an algebraically closed field $\mathbb{F}$, $\chi \in L^*$, and M, N be simple $u(L, \chi)$-modules. Then there are the implications*

$$(\mathrm{a}) \Leftrightarrow (\mathrm{b}) \Leftrightarrow (\mathrm{c}) \Leftarrow (\mathrm{d})$$

among the following statements:

(a) $M_{|T_p(L)} \cong N_{|T_p(L)}$.

(b) $M_{|C(L)} \cong N_{|C(L)}$.

(c) *There exists a simple module S in the principal block of L such that $N \cong S \otimes_{\mathbb{F}} M$.*

(d) *M and N belong to the same block of $u(L, \chi)$.*

Proof. (a) ⇒ (b): The assertion follows from

$$\mathrm{Hom}_{T_p(L)}(M, N) = \mathrm{Hom}_{C(L)}(M, N).$$

By virtue of the Jordan−Chevalley−Seligman decomposition for the center $C(L)$ of L, it is thus enough to show that the action of any p-nilpotent element in $C(L)$ on a simple $u(L, \chi)$-module depends *only* on χ which is an immediate consequence of [21, Formula (1)].

(b) ⇒ (c) (cf. [22, Theorem 5.8.7(1)]): Let $\sigma: L \to C(L)$ be a p-semilinear mapping such that $\sigma_{|C(L)} = ?^{[p]}$ and set

$$x^{[p]'} := x^{[p]} - \sigma(x) \qquad \forall x \in L.$$

Then $(L, ?^{[p]})$ is a restricted Lie algebra (cf. [22, Proposition 2.2.1]) with $C(L)^{[p]} = 0$. According to [22, Theorem 5.2.7(1)], $X := \mathrm{Hom}_{\mathbb{F}}(M, N)$ is a finite dimensional restricted L-module for both p-mappings of L. By hypothesis, $Y := \mathrm{Hom}_{C(L)}(M, N) = X^{C(L)}$ is a non-zero L-submodule of X. Hence Y has a non-zero socle, i.e., there exists a simple L-submodule S of Y. From the adjointness of Hom and ⊗ we obtain $\mathrm{Hom}_L(S \otimes_{\mathbb{F}} M, N) \cong \mathrm{Hom}_L(S, X) \neq 0$. Since $[L, L]$ is $[p]'$-nilpotent, S is one-dimensional (cf. [22, Lemma 5.8.6(a)]), and thus $S \otimes_{\mathbb{F}} M$ is simple. Therefore we conclude from Schur's Lemma that $N \cong S \otimes_{\mathbb{F}} M$. As we will see later in Theorem 5, the inclusion $T_p(L) \cdot S \subseteq C(L) \cdot Y = 0$ implies that S belongs to the principal block of L.

Since the implication (c) ⇒ (a) is an immediate consequence of Proposition 3, it remains to prove (d) ⇒ (c). Set again $X := \mathrm{Hom}_{\mathbb{F}}(M, N)$. Without loss of generality we may assume $H^1_*(L, X) \cong \mathrm{Ext}^1_{u(L,\chi)}(M, N) \neq 0$, and by virtue of Lemma 2, X possesses a simple submodule S belonging to $\mathbb{B}_0$. According to Theorem 1, S is one-dimensional and thus $S \otimes_{\mathbb{F}} M$ is simple. Hence the assertion follows from Schur's Lemma as in the proof of (b) ⇒ (c). ▮

Remark. The proof of the implication (d) ⇒ (c) of Theorem 3 is valid for an *arbitrary* ground field. Note also that its analogue in the modular representation theory of finite p-supersolvable groups (cf. [20, Corollary 3 and its proof]) can be simplified slightly by using the analogue of Lemma 2.

Unfortunately, the implication (c) ⇒ (d) in Theorem 3 is *not* true in general as the following example shows:

EXAMPLE 1. Consider the restricted *diamond algebra*

$$L := \mathbb{F}t \oplus \mathbb{F}x \oplus \mathbb{F}y \oplus \mathbb{F}z,$$

$$[t, x] = x, \qquad [t, y] = -y, \qquad [x, y] = z, \qquad [z, L] = 0,$$

$$t^{[p]} = t, \qquad x^{[p]} = y^{[p]} = z^{[p]} = 0$$

over an algebraically closed ground field $\mathbb{F}$. Then the largest nilpotent ideal $\mathrm{Nil}(L)$ and the largest p-nilpotent ideal $\mathrm{Rad}_p(L)$ of L coincide, namely

$$\mathrm{Nil}(L) = \mathbb{F}x \oplus \mathbb{F}y \oplus \mathbb{F}z = \mathrm{Rad}_p(L)$$

is a three-dimensional *p-nilpotent Heisenberg algebra*. In particular, the center $C(L) = \mathbb{F}z$ of L is p-nilpotent and thus $T_p(L) = 0$.

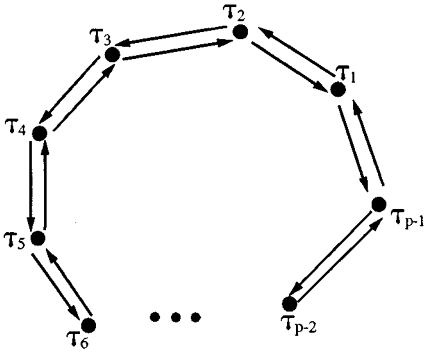
Let $\chi \in L^*$. If $\chi_{|\mathrm{Nil}(L)} = 0$, then

$$\mathrm{Irr}(L, \chi) = \left\{[F_\tau] \middle| \tau^p - \tau = \chi(t)^p \cdot 1\right\},$$

where the eigenvalue of t on F_τ is τ and the action of $\mathrm{Nil}(L)$ on F_τ is trivial. An easy computation shows that

$$\dim_{\mathbb{F}} \mathrm{Ext}^1_{u(L,\chi)}(F_\tau, F_\theta) = \delta_{1,\theta-\tau} + \delta_{-1,\theta-\tau}$$

holds for all roots τ, θ of $X^p - X - \chi(t)^p \cdot 1$. Hence $u(L, \chi)$ has the following Gabriel quiver where the vertices corresponding to the one-dimensional $u(L, \chi)$-modules F_τ are labeled by the respective eigenvalues



τ of t (cf. [6, Beispiel II.4.2]). In particular, the restricted universal enveloping algebra $u(L,0)$ has a unique block (see also Theorem 4 below).

If $\chi(z) = 1$ and $\chi(x) = 0 = \chi(y)$ (i.e., $\chi_{|\mathrm{Nil}(L)} \neq 0$), then we obtain from [22, Corollary 5.7.6] (see also [22, Example 5.9.2(b)]):

$$\mathrm{Irr}(L,\chi) = \Big\{\big[\mathrm{Ind}_K^L(F_\lambda,\chi)\big]\Big|\lambda(x) = 0,\ \lambda(z) = 1,\\ \lambda(t)^p - \lambda(t) = \chi(t)^p \cdot 1\Big\},$$

where $K := \mathbb{F}t \oplus \mathbb{F}x \oplus \mathbb{F}z$, $\lambda \in K^*$, and $k \cdot 1_\lambda = \lambda(k)\cdot 1_\lambda$ $\forall k \in K$. Set $b_n^{(\lambda)} := y^n \otimes 1_\lambda$ for $0 \le n \le p-1 (b_{-1}^{(\lambda)} := 0 =: b_p^{(\lambda)})$. Then $\{b_n^{(\lambda)} | 0 \le n \le p-1\}$ is a basis of $S_\lambda := \mathrm{Ind}_K^L(F_\lambda,\chi)$ with L-action given by

$$t \cdot b_n^{(\lambda)} = (\lambda - n)\cdot b_n^{(\lambda)}, \qquad x \cdot b_n^{(\lambda)} = n \cdot b_{n-1}^{(\lambda)}, \qquad y \cdot b_n^{(\lambda)} = b_{n+1}^{(\lambda)},\\ z \cdot b_n^{(\lambda)} = b_n^{(\lambda)}.$$

A straightforward computation shows that $c := zt + xy \in C(u(L,0))$ and

$$(c)_{\mathrm{Hom}_{\mathbb{F}}(S_\lambda, S_\mu)} = (\mu - \lambda)\cdot \mathrm{id}_{\mathrm{Hom}_{\mathbb{F}}(S_\lambda, S_\mu)} \qquad \forall \lambda, \mu.$$

Hence [7, Theorem 2.4] in conjunction with [5, Corollary 5.1] yields for $\lambda \neq \mu$:

$$\mathrm{Ext}^1_{u(L,\chi)}(S_\lambda, S_\mu) \cong H^1_*\big(L, \mathrm{Hom}_{\mathbb{F}}(S_\lambda, S_\mu)\big) = 0.$$

As a consequence of Lemma 1(b) we have in this case $b(L,\chi) = p = |\mathrm{Irr}(L,\chi)|$. From the action of L on S_λ we obtain for every λ that

$$F_\tau \otimes_{\mathbb{F}} S_\lambda \cong S_{\lambda+\tau} \qquad \forall \tau \in \mathbb{F}_p,$$

i.e., the blocks of $u(L,\chi)$ are permuted (faithfully) by the principal block (via $\otimes$). Therefore the implication (c) ⇒ (d) in Theorem 3 is not true in general. Moreover, one can read off from the Gabriel quiver $\mathbf{Q}(L,0)$ (or [7, Proposition 2.7]) that $H^1_*(L,\mathbb{F}) = 0$. Hence the simple module S in statement (c) of Theorem 3 does *not* always satisfy $H^1_*(L,S) \neq 0$ (even if $\mathrm{Ext}^1_{u(L,\chi)}(M,N) \neq 0$).

The implication (d) ⇒ (c) of Theorem 3 in conjunction with the implication (a) ⇒ (b) of Theorem 1 immediately yields an upper bound for the number of simple modules in an arbitrary block of a finite dimensional supersolvable restricted Lie algebra (cf. Problem I.2):

COROLLARY 1. *Let L be a finite dimensional supersolvable restricted Lie algebra and $\chi \in L^*$. Then $|\mathbb{B}| \leq |\mathbb{B}_0|$ holds for any block $\mathbb{B}$ of $u(L,\chi)$.*

As already mentioned in Section 2, we are now able to show that $(*)$ is true in the supersolvable case (cf. also [23, Satz 2.5 and Bemerkung 2.6] for $\chi = 0$ and an algebraically closed ground field):

COROLLARY 2. *Let L be a finite dimensional supersolvable restricted Lie algebra and $\chi \in L^*$. Then all simple modules in the same block of $u(L,\chi)$ have the same dimension. In particular, every block ideal of $u(L,\chi)$ is a full matrix algebra over a basic algebra.*

Proof. The first statement follows from the implication (d) ⇒ (c) in Theorem 3 and Theorem 1. In order to prove the second statement, we consider an arbitrary block ideal B of $u(L,\chi)$. Let d denote the common dimension of the simple B-modules and let $P_1,\ldots,P_s$ be a representative set of the isomorphism classes of the projective indecomposable B-modules. Then $B = \bigoplus_{i=1}^s dP_i$ and therefore we obtain (cf. [19, Corollary 3.4a])

$$B \cong \mathrm{End}_B(B)^{\mathrm{op}} \cong \mathrm{End}_B(dP)^{\mathrm{op}} \cong \mathrm{Mat}_d\left(\mathrm{End}_B(P)^{\mathrm{op}}\right),$$

where $P := \bigoplus_{i=1}^s P_i$. Since P is multiplicity-free, $\mathrm{End}_B(P)^{\mathrm{op}}$ is a basic algebra (cf. [19, Lemma 6.6a]) and the second statement is also completely proved. ∎

By virtue of Corollary 2 and Lemma 1(a), we obtain the following generalization of [7, Theorem 5.10]:

COROLLARY 3. *Let L be a finite dimensional supersolvable restricted Lie algebra and $\chi \in L^*$. If M and N are simple $u(L,\chi)$-modules such that $\dim_{\mathbb{F}} M \neq \dim_{\mathbb{F}} N$, then $\mathrm{Ext}^n_{u(L,\chi)}(M,N) = 0$ for every integer $n \geq 0$.*

If we proceed analogously to the proof of Corollary 2, we obtain from the implication (d) $\Rightarrow$ (c) in Theorem 3 and Theorem 2 the following special case of a result due to Curtis [2, Theorem 1] (cf. also [25, 26]):

COROLLARY 4. *Let L be a finite dimensional nilpotent restricted Lie algebra and $\chi \in L^*$. Then every block of $u(L, \chi)$ contains only one isomorphism class of simple modules. In particular, every block ideal of $u(L, \chi)$ is a full matrix algebra over a local algebra.*

Corollary 2 (resp. Corollary 4) solve Problem II and generalize the main result of [15] to *supersolvable* (resp. *nilpotent*) restricted Lie algebras. Corollary 4 in conjunction with [21, Satz 6] shows that for an *arbitrary* character of a *nilpotent* restricted Lie algebra equality holds in Proposition 1. Moreover, Corollary 4 enables us to obtain the following stronger version of Theorem 3 in the nilpotent case, i.e., more precisely we have

PROPOSITION 4. *Let L be a finite dimensional nilpotent restricted Lie algebra, $\chi \in L^*$, and M, N be non-projective simple $u(L, \chi)$-modules. Then the following statements are equivalent*:

(a) $\mathrm{Ext}^1_{u(L,\chi)}(M, N) \neq 0$.

(b) *M and N belong to the same block of $u(L, \chi)$.*

(c) $M \cong N$.

Proof. According to Lemma 1(b) and Corollary 4, it only remains to show the implication (c) $\Rightarrow$ (a). Suppose that $\mathrm{Ext}^1_{u(L,\chi)}(M, M) = 0$ and let P denote the *projective cover* of M (i.e., there is an L-module epimorphism from P onto M such that the kernel is contained in the radical of P). Since M is simple, P is indecomposable. Hence every composition factor of P belongs to the same block of $u(L, \chi)$ as M and is therefore by Corollary 4 isomorphic to M. Then it follows from our assumption that P is a direct sum of copies of M and thus the indecomposability of P implies that $M \cong P$ is projective in contradiction to our hypothesis. ∎

Remark. A comparison of Proposition 4 and [13, Theorem 4.3(a)] shows that the block structure of reduced universal enveloping algebras of classical Lie algebras for a regular nilpotent character and reduced universal enveloping algebras of nilpotent restricted Lie algebras for an *arbitrary* character are the same.

EXAMPLE 2 (cf. [6, Beispiel II.4.3] or [9] for a different approach using projective covers). Consider the three-dimensional *Heisenberg algebra*

$$L := \mathbb{F}e_- \oplus \mathbb{F}z \oplus \mathbb{F}e_+, \qquad [e_+, e_-] = z, \qquad [z, e_\pm] = 0,$$

$$e_\pm^{[p]} = 0, \qquad z^{[p]} = z$$

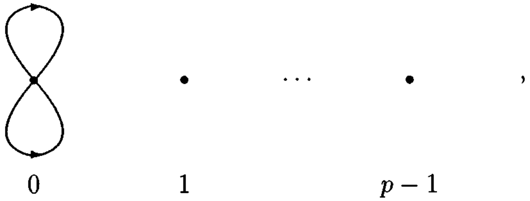
over an algebraically closed ground field $\mathbb{F}$. Put $I := \mathbb{F}e_- \oplus \mathbb{F}z$ and let $\lambda \in I^*$. If $\chi \in L^*$ with $\chi(z) = 0$, we derive from [22, Corollary 5.7.6(2)]:

$$\operatorname{Irr}(L, \chi) = \{[F_\chi]\} \cup \{[\operatorname{Ind}_I^L(F_\lambda, \chi)] \mid \lambda(e_-) = \chi(e_-), 0 \neq \lambda(z) \in \mathbb{F}_p\}.$$

It can immediately be read off from [7, Proposition 2.7] that $\operatorname{Ext}^1_{u(L,\chi)}(F_\chi, F_\chi) \cong H^1_*(L, \mathbb{F})$ is two-dimensional. Let $V(\lambda)$ denote the restricted I-module $F_{-\lambda} \otimes \operatorname{Ind}_I^L(F_\lambda, \chi)_{|I}$. Then e_- acts on $V(\lambda)$ as on $\operatorname{Ind}_I^L(F_\lambda, \chi)$ but z acts trivially. Using Frobenius reciprocity (cf. [7, Corollary 1.3b)]) and the five-term exact sequence for restricted cohomology in conjunction with the main result of [15] (cf. also [7, Corollary 3.6]) we obtain

$$\begin{aligned} &\operatorname{Ext}^1_{u(L,\chi)}\big(\operatorname{Ind}_I^L(F_\lambda, \chi), \operatorname{Ind}_I^L(F_\lambda, \chi)\big) \\ &\quad \cong \operatorname{Ext}^1_{u(I,\chi_{|I})}\big(F_\lambda, \operatorname{Ind}_I^L(F_\lambda, \chi)_{|I}\big), \\ &\quad \cong H^1_*(I, V(\lambda)) \cong H^1_*\big(\mathbb{F}e_-, V(\lambda)_{|\mathbb{F}e_-}\big). \end{aligned}$$

According to [7, Proposition 5.5], it is enough to show that the 0th *complete* cohomology space $\hat{H}^0_*(\mathbb{F}e_-, V(\lambda))$ vanishes. But this is a consequence of [7, Proposition 2.6a] because the trace element of $u(\mathbb{F}e_-, 0)$ is e_-^{p-1} (cf. [7, p. 2875]) and $\dim_{\mathbb{F}} V(\lambda)^{\mathbb{F}e_-} = 1 = \dim_{\mathbb{F}} e_-^{p-1} V(\lambda)$. Hence $u(L, \chi)$ has the following Gabriel quiver



where the vertices corresponding to the simple $u(L, \chi)$-modules are labelled by the respective eigenvalues $\lambda(z)$ of z. In particular, it follows from Proposition 4 that $\operatorname{Ind}_I^L(F_\lambda, \chi)$ is projective. Finally, a simple dimension counting argument shows that the projective cover $P_L(F_\chi)$ of F_χ is p^2-dimensional. In fact, $P_L(F_\chi)$ is isomorphic to $\operatorname{Ind}_{\mathbb{F}z}^L(F_0, \chi)$.

If $\chi(z) \neq 0$, then *every* simple $u(L,\chi)$-module is (properly) induced, i.e.,

$$\operatorname{Irr}(L,\chi) = \left\{ \left[\operatorname{Ind}_I^L(F_\lambda,\chi) \right] \middle| \lambda(e_-) = \chi(e_-),\ \lambda(z)^p - \lambda(z) = \chi(z)^p \cdot 1 \right\},$$

and by the same computation (or, more directly, by comparison of dimensions) as above we conclude that $u(L,\chi)$ is semisimple.

Remark. Note that $u(L,\chi) \cong u(L,0)$ (as associative $\mathbb{F}$-algebras) if $\chi(z) = 0$ (cf. [22, Exercise 5.3.4]). Moreover, it should be pointed out that, more generally, the same global picture remains true for the $(2n+1)$-dimensional *Heisenberg algebra*

$$H_n(\mathbb{F}) := \mathbb{F}e_{-n} \oplus \cdots \oplus \mathbb{F}e_{-1} \oplus \mathbb{F}z \oplus \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_n,$$

$$[e_{-i}, e_{-j}] = [e_i, e_j] = [e_{\pm i}, z] = 0, \qquad [e_i, e_{-j}] = \delta_{ij} \cdot z, \qquad e_{\pm i}^{[p]} = 0,$$

$$z^{[p]} = z \qquad \forall 1 \leq i, j \leq n,$$

if we use $I := \mathbb{F}e_{-n} \oplus \cdots \oplus \mathbb{F}e_{-1} \oplus \mathbb{F}z$ (e.g. the vertex corresponding to F_χ has $2n$ loops, the p^n-dimensional simple $u(H_n(\mathbb{F}),\chi)$-modules are projective and thus $P_{H_n(\mathbb{F})}(F_\chi)$ is isomorphic to $\operatorname{Ind}_{\mathbb{F}z}^{H_n(\mathbb{F})}(F_0,\chi)$ as in the case $n = 1$).

In order to deal with Problem I.2, we consider for any finite dimensional restricted Lie algebra L the (abelian p-) subgroup

$$G^L := \left\{ \gamma \in L^* | \gamma([L,L]) = 0,\ \gamma(x^{[p]}) = \gamma(x)^p\ \forall x \in L \right\}$$

of the (additive) group L^* (cf. [22, p. 242]). As a consequence of the Jordan–Chevalley–Seligman decomposition, we obtain that G^L is finite (cf. [22, Proposition 5.8.8(1)]). For every $\gamma \in G^L$ the one-dimensional vector space F_γ is a restricted L-module and thus for any $\chi \in L^*$ the group G^L acts on $\operatorname{Irr}(L,\chi)$ via $\gamma \cdot [S] := [F_\gamma \otimes_{\mathbb{F}} S]$.

Consider $G_0^L := \{\gamma \in G^L \| [F_\gamma] \in \mathbb{B}_0\}$. By virtue of Lemma 1(b) and the fact that (isomorphism classes of) one-dimensional restricted L-modules are invertible with respect to the tensor product (over $\mathbb{F}$), this is a subgroup of G^L. According to Theorem 1, G_0^L is as large as possible (i.e., $|G_0^L| = |\mathbb{B}_0|$) if and only if L is supersolvable. Moreover, note that the other extreme case $G_0^L = 0$ $(= G^L)$ holds for perfect restricted (e.g. simple) Lie algebras. Of course, both cases coincide by virtue of Theorem 2 if L is nilpotent and the solvable case lies somewhere in between. Example 1 shows that, in general, the blocks of L need not be G_0^L-invariant. Nevertheless, in the following we will find conditions on L and χ under which each block of $u(L,\chi)$ is G_0^L-invariant (see Proposition 6 and its proof).

EXAMPLE 3 (cf. [6, Beispiel II.4.1]). Let $L = T \oplus \mathbb{F}e$ with T a (maximal) torus, $e^{[p]} = 0$, and $[t, e] = \alpha(t) \cdot e$ for $\alpha \in T^*$. If $\mathbb{F}$ is algebraically closed, then every simple restricted L-module is one-dimensional and by the five-term exact sequence for restricted cohomology in conjunction with the main result of [15] (cf. also [7, Corollary 3.6]) and [7, Proposition 2.7], we obtain

$$\operatorname{Ext}^1_{u(L,0)}(F_\lambda, F_\mu) \cong H^1_*(L, F_{\mu-\lambda}) \cong H^1_*(\mathbb{F}e, F_{\mu-\lambda})^T \cong \operatorname{Hom}_T(\mathbb{F}e, F_{\mu-\lambda}) \quad \forall \lambda, \mu \in G^L.$$

Hence we can conclude from Lemma 1(b) that $G_0^L = \mathbb{F}_p \cdot \gamma_\alpha$, where $\mathbb{F}_p$ denotes the prime field of $\mathbb{F}$ and γ_α is the (unique) lifting of $\alpha \in T^*$ to L such that $\gamma_\alpha(e) = 0$.

Following Schue, we call a restricted Lie algebra L *strongly solvable* if L is the semidirect product of a (maximal) torus T and a p-nilpotent ideal N. It is well-known that every strongly solvable Lie algebra over an algebraically closed field is supersolvable (cf. e.g. [4, Theorem 3]). Then Example 3 can be generalized to

PROPOSITION 5. *Let L be a finite dimensional strongly solvable restricted Lie algebra over an algebraically closed field $\mathbb{F}$. If $N = \bigoplus_{\alpha \in R} N_\alpha$ for some $R \subseteq T^*$ is the root space decomposition of N relative to T, then $G_0^L = \sum_{\alpha \in R} \mathbb{F}_p \cdot \gamma_\alpha$, where γ_α denotes the (unique) lifting of $\alpha \in T^*$ to L such that $\gamma_\alpha|_N = 0$.*

Proof. First, $\gamma_\alpha \in G^L$ for any $\alpha \in R$. Indeed, $\gamma_\alpha([L, L]) \subseteq \gamma_\alpha(N) = 0$, and for $x \in L$ we have $\gamma_\alpha(x^{[p]}) = \alpha(t_x^{[p]}) = \alpha(t_x)^p = \gamma_\alpha(x)^p$, where $x \equiv t_x \bmod(N)$ (cf. [22, Lemma 2.1.2] and [22, Theorem 2.3.6(1)]. Moreover, it is clear that G^L is an $\mathbb{F}_p$-subspace of L^*. According to Theorem 1, every simple module in the principal block of L is one-dimensional and therefore is isomorphic to F_λ for some $\lambda \in G^L$. Assume that $\operatorname{Ext}^1_{u(L,0)}(F_\lambda, F_\mu) \neq 0$ for $\lambda, \mu \in G^L$. Since by hypothesis N is p-nilpotent, we obtain by the same arguments as in Example 3 in conjunction with the left exactness of $\operatorname{Hom}_T(?, F_{\mu-\lambda})$ that

$$\operatorname{Ext}^1_{u(L,0)}(F_\lambda, F_\mu) \cong \operatorname{Hom}_T\big(N/([N, N] + \langle N^{[p]} \rangle_{\mathbb{F}}), F_{\mu-\lambda}\big) \hookrightarrow \operatorname{Hom}_T(N, F_{\mu-\lambda}),$$

and similarly to the above we conclude from Lemma 1(b) that $G_0^L \subseteq \sum_{\alpha \in R} \mathbb{F}_p \cdot \gamma_\alpha$.

Since G_0^L is an $\mathbb{F}_p$-subspace of G^L, for the other inclusion it is enough to show that any F_{γ_α} is a composition factor of the adjoint module of L (cf.

Proposition 2). In order to do this, we consider the following prolongation of the descending central series of N to L:

$$L = N^0 \supset N^1 \supset N^2 \supset N^3 \supset \cdots \supset N^k \supset N^{k+1} = 0,$$

where $N^0 := L$, $N^1 := N$, and $N^i := [N, N^{i-1}]$ if $i > 1$. By definition, every factor $N_i := N^i/N^{i+1}$ is a trivial N-module, and the main result of [15] implies that every restricted T-module is semisimple. This shows that N_i $(0 \leq i \leq k)$ is a semisimple restricted L-module. For any $\alpha \in R$ and $i_\alpha := \max\{0 \leq i \leq k | N_\alpha \cap N^i \neq 0\}$ there exists an element $y_\alpha \in N_\alpha \cap N^{i_\alpha}$ such that $\bar{y}_\alpha \in N_{i_\alpha}$ is non-zero. Hence $F_{\gamma_\alpha} \cong \mathbb{F} \cdot \bar{y}_\alpha$ is a (-n L-) direct summand of N_{i_α} and thus a composition factor of the adjoint module of L. Finally, as already mentioned above, Proposition 2 implies that $\sum_{\alpha \in R} \mathbb{F}_p \cdot \gamma_\alpha \subseteq G_0^L$. ∎

Remark. It is immediately clear that G_0^L is isomorphic to $\sum_{\alpha \in R} \mathbb{F}_p \cdot \alpha$, i.e., G_0^L is isomorphic to the $\mathbb{F}_p$-vector space generated by the roots R. Hence the minimal number of generators of G_0^L is the number of "simple" roots (i.e., the generators of $\langle R \rangle_{\mathbb{F}_p}$ as a p-group).

Since G_0^L is a p-group, $|\mathbb{B}_0|$ is always a p-power in the supersolvable case (see also Theorem 6 for the precise result). This is *not* the case in general—consider e.g. the case of a three-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 2$, where the principal block contains two isomorphism classes of simple modules (see e.g. [13, Proposition 2.4]).

QUESTION 1. *Let L be a finite dimensional supersolvable restricted Lie algebra, $\chi \in L^*$, and $\mathbb{B}$ be a block of $u(L, \chi)$. For which simple modules S in $\mathbb{B}_0$ and for which simple modules M in $\mathbb{B}$ does $\otimes_{\mathbb{F}} M$ again belong to $\mathbb{B}$? Under which conditions on L (resp. χ) is this satisfied for every simple module in $\mathbb{B}_0$ and every simple module in $\mathbb{B}$?*

In the case that the block $\mathbb{B}$ is G_0^L-invariant, we would obtain from the implication (d) ⇒ (c) in Theorem 3 that G_0^L acts transitively on $\mathbb{B}$. Let $G_0^L(M) := \{\gamma \in G_0^L | \gamma \cdot [M] = [M]\}$ denote the *stabilizer* (in G_0^L) of (the isomorphism class of) some simple module M belonging to $\mathbb{B}$. Then it is clear that the same conditions as in the second part of Question 1 on L (resp. χ) would also provide an answer to

QUESTION 2. *Let L be a finite dimensional supersolvable restricted Lie algebra, $\chi \in L^*$, and $\mathbb{B}$ be a block of $u(L, \chi)$. When does $|\mathbb{B}| = |G_0^L / G_0^L(M)|$ hold for every simple module M belonging to $\mathbb{B}$?*

In particular, $|\mathbb{B}|$ would always be a p-power and in the case that Question 2 has an affirmative answer for *every* block of $u(L, \chi)$, $(**)$ at the end of Section 2 is satisfied if and only if $|G_0^L / G_0^L(M)| = \text{const.}$ for *every* simple $u(L, \chi)$ module M. The latter condition does not seem to appear very often. Nevertheless it is satisfied in the next result (partly due to Farnsteiner) which generalizes the case $\chi_{|\mathrm{Nil}(L)} = 0$ in Example 1.

PROPOSITION 6. *Let L be a finite dimensional strongly solvable restricted Lie algebra over an algebraically closed field $\mathbb{F}$ and $\chi \in L^*$ such that χ vanishes on the p-nilpotent radical of L. If T denotes a maximal torus of L, then the following statements hold*:

(a) $|\mathbb{B}| = p^{\dim_{\mathbb{F}} T/(T \cap C(L))}$ *for any block $\mathbb{B}$ of $u(L, \chi)$.*

(b) $b(L, \chi) = p^{\dim_{\mathbb{F}} T \cap C(L)}$.

Proof. Let us begin by proving (a) in the special case $\mathbb{B} = \mathbb{B}_0$. We keep the notation of Proposition 5. Consider the canonical (non-degenerate) pairing of T^* with T and denote by $\Gamma^{\perp} := \{t \in T | \gamma(t) = 0\ \forall \gamma \in \Gamma\}$ the orthogonal space of $\Gamma \subseteq T^*$ in T. From the root decomposition of N, it is immediately clear that $\langle R \rangle_{\mathbb{F}}^{\perp} = T \cap C(L)$. Because of $\dim_{\mathbb{F}_p} \langle R \rangle_{\mathbb{F}_p} = \dim_{\mathbb{F}} \langle R \rangle_{\mathbb{F}} = \dim_{\mathbb{F}} T/\langle R \rangle_{\mathbb{F}}^{\perp} = \dim_{\mathbb{F}} T/(T \cap C(L))$, Theorem 1 and Proposition 5 give

$$|\mathbb{B}_0| = |G_0^L| = p^{\dim_{\mathbb{F}_p} \langle R \rangle_{\mathbb{F}_p}} = p^{\dim_{\mathbb{F}} T/(T \cap C(L))}.$$

Since by hypothesis χ vanishes on the p-nilpotent radical of L and the ground field is algebraically closed, every simple $u(L, \chi)$-module is one-dimensional and thus invertible with respect to $\otimes_{\mathbb{F}}$. Then we conclude from Lemma 1(b) that every block of $u(L, \chi)$ is G_0^L-invariant and G_0^L acts transitively (cf. the remarks after Question 1) and faithfully on every block $\mathbb{B}$ of $u(L, \chi)$. Hence we have $|\mathbb{B}| = |\mathbb{B}_0|$ (cf. Question 2) and (a) follows from the result for $\mathbb{B}_0$. Finally, (b) is an immediate consequence of (a) and $b(L, \chi) = |\mathrm{Irr}(L, \chi)|/|\mathbb{B}_0|$. ∎

The next two results are motivated by analogous results of Fong and Gaschütz in the modular representation theory of finite solvable groups (cf. also the conjecture at the end of [16]). In particular, we obtain from Theorem 4 a characterization of supersolvable restricted Lie algebras with exactly one block (cf. [12, Theorem 2.1] or [18, Theorem VII.13.5] for the analogue in the modular representation theory of solvable groups).

THEOREM 4. *Let L be a finite dimensional supersolvable restricted Lie algebra over $\mathbb{F}$ and $\chi \in L^*$ such that χ vanishes on the largest nilpotent ideal of L. Then the following statements are equivalent*:

(a) *$u(L, \chi)$ has precisely one block.*

(b) *L has no non-zero toral ideals.*

(c) $T_p(L) = 0$.

(d) *$C(L)$ is p-nilpotent.*

Proof. By the characterization of $T_p(L)$ as the largest toral ideal of L, the equivalence (b) ⇔ (c) is clear. Moreover, (c) ⇒ (d) is an immediate consequence of [22, Theorem 2.3.4] and the converse implication is trivial.

In order to show the implication (b) ⇒ (a), we assume first that $\mathbb{F}$ is algebraically closed. Let $\mathrm{Nil}(L)$ denote the largest nilpotent ideal of L. Since L is supersolvable, we have $[L, L] \subseteq \mathrm{Nil}(L)$ and therefore the factor algebra $\overline{L} := L/\mathrm{Nil}(L)$ is abelian. By virtue of the Jordan−Chevalley−Seligman decomposition, $\overline{L}$ is a direct sum of a torus and a p-nilpotent ideal $\overline{N} = N/\mathrm{Nil}(L)$ of $\overline{L}$, where $N \supseteq \mathrm{Nil}(L)$ is a p-ideal of L and there exists an integer m with $N^{[p]^m} \subseteq \mathrm{Nil}(L)$. As every element of $\mathrm{Nil}(L)$ is ad-nilpotent, for any $y \in N$ there is an integer n such that

$$\begin{aligned}(\mathrm{ad}\, y)^{p^{m+n}}(x) &= \left(\mathrm{ad}\, y^{[p]^{m+n}}\right)(x) = \left(\mathrm{ad}\left(y^{[p]^m}\right)^{[p]^n}\right)(x)\\ &= \left(\mathrm{ad}\, y^{[p]^m}\right)^{p^n}(x) = 0 \qquad \forall x \in L,\end{aligned}$$

i.e., ad y is nilpotent. Then Engel's Theorem implies that N is nilpotent and we obtain $N = \mathrm{Nil}(L)$. Hence $\overline{L}$ is a torus and we conclude from [22, Lemma 2.4.4(2)] that L is the semidirect product of a torus T with $\mathrm{Nil}(L)$. But the unique maximal torus $T_p(\mathrm{Nil}(L))$ of $\mathrm{Nil}(L)$ is an ideal of L and thus by hypothesis $T_p(\mathrm{Nil}(L)) = 0$. Hence $\mathrm{Nil}(L)$ is p-nilpotent and by virtue of $T_p(L) = T \cap C(L)$ the assertion is a special case of Proposition 6(b). In order to deal with the general case, let $\overline{\mathbb{F}}$ denote an algebraic closure of $\mathbb{F}$. Then it is clear that $L \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is again supersolvable and in view of the already established equivalence (b) ⇔ (d) also has no non-zero toral ideals. Finally, an application of Lemma 3(a) in conjunction with the already established case of an algebraically closed ground field yields $b(L, \chi) \leq b(L \otimes_{\mathbb{F}} \overline{\mathbb{F}}, \chi \otimes \mathrm{id}_{\overline{\mathbb{F}}}) = 1$.

It remains to show the implication (a) ⇒ (b). Suppose that L possesses a non-zero toral ideal T_0. Since T_0 is contained in the largest nilpotent ideal of L, our assumption implies $\chi_{|T_0} = 0$ and therefore the restricted universal enveloping algebra $u(T_0, 0)$ can be embedded into $u(L, \chi)$. $u(T_0, 0)$ is an *augmented* algebra (cf. [1, Definition 2.4.4]). Hence $0 \neq \varepsilon_0(s_0)^{-1} \cdot s_0 \neq 1$ is a central idempotent of $u(L, \chi)$, where ε_0 (resp. s_0) denotes the *augmentation mapping* (resp. *trace element*) of $u(T_0, 0)$ (cf. [7, p. 2875]). Then it is clear from the primitive central idempotent decomposition of the identity element of $u(L, \chi)$ that L has at least two blocks. ∎

Remark. Note that the implication (a) ⇒ (b) in Theorem 4 is an immediate consequence of Proposition 1 if the ground field is algebraically closed. Moreover, Theorem 4 can be applied to the *Borel subalgebras* of *semisimple Lie algebras of classical type* (as was already observed by Farnsteiner for Proposition 6) and to the maximal solvable subalgebra of the *Witt algebra* which both have zero center and therefore are block degenerate.

EXAMPLE 4. Consider again the *diamond algebra* from Example 1. If $\chi(z) = 1$ and $\chi(x) = 0 = \chi(y)$, then $b(L, \chi) = p > 1$, but $T_p(L) = 0$ which shows that the implication (b) ⇒ (a) in Theorem 4 does *not* hold in general if $\chi_{|\mathrm{Nil}(L)} \neq 0$.

If we modify the p-mapping on the center (i.e., $z^{[p]} = z$) and consider $\chi \in L^*$ such that $\chi_{|\mathrm{Nil}(L)} = 0$, we even obtain that $b(L, \chi)$ is *not* a p-power, namely $b(L, \chi) = p \cdot (p - 1) + 1$. One can see this by considering the eigenspace decomposition of $u(L, \chi)$ with respect to the action of the semisimple element z, i.e.,

$$u(L, \chi) = \bigoplus_{\gamma \in \mathbb{F}_p} I_\gamma,$$

where $I_\gamma \cong u(L, \chi)/u(L, \chi)(z - \gamma)$ are two-sided ideals of $u(L, \chi)$. If $\gamma \neq 0$, then we are in the same situation as in Example 1 with $\chi_{|\mathrm{Nil}(L)} \neq 0$ and therefore I_γ has p blocks. In the remaining case we have $I_0 \cong u(\overline{L}, \bar{\chi})$, where

$$\overline{L} := \mathbb{F}\bar{t} \oplus \mathbb{F}\bar{x} \oplus \mathbb{F}\bar{y},$$

$$[\bar{t}, \bar{x}] = \bar{x}, \qquad [\bar{t}, \bar{y}] = -\bar{y}, \qquad [\bar{x}, \bar{y}] = 0, \qquad \bar{t}^{[p]} = \bar{t},$$

$$\bar{x}^{[p]} = \bar{y}^{[p]} = 0, \quad \text{and} \quad \bar{\chi}(\bar{x}) = \bar{\chi}(\bar{y}) = 0.$$

Hence $\overline{L}$ is *strongly solvable* and $\bar{\chi}_{|\mathrm{Nil}(\overline{L})} = 0$. Therefore Theorem 4 implies that $\overline{L}$ has a unique block and $u(L, \chi)$ has the number of blocks as mentioned above. Note that $b(L, \chi)$ again is strictly larger than $p^{\dim_{\mathbb{F}} T_p(L)}$ (cf. Proposition 1)!

In particular, Theorem 4 implies that the condition in Proposition 3 is also sufficient for belonging to the principal block in case L is supersolvable (cf. [12, Lemma 2.2] or [18, Theorem VII.13.7] for the analogue in the representation theory of finite modular group algebras):

THEOREM 5. *Let L be a finite dimensional supersolvable restricted Lie algebra and S be a simple restricted L-module. Then S belongs to the principal block of L if and only if $T_p(L) \subseteq \mathrm{Ann}_L(S)$.*

Proof. By Proposition 3 it is enough to show that "if"-part of the assertion. The hypothesis implies that S is a simple restricted $L/T_p(L)$-

module. Since factor algebras of supersolvable Lie algebras are also supersolvable, Theorem 4 in conjunction with $T_p(L/T_p(L)) = 0$ implies that $L/T_p(L)$ has precisely one block and therefore S necessarily belongs to the principal block of $L/T_p(L)$. Finally, Lemma 4 shows that then S also belongs to the principal block of L. ∎

Theorem 5 enables us to generalize the implication (a) ⇒ (b) of Theorem 2 to the supersolvable case and by Corollary 1 we also obtain a weak generalization of (a part of) Corollary 4 in the form of an upper bound for the number of isomorphism classes of simple modules in arbitrary blocks:

THEOREM 6. *The number of isomorphism classes of simple modules in the principal block of a finite dimensional supersolvable restricted Lie algebra L over an algebraically closed field $\mathbb{F}$ is $p^{\dim_{\mathbb{F}} T_{\max}/T_p(L)}$, where $T_{\max}$ is any maximal torus of L.*

Remark. By an old result of Winter [24, Proposition 2.17], $\dim_{\mathbb{F}} T_{\max} =$ const. for every maximal torus $T_{\max}$ of a finite dimensional solvable restricted Lie algebra L. In the supersolvable case this will also follow from the proof of Theorem 6.

Proof. As already used in the proof of the implication (b) ⇒ (a) of Theorem 4, there exists a torus T such that L is the semidirect product of T and $\mathrm{Nil}(L)$. Since $L/T_p(L) \cong T \oplus \mathrm{Nil}(L)/T_p(\mathrm{Nil}(L))$ is strongly solvable, $(T \oplus T_p(L))/T_p(L)$ $(\cong T)$ is a maximal torus of $L/T_p(L)$ and an application of [22, Theorem 2.4.5(2)] shows that $T \oplus T_p(L)$ is a maximal torus of L. By virtue of Theorem 5, there is a one-to-one correspondence between the principal block of L and the isomorphism classes of simple restricted $L/T_p(L)$-modules. Hence the assertion follows from

$$|\mathbb{B}_0| = \left|\mathrm{Irr}\big(L/T_p(L), 0\big)\right| = p^{\dim_{\mathbb{F}} T}$$

using the maximal tori of L resp. $L/T_p(L)$ established above (cf. [22, Exercise 5.8.4] and the remarks after [22, Lemma 5.8.6]). ∎

Remark. Note that the proof of

$$|\mathbb{B}_0| \le p^{\dim_{\mathbb{F}} T_{\max}/T_p(L)}$$

in Theorem 6 is independent of Proposition 6! Moreover, if $\mathrm{Nil}(L)$ denotes the largest nilpotent ideal of L, the above proof shows that Theorem 6 can also be formulated as

$$|\mathbb{B}_0| = p^{\dim_{\mathbb{F}} L/\mathrm{Nil}(L)},$$

which does not use tori, but immediately makes clear how Theorem 6 can be considered as a generalization of Theorem 2 to the supersolvable case.

COROLLARY 5. *Let L be a finite dimensional supersolvable restricted Lie algebra over an algebraically closed field $\mathbb{F}$, $\chi \in L^*$, and $\mathbb{B}$ be a block of $u(L,\chi)$. Then for any maximal torus $T_{\max}$ of L we have*

$$|\mathbb{B}| \leq p^{\dim_{\mathbb{F}} T_{\max}/T_p(L)} = p^{\dim_{\mathbb{F}} L/\mathrm{Nil}(L)}.$$

ACKNOWLEDGMENTS

It is a pleasure to thank Yuri Bahturin, Joe Ferrar, Dan Nakano, and Helmut Strade for some very useful comments on various preliminary versions of this paper. In particular, Dan Nakano pointed out the simplified proof of Lemma 2 which *uses* Lemma 1(a) instead of repeating its proof again and Example 4 (in a slightly different form) is due to Helmut Strade. Finally, I thank Rolf Farnsteiner for showing me his result on the number of blocks of strongly solvable Lie algebras which inspired Proposition 5, resp. the proof of Proposition 6(b), and the referee for pointing out a simplification of my original proof of Proposition 6.

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