A Characterization of Finite Chebyshev Sequences in R^n

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ABSTRACT

Given an arbitrary totally ordered set Γ, we distinguish three types of sequences $A=(a_i)_{i\in\Gamma}\subset R^n$, called LI-, H-, and T-sequences according as $\det[a_{i_1},a_{i_2},\dots,a_{i_n}]$ is non-zero for some $i_1<i_2<\cdots<i_n$, is non-zero for all $i_1<i_2<\cdots<i_n$, and is of constant non-zero sign for all $i_1<i_2<\cdots<i_n$. In this paper we show that if Γ is of finite cardinality, then the inhomogeneous system of linear inequalities

$$\langle a_i,x\rangle=\alpha_i,\qquad i\in\Gamma\backslash B(\alpha),$$

$$\operatorname{sgn}\langle a_i,x\rangle=\operatorname{sgn}\alpha_i,\qquad i\in B(\alpha),$$

is solvable for all saturated $\alpha\stackrel{\Delta}{=}(\alpha_i)_{i\in\Gamma}\subset R^1$ with $S^+(\alpha)\leqslant n-1$ iff A is a T-sequence. Here saturated means that there is a unique way of replacing the zeros of α by $+1$ and -1, in order to reach $S^+(\alpha)$. $B(\alpha)$ is the union of those "intervals" of Γ of maximal cardinality $\geqslant 2$ (called blocks), on which α has a constant non-zero sign.

1. INTRODUCTION

Let Γ be an arbitrary totally (linearly) ordered set with $<$ as order relation. We shall use $<$ for real numbers as well; the meaning will be clear from the context. Γ will be used as an "index set" of sequences. Denote by

$$A\stackrel{\Delta}{=}(a_i)_{i\in\Gamma}\quad\text{and}\quad\alpha\stackrel{\Delta}{=}(\alpha_i)_{i\in\Gamma}$$

a sequence of vectors $a_i=(a_{1i},a_{2i},\dots,a_{ni})\in R^n$ and real numbers $\alpha_i\in R^1$, $i\in\Gamma$, respectively. For $p<\infty$, we write $(i_k)_{k=1}^p$ for a subsequence $(i_1,i_2,\dots,i_p)\subset\Gamma$ with $i_1<i_2<\cdots<i_p$.

Assume that $|\Gamma| \geqslant n \geqslant 1$, where $|\Gamma|$ is the cardinality of Γ, and for $(i_k)_{k=1}^n \subseteq \Gamma$ denote

$$a(i_1, i_2, \dots, i_n) \overset{\Delta}{=} \det \begin{bmatrix} a_{1i_1} & a_{1i_2} & \cdots & a_{1i_n} \\ a_{2i_1} & a_{2i_2} & \cdots & a_{2i_n} \\ \vdots & \vdots & & \vdots \\ a_{ni_1} & a_{ni_2} & \cdots & a_{ni_n} \end{bmatrix}. \tag{1}$$

The sequence A is called a

linearly independent (LI) sequence in R^n if

$$a(i_1, i_2, \dots, i_n) \neq 0 \qquad \text{for some} \quad (i_k)_{k=1}^n \subseteq \Gamma,$$

Haar (H) sequence in R^n if

$$a(i_1, i_2, \dots, i_n) \neq 0 \qquad \text{for all} \quad (i_k)_{k=1}^n \subseteq \Gamma,$$

Chebyshev (T) sequence in R^n if

$$\operatorname{sgn} a(i_1, i_2, \dots, i_n) = \text{const} \neq 0 \quad \text{for all} \quad (i_k)_{k=1}^n \subseteq \Gamma.$$

Then the following statements trivially hold:

(a) A is an LI-sequence iff there is an $(i_k)_{k=1}^n \subseteq \Gamma$ such that the system of equations

$$\langle a_{i_k}, x \rangle \overset{\Delta}{=} \sum_{j=1}^{n} a_{ji_k} x_j = \alpha_{i_k}, \qquad k = 1, 2, \dots, n, \tag{2}$$

is uniquely solvable for all α.

(b) A is an H-sequence iff the system (2) is uniquely solvable for all $(i_k)_{k=1}^n \subseteq \Gamma$ and all α.

(c) If A is a T-sequence then the system (2) is uniquely solvable for all $(i_k)_{k=1}^n \subseteq \Gamma$ and all α.

The unique solvability of (2) for all $(i_k)_{k=1}^n \subseteq \Gamma$ and all α characterizes H-sequences but not T-sequences. Property (b) is sometimes called the "universal interpolation property" of the sequence A.

The purpose of this paper is to introduce an "extended interpolation property" and show that this property characterizes T-sequences (in case $|\Gamma| < \infty$).

For the formulation of our theorem we recall the following well-known concept (see [4], [5], and [3] for its generalization). For a finite sequence $\beta_1, \beta_2, \ldots, \beta_p$ of real numbers we denote by $S^+(\beta_1, \beta_2, \ldots, \beta_p)$ the maximal number of sign changes in the sequence, where the zeros are given arbitrary non-zero signs. For example, $S^+(0,0,+1,-2,-3,0,+4) = 4$.

Define

$$S^+(\alpha) \stackrel{\Delta}{=} \sup S^+(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_p}),$$

where the supremum is taken over all positive integers p and all $(i_k)_{k=1}^p \subseteq \Gamma$. In the sequel we shall use the notation

$$I_0(\alpha) \stackrel{\Delta}{=} \{i \in \Gamma : \alpha_i = 0\}, \quad I_+(\alpha) \stackrel{\Delta}{=} \{i \in \Gamma : \alpha_i > 0\}, \quad I_-(\alpha) = \{i \in \Gamma : \alpha_i < 0\}.$$

Two sequences $\alpha = (\alpha_i)_{i \in \Gamma}$, $\beta = (\beta_i)_{i \in \Gamma}$ are considered different iff $\alpha_i \neq \beta_i$ for some $i \in \Gamma$.

Suppose now that $|\Gamma| < \infty$, say

$$\Gamma \stackrel{\Delta}{=} (1, 2, \ldots, m).$$

Then $S^+(\alpha) = S^+(\alpha_1, \alpha_2, \ldots, \alpha_m)$, and in general there are many different "fillings" of the zeros of α by numbers $+1$ and -1 to reach $S^+(\alpha)$. More precisely, there are many different sequences $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ such that

$$S^+(\beta) = S^+(\alpha) \quad \text{and} \quad \beta_i = \begin{cases} \alpha_i & \text{for } i \in I_+(\alpha) \cup I_-(\alpha), \\ +1 \text{ or } -1 & \text{for } i \in I_0(\alpha). \end{cases} \tag{3}$$

DEFINITION 1. The sequence α is said to be *saturated* if there is only one β for which (3) holds.

We also need the following simple concept:

The set of integers

$$J \stackrel{\Delta}{=} \{i : 1 \leqslant j \leqslant i \leqslant k \leqslant m\}$$

will be called a *block* of α if $j < k$, $\operatorname{sgn} \alpha_i = \epsilon \neq 0$ for $i \in J$, $\operatorname{sgn} \alpha_{j-1} \neq \epsilon$ (if $j > 1$), and $\operatorname{sgn} \alpha_{k+1} \neq \epsilon$ (if $k < m$).

I.e., a block is an interval of maximal "length" ($\geqslant 2$) on which α has constant non-zero sign.

The union of all blocks of α will be denoted by $B(\alpha)$.

Now, our theorem is the following [here $\Gamma \stackrel{\Delta}{=} (1,2,\dots,m)$]:

THEOREM. *Let*

$$A \stackrel{\Delta}{=} (a_1, a_2, \dots, a_m) \subset R^n, \qquad 2 \leqslant n+1 \leqslant m < \infty,$$

be a sequence of vectors $\theta \neq a_i \in R^n$ $(i = 1,2,\dots,m)$. *A is a* T-*sequence iff the system*

$$\begin{aligned} \langle a_i, x\rangle &= \alpha_i, \qquad i \in \Gamma \setminus B(\alpha), \\ \operatorname{sgn}\langle a_i, x\rangle &= \operatorname{sgn}\alpha_i, \qquad i \in B(\alpha), \end{aligned} \tag{4}$$

is consistent for all saturated sequences of real numbers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ *for which* $S^+(\alpha) \leqslant n-1$.

Using this theorem we can prove an analogous result for an arbitrary non-finite Γ, but it still remains essentially a finite statement (see [9] and the remarks in Sec. 5).

Obviously our theorem is meaningful only when there are T-sequences at all. The classical example for a T-sequence is the sequence

$$a_t \stackrel{\Delta}{=} (1, t, t^2, \dots, t^{n-1}) \in R^n, \qquad t \in \Gamma \subseteq R^1.$$

More generally, any linearly independent system [or Haar system or Chebyshev system] $\{u_k(i)\}_{k=1}^n$, $i \in \Gamma$, of real valued functions defined on an arbitrary totally ordered set Γ determines an LI- [or H- or T-] sequence by

$$\hat{u}_i \stackrel{\Delta}{=} (u_1(i), u_2(i), \dots, u_n(i)) \in R^n, \qquad i \in \Gamma.$$

This "sequence" $\hat{u}_i$ is usually called the moment curve of the system. Now each $x \in R^n$ defines a function $u(i) = \langle \hat{u}_i, x\rangle$, $i \in \Gamma$, called the generalized (g-) polynomial of the system. In this terminology (2) expresses a well-known fact about the interpolation properties of g-polynomials of an H- [or T- or LI-] system. On the other hand there are also well-known theorems on the existence of g-polynomials of a T-system having prescribed zeros and sign change properties. The condition (4) is a kind of "extended interpolation property" for g-polynomials.

It is obvious that an LI- [H-, T-] sequence $(a_i)_{i\in\Gamma}\subseteq R^n$ defines an LI- [H-, T-] system on Γ by

$$u_k(i) \overset{\Delta}{=} a_{ki}, \qquad i\in\Gamma, \quad k=1,2,\dots,n.$$

Thus we could have formulated our theorem equivalently in terms of T-systems and g-polynomials. We have chosen rather the terminology of T-sequences because this shows more clearly that any statement about the existence of g-polynomials is a consistency statement for (finite or infinite) systems of linear inequalities. In this connection the methods of convex analysis can be applied (see, e.g., Rockafellar [2]). In fact, it is the well-known theorem of Helly on the intersection of convex sets in R^n which (together with some purely combinatorial considerations) lies in the core of our proof (for the Helly's theorem see, e.g., Valentine [1]).

We divide our material into four sections. Section 2 contains a survey of some known results about the existence of g-polynomials (of T-systems) with prescribed zeros and signs, formulated in our terminology of T-sequences. In Sec. 3 we deal with some combinatorial properties of finite sequences of numbers 0, $+1$, and -1. In Sec. 4 the proof of our theorem is given. Section 5 contains some concluding remarks.

For the basic properties of T-systems and g-polynomials see the book of Karlin and Studden [3], or a newer book of Krein and Nudelman [7].

2. SOME PRELIMINARY RESULTS

Here Γ, A, and α are, as in Sec. 1, an arbitrary totally ordered set, a sequence of vectors

$$A \overset{\Delta}{=} (a_i)_{i\in\Gamma}\subseteq R^n,$$

and a sequence of real numbers

$$\alpha \overset{\Delta}{=} (\alpha_i)_{i\in\Gamma}\subset R^1.$$

Assume in this section that $|\Gamma|\geqslant n+1\geqslant 2$, and A is an arbitrary T-sequence in R^n.

It can be shown easily (see [3]) that using the notation

$$\beta_x \overset{\Delta}{=} (\langle a_i, x\rangle)_{i\in\Gamma}, \qquad x\in R^n,$$

we have

$$S^+(\beta_x) \leqslant n-1 \qquad \text{for all} \quad \theta \neq x \in R^n. \tag{5}$$

This relation suggests that if we try to characterize those sequences α for which the homogeneous system

$$\operatorname{sgn}\langle a_i, x\rangle = \operatorname{sgn}\alpha_i, \qquad i \in P, \tag{6}$$

is consistent for (as large as possible) $P \subseteq \Gamma$, then, probably, we have to assume that $S^+(\alpha) \leqslant n-1$. Indeed, for a wide class of Γ and A the following statement has been proved:

If $S^+(\alpha) \leqslant n-1$, then (6) *is consistent for some $P \subseteq \Gamma$, where P depends on Γ and A but, in general, does not depend on α.*

For example, a classical result of Krein [6, 7] (in its modified form due to Karlin and Studden [3]) states:

Let

$$\Gamma \triangleq [a,b] \subset R^1, \qquad -\infty < a < b < +\infty,$$

$$a_i \triangleq (u_1(i), u_2(i), \ldots, u_n(i)) \in R^n, \qquad i \in \Gamma,$$

where $\{u_k\}_{k=1}^n \subset C([a,b])$ is a (continuous) T-system and $\alpha \in C([a,b])$. If $S^+(\alpha) \leqslant n-1$, then (6) *is consistent for $P=(a,b)$. Here $C(\Gamma)$ denotes the set of real valued functions continuous on Γ (if Γ has some topology).*

Another well-known result is due to Krein and Rehtman [5] (see also [3] or [7]):

Let

$$\Gamma \triangleq \bar{K} \triangleq K \cup \{+\infty\},$$

where K is the set of non-negative integers. Let $\{u_k\}_{k=1}^n \subset C(\bar{K})$ be a T-system,

$$a_i \triangleq (u_1(i), u_2(i), \ldots, u_n(i)) \in R^n, \qquad i \in \Gamma,$$

and $\alpha \in C(\bar{K})$. If $\alpha_i \geqslant 0$ for all $i \in \Gamma$, and $S^+(\alpha) \leqslant n-1$, then (6) *is consistent for $P = \Gamma = K$.*

Karlin and Studden [3] gave a more general (but weaker) statement:

Let

$$\Gamma \triangleq \bar{K}, \qquad \alpha \in C(\bar{K}), \quad \textit{and} \quad a_i \in R^n, i \in \Gamma,$$

be the same as in the previous statement. If $S^+(\alpha) \leqslant n-1$, *then there is a* $\theta \neq x \in R^n$ *such that* $\langle a_i, x\rangle \alpha_i \geqslant 0$ *for all* $i \in \Gamma$.

This result has been sharpened recently by Tihanyi and Uhrin [8], who proved:

Let

$$\Gamma \triangleq \bar{K}, \qquad \alpha \in C(\bar{K}) \quad \textit{and} \quad a_i \in R^n, i \in \Gamma,$$

be as previously. If $S^+(\alpha) \leqslant n-1$, *then* (6) *is consistent for* $P = K$ (*see also Uhrin* [9]).

The latter result followed, by standard constructions, from the following finite

PROPOSITION (see [8], [9]). *Let* $2 \leqslant n+1 \leqslant |\Gamma| < +\infty$,

$$A \triangleq (a_i)_{i \in \Gamma}$$

be a T-sequence in R^n, *and*

$$\alpha \triangleq (\alpha_i)_{i \in \Gamma} \subset R^1.$$

If $S^+(\alpha) \leqslant n-1$, *then* (6) *is consistent for* $P = \Gamma$.

Our Theorem shows that in the finite case A is a T-sequence iff the system (4) is consistent for all α in a particular subclass of the α-fulfilling $S^+(\alpha) \leqslant n-1$.

3. FINITE SEQUENCES OF NUMBERS 0, +1, AND −1

In this section

$$\Gamma \triangleq (1, 2, \ldots, m), \qquad m < +\infty,$$

and $w = (w_1, w_2, \ldots, w_m)$ is a sequence of numbers 0, +1, and −1.

DEFINITION 2. $i \in \Gamma$ is a 0-multiple zero of w of type 1 or 2, if $i+1 \in \Gamma$ and $w_i w_{i+1} < 0$ or $w_i w_{i+1} > 0$, respectively.

DEFINITION 3. $i \in \Gamma$ is a k-multiple zero of w of type 1 or 2 (for $k \geqslant 1$), if $i-1, i+k \in \Gamma$, $w_i = w_{i+1} = \cdots = w_{i+k-1} = 0$, and $w_{i-1} w_{i+k} < 0$ or $w_{i-1} w_{i+k} > 0$, respectively.

Denote by $Z_k^j(w)$ ($k \geqslant 0$, $j = 1,2$) the set of k-multiple zeros of w of type j. Easy considerations show that the numbers $|Z_k^1(w)|$, $|Z_k^2(w)|$, and $S^+(w) = S^+(w_1, w_2, \ldots, w_m)$ are related through the following identity (see [9], [10]):

$$S^+(w) + \sum_{k \geqslant 0} \left[|Z_{2k}^2(w)| + |Z_{2k+1}^1(w)| \right] = |\Gamma| - 1. \tag{7}$$

($|H|$ stands for the cardinality of the set H.)

Assume that w has $r(w)$ blocks (see Sec. 1), $r(w) \geqslant 0$, and denote its blocks by $B_1(w), B_2(w), \ldots, B_{r(w)}(w)$. Further denote

$$B(w) \stackrel{\Delta}{=} \bigcup_{k=1}^{r(w)} B_k(w) \qquad (\text{if } r(w) = 0, \text{ then take } B(w) = \varnothing).$$

Clearly we have

$$B(w) = \bigcup_{i \in Z_0^2(w)} \{i, i+1\},$$

which implies

$$|B(w)| = |Z_0^2(w)| + r(w). \tag{8}$$

The following lemma is of basic importance in our investigations.

LEMMA. *Let $w = (w_1, w_2, \ldots, w_m)$ be a sequence of numbers* 0, +1, *and* −1, *where $m \geqslant 2$. If $S^+(w) \leqslant m-2$ and w is saturated, then $r(w) \geqslant 1$ and*

$$|P \cap B(w)| \geqslant r(w) + 1$$

for all subsets $P \subseteq \Gamma$ such that $|P| \geqslant S^+(w) + 2$.

Proof. First we prove that the assumption that w is saturated implies

$$\bigcup_{k \geqslant 1} \left(Z_{2k}^2(w) \cup Z_{2k-1}^1(w) \right) = \varnothing. \tag{10}$$

Suppose that the set in (10) is not empty, and let $i \in Z_{2k}^2(w)$ for some $k \geqslant 1$. The part of the sequence defined by i is of the type

$$\begin{matrix} +1, & 0, & 0, & \ldots, & 0, & +1 \\ i-1, & i, & i+1, & \cdots, & i+2k-1, & i+2k \end{matrix}.$$

If we fill in this part by $+1$ and -1 in an arbitrary way, the maximal number of sign changes will be $2k$. This can be reached by at least two different fillings, for example: $w_i = -1, w_{i+1} = +1, \ldots, w_{i+2k-1} = +1$ or $w_i = +1, w_{i+1} = -1, \ldots, w_{i+2k-1} = -1$. This implies that w is not saturated, contradicting our assumption. We would similarly come to a contradiction if we assumed that $Z_{2k-1}^1(w) \neq \varnothing$ for some $k \geqslant 1$.

Using (7), the condition (10) implies

$$S^+(w) + |Z_0^2(w)| = |\Gamma| - 1 = m - 1, \tag{11}$$

and this shows, using the assumption $S^+(w) \leqslant m-2$, that $|Z_0^2(w)| = m - 1 - S^+(w) \geqslant 1$, i.e., $r(w) \geqslant 1$.

Assume that for some $P \subseteq \Gamma$ with $|P| \geqslant S^+(w) + 2$ we have $|P \cap B(w)| \leqslant r(w)$. Then using the identities (8) and (11), we get

$$\begin{aligned} |P| &= |P \cap (\Gamma \setminus B(w))| + |P \cap B(w)| \leqslant |\Gamma \setminus B(w)| + r(w) \\ &= |\Gamma| - |B(w)| + r(w) = S^+(w) + 1, \end{aligned}$$

and this contradicts the condition $|P| \geqslant S^+(w) + 2$. Hence the lemma is proved. ∎

For the proof of our theorem, besides the lemma, we shall need the following simple constructions.

Consider a subsequence $P = (i_k)_{k=1}^p \subseteq \Gamma$ and let $w_{i_1}, w_{i_2}, \ldots, w_{i_p}$ be an arbitrary sequence of numbers $+1$ and -1. Define the sequence $w = (w_1, w_2, \ldots, w_m)$ as follows:

$$w_i \stackrel{\Delta}{=} \begin{cases} w_{i_1} & \text{if} \quad 1 \leqslant i \leqslant i_1, \\ w_{i_k} & \text{if} \quad i_k \leqslant i < i_{k+1} \text{ for } k = 1, 2, \ldots, p-1, \\ w_{i_p} & \text{if} \quad i_p \leqslant i \leqslant m. \end{cases} \tag{12}$$

It is clear that w is saturated, and we can easily see that for this sequence we have

$$\Gamma \setminus B(w) \subseteq P \quad \text{and} \quad S^+(w) = S^+(w_{i_1}, w_{i_2}, \ldots, w_{i_p}).$$

It is also clear that

$$P \neq \Gamma \quad \Rightarrow \quad B(w) \neq \varnothing \text{ and } P \cap B(w) \neq \varnothing. \tag{13}$$

Define a new sequence $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_m)$:

$$\tilde{w}_i \stackrel{\Delta}{=} \begin{cases} w_i & \text{for} \quad i \in B(w) \\ 0 & \text{for} \quad i \in \Gamma \setminus B(w). \end{cases} \tag{14}$$

First we prove that

$$\bigcup_{k \geq 1} \left[Z_{2k}^2(\tilde{w}) \cup Z_{2k-1}^1(\tilde{w}) \right] = \varnothing. \tag{15}$$

If $\Gamma \setminus B(w) = \varnothing$, then (15) holds trivially. Let $\Gamma \setminus B(w) = (j_1, j_2, \dots, j_s)$, $j_1 < j_2 < \cdots < j_s$. The definition of $B(w)$ implies that if $j_i = j_{i+1} - 1$, then $w_{j_i} \neq w_{j_{i+1}}$. This shows that in each subset of $\Gamma \setminus B(w)$ which consists of adjacent (in Γ) elements, the numbers w_j alternate in sign. The set $\Gamma \setminus B(w)$ is equal to the union of subsets of the above type (of maximal "length"), and of such j_i's whose both neighbors (in Γ) belong to $B(w)$. Clearly, the latter elements are members of $Z_1^2(\tilde{w})$, and the first elements of the above mentioned subsets are the elements of either $Z_{2k+1}^2(\tilde{w})$ for some $k \geq 1$ or $Z_{2k}^1(\tilde{w})$ for some $k \geq 1$. This proves (15).

Now using the identity (7) and the trivial relation $Z_0^2(\tilde{w}) = Z_0^2(w)$, we have $S^+(w) = S^+(\tilde{w})$.

It is not hard to show that (15) also implies that $\tilde{w}$ is saturated [see also the identity (17) below].

Thus, we have the following relations for $\tilde{w}$:

$$B(\tilde{w}) = B(w), \qquad S^+(\tilde{w}) = S^+(w), \quad \text{and} \quad \tilde{w} \text{ is saturated.} \tag{16}$$

Let us note finally that all concepts and statements of this section are valid in an unchanged form for an arbitrary sequence $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ of real numbers, when applied to the sequence

$$\operatorname{sgn} \alpha \stackrel{\Delta}{=} (\operatorname{sgn} \alpha_1, \dots, \operatorname{sgn} \alpha_m).$$

For notational simplicity, instead of $S^+(\operatorname{sgn} \alpha)$, $B(\operatorname{sgn} \alpha)$, etc., we shall write $S^+(\alpha)$, $B(\alpha)$, etc.

REMARK. That $\tilde{w}$ is saturated follows also from a more general identity proved in [9] (see also [10]):

Denote by $M(w)$ the number of different sequences β fulfilling (3) *of Sec.* 1 *with $\alpha = w$. Then we have*

$$M(w)=\begin{cases}2 & \text{if} \quad w=\theta,\\ \prod\limits_{k\geq 1}(2k+1)^{|Z^2_{2k}(w)|}(2k)^{|Z^1_{2k-1}(w)|} & \text{if} \quad w\neq\theta,\end{cases} \tag{17}$$

where θ is the zero sequence.

4. PROOF OF THE THEOREM

The fact α is saturated is equivalent to the equality $M(\alpha)=1$. First we prove the "only if" part of the theorem.

Let A be a T-sequence in R^n, and let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_m)$ be such that $S^+(\alpha)\leqslant n-1$ and $M(\alpha)=1$. Denote

$$C_i \overset{\Delta}{=} \begin{cases}\{x\in R^n:\langle a_i,x\rangle=\alpha_i\} & \text{if} \quad i\in\Gamma\setminus B(\alpha)\\ \{x\in R^n:\operatorname{sgn}\langle a_i,x\rangle=\operatorname{sgn}\alpha_i\} & \text{if} \quad i\in B(\alpha).\end{cases} \tag{18}$$

We have to prove that

$$\bigcap_{i\in\Gamma} C_i\neq\varnothing. \tag{19}$$

The sets C_i are convex subsets of R^n (hyperplanes and open halfspaces), and according to Helly's theorem it is enough to prove that

$$\bigcap_{i\in I} C_i\neq\varnothing \qquad \text{for all} \quad I=(i_k)_{k=0}^{n}\subseteq\Gamma. \tag{20}$$

According to our Lemma, $S^+(\alpha)\leqslant n-1$ and $M(\alpha)=1$ implies that $r(\alpha)\geqslant 1$ and

$$|I\cap B(\alpha)|\geqslant r(\alpha)+1, \tag{21}$$

where $B(\alpha)=\cup_{j=1}^{r(\alpha)}B_j(\alpha)$, and $B_j(\alpha)$ are the blocks of α. The inequality in (21) implies that

$$|I\cap B_s(\alpha)|\geqslant 2 \tag{22}$$

for some $1 \leqslant s \leqslant r(\alpha)$ (depending on I). Clearly if $i_p, i_q \in B_s(\alpha)$, $0 \leqslant p < q \leqslant n$, then also $i_p, i_{p+1}, \ldots, i_q \in B_s(\alpha)$. Since the subdeterminants $a(j_1, j_2, \ldots, j_n)$ are of constant non-zero sign for all $(j_k)_{k=1}^n \subseteq \Gamma$, it is easily seen that the equation

$$\det\begin{bmatrix} y_0 & y_1 & \cdots & y_p & y_{p+1} & \cdots & y_n \\ a_{i_0} & a_{i_1} & \cdots & a_{i_p} & a_{i_{p+1}} & \cdots & a_{i_n} \end{bmatrix} = 0 \tag{23}$$

has a solution $\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_p, \bar{y}_{p+1}, \ldots, \bar{y}_n$ such that

$$\begin{aligned} \operatorname{sgn} \bar{y}_p &= \operatorname{sgn} \alpha_{i_p}, \\ \operatorname{sgn} \bar{y}_{p+1} &= \operatorname{sgn} \alpha_{i_{p+1}}, \\ \bar{y}_k &= \alpha_{i_k} \qquad \text{if} \quad i_k \in \Gamma \setminus B(\alpha), \\ \operatorname{sgn} \bar{y}_k &= \operatorname{sgn} \alpha_{i_k} \qquad \text{if} \quad i_k \in B(\alpha). \end{aligned} \tag{24}$$

Clearly, the last n rows of the determinant in (23) are linearly independent (as vectors in R^{n+1}) because A is a T-sequence in R^n and $m \geqslant n+1$. Hence the solution row $(\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_n)$ of the equation (23) must be equal to a linear combination of the last n rows:

$$\bar{y}_k = \sum_{j=1}^{n} x_j a_{j i_k} = \langle a_{i_k}, x \rangle, \qquad k = 0, 1, \ldots, n, \tag{25}$$

where $a_{i_k} = (a_{1i_k}, a_{2i_k}, \ldots, a_{ni_k})$ and $x = (x_1, x_2, \ldots, x_n)$. But this means $x \in \bigcap_{i \in I} C_i$, and this proves the "only if" part of the theorem.

To prove the "if" part of the theorem, assume first that

$$a(j_1, j_2, \ldots, j_n) = 0 \qquad \text{for some} \quad (j_k)_{k=1}^n \subseteq \Gamma. \tag{26}$$

This implies that the equation

$$\sum_{k=1}^{n} y_{j_k} a_{j_k} = \theta \tag{27}$$

has a non-identically-zero solution $\bar{y}_{j_1}, \bar{y}_{j_2}, \ldots, \bar{y}_{j_n}$. Introduce the notation $K^+ = \{ j_k : \bar{y}_{j_k} > 0 \}$ and $K^- = \{ j_k : \bar{y}_{j_k} < 0 \}$. Clearly $|K^+ \cup K^-| \geqslant 2$ (one of them may be empty), and denoting $\eta_i = \bar{y}_i$, $i \in K^+$ and $\nu_i = -\bar{y}_i$, $i \in K^-$, we can write

$$\sum_{i\in K^+} \eta_i a_i - \sum_{i\in K^-} \nu_i a_i = \theta. \tag{28}$$

Let $K^+\cup K^- = \{i_1, i_2, \dots, i_p\}$, $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant m$, $2 \leqslant p \leqslant n$, and set

$$w_{i_k} = \begin{cases} +1 & \text{if} \quad i_k \in K^+, \\ -1 & \text{if} \quad i_k \in K^-, \end{cases} \qquad k = 1, 2, \dots, p.$$

Construct the sequences $w = (w_1, w_2, \dots, w_m)$ and $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_m)$ in the same way as in (12) and (14) of Sec. 3. According to (13), $p \leqslant n < m$ implies that $B(w) \neq \varnothing$ and $B(w)\cap(K^+\cup K^-)\neq\varnothing$. By (16), the same is true for $\tilde{w}$, i.e.,

$$B(\tilde{w}) \neq \varnothing \quad \text{and} \quad B(\tilde{w})\cap(K^+\cup K^-)\neq\varnothing. \tag{29}$$

Assume that the system

$$\begin{aligned} \langle a_i, x\rangle = \tilde{w}_i = 0 \quad & \text{if} \quad i\in\Gamma\setminus B(\tilde{w}), \\ \langle a_i, x\rangle < 0 \quad & \text{if} \quad i\in B(\tilde{w}) \text{ and } \tilde{w}_i < 0, \\ \langle -a_i, x\rangle < 0 \quad & \text{if} \quad i\in B(\tilde{w}) \text{ and } \tilde{w}_i > 0 \end{aligned} \tag{30}$$

has a solution $x\in R^n$. Multiplying the corresponding rows of (30) by η_i and ν_i, we get from (29)

$$-\sum_{i\in K^+} \eta_i\langle a_i, x\rangle + \sum_{i\in K^-} \nu_i\langle a_i, x\rangle < 0,$$

and this contradicts (28). But the construction of w shows that $S^+(w) \leqslant n-1$, and according to (16) we have $S^+(\tilde{w}) \leqslant n-1$ and $M(\tilde{w}) = 1$. Thus, $\tilde{w}$ is a sequence fulfilling the assumptions of the theorem and (30) has no solutions, i.e., the assumption (26) led to a contradiction.

Now let us assume that A is an H-sequence but not a T-sequence, i.e., assume that

$$a(j_1, j_2, \dots, j_n) > 0 \quad \text{and} \quad a(k_1, k_2, \dots, k_n) < 0 \tag{31}$$

for some $(j_i)_{i=1}^n, (k_i)_{i=1}^n \subseteq \Gamma$. We can see easily that there is a sequence of $n+1$ different indices $(i_1, i_2, \dots, i_{n+1}) \subset \{j_1, j_2, \dots, j_n\} \cup \{k_1, k_2, \dots, k_n\}$ such that

$$0 \neq \operatorname{sgn} a(i_1,\dots,i_{p-1},i_{p+1},\dots,i_{n+1}) \neq \operatorname{sgn} a(i_1,\dots,i_p,i_{p+2},\dots,i_{n+1}) \neq 0 \quad (32)$$

for some $1 \leqslant p \leqslant n+1$. Put $A_{i_k}=a(i_1,i_2,\dots,i_{k-1},i_{k+1},\dots,i_{n+1})$, $\alpha_{i_k}=(-1)^k A_{i_k}$, and $\eta_{i_k}=-\alpha_{i_k}$ for $k=1,2,\dots,n+1$. The condition (32) shows that $\operatorname{sgn}\alpha_{i_p}=\operatorname{sgn}\alpha_{i_{p+1}}\neq 0$; hence $S^+(\alpha_{i_1},\alpha_{i_2},\dots,\alpha_{i_{n+1}})\leqslant n-1$. This implies that for the sequence $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_m)$ defined by (12) (with $p=n+1$) we have

$$S^+(\alpha)=S^+(\alpha_{i_1},\alpha_{i_2},\dots,\alpha_{i_{n+1}})\leqslant n-1 \quad (33)$$

The relation $M(\alpha)=1$ holds trivially, because α has no zeros. Further, we have

$$\det\begin{bmatrix} \alpha_{i_1} & \alpha_{i_2} & \cdots & \alpha_{i_{n+1}} \\ a_{i_1} & a_{i_2} & \cdots & a_{i_{n+1}} \end{bmatrix} = \sum_{k=1}^{n+1}(-1)^{k-1}\alpha_{i_k}A_{i_k} = -\sum_{k=1}^{n+1}\alpha_{i_k}^2=\delta<0, \quad (34)$$

and clearly the system of equations

$$\sum_{k=1}^{n+1} y_k \begin{pmatrix} \alpha_{i_k} \\ a_{i_k} \end{pmatrix} = \begin{pmatrix} \delta \\ \theta \end{pmatrix} \quad (35)$$

has the unique solution $y_k=\eta_{i_k}$ $(k=1,2,\dots,n+1)$.

Assume now that the system

$$\begin{aligned} \langle a_i,x\rangle &= \alpha_i && \text{for} \quad i\in\Gamma\setminus B(\alpha), \\ \langle a_i,x\rangle &< 0 && \text{for} \quad i\in B(\alpha) \text{ and } \alpha_i<0, \\ \langle -a_i,x\rangle &< 0 && \text{for} \quad i\in B(\alpha) \text{ and } \alpha_i>0 \end{aligned} \quad (36)$$

has a solution $x\in R^n$. [It may happen now that $B(\alpha)=\varnothing$, but this does not affect the proof.] Multiplying the corresponding rows of (36) by η_{i_k}, $k=1,2,\dots,n+1$, we get

$$\begin{aligned} \eta_{i_k}\langle a_{i_k},x\rangle &= -\alpha_{i_k}^2 && \text{if} \quad i_k\in\Gamma\setminus B(\alpha), \\ \eta_{i_k}\langle a_{i_k},x\rangle &< 0 && \text{if} \quad i_k\in B(\alpha) \text{ and } \alpha_{i_k}<0, \\ \eta_{i_k}\langle a_{i_k},x\rangle &< 0 && \text{if} \quad i_k\in B(\alpha) \text{ and } \alpha_{i_k}>0. \end{aligned} \quad (37)$$

Adding up these inequalities, we obtain

$$\sum_{k=1}^{n+1} \eta_{i_k} \langle a_{i_k}, x \rangle < 0,$$

which contradicts (35).

We see that (36) cannot have any solution, and $S^+(\alpha) \leqslant n-1$, $M(\alpha)=1$. Thus, (31) led to a contradiction. This proves the "if" part of the theorem and the whole theorem is proved. ■

5. CONCLUDING REMARKS

(1) For the proof of existence results such as our Theorem or the Proposition of Sec. 2, Helly's theorem was used in [9], where in addition to these results, some other more general results were proved. In [9] the identity (7) played an important role also in the proof of the Proposition of Sec. 2. As A. Tihanyi observed, the proof of the Proposition can be established directly from the assumption $S^+(\alpha) \leqslant n-1$. This proof is given in [8].

(2) In [9] a more systematic study of finite sequences of numbers $0, +1, -1$ and the quantities $|Z_k^j(w)|$, $M(w)$, and $S^+(w)$ can be found. It is proved, for example, that the relation $M(w)=1$ (which means w is saturated) is equivalent to the following property of w (if $w \neq \theta$ and $I_0(w) \neq \varnothing$):

$$S^+(w_1, w_2, \dots, w_{i-1}, w_{i+1}, \dots, w_m) < S^+(w) \qquad \text{for all} \quad i \in I_0(w). \tag{38}$$

This in turn is equivalent to

$$\bigcup_{k \geqslant 1} \left[Z_{2k}^2(w) \cup Z_{2k-1}^1(w) \right] = \varnothing \qquad (\text{if} \quad w \neq \theta). \tag{39}$$

(3) It can also be shown that the concepts introduced here for finite sequences of numbers, can, to some extent, be generalized to arbitrary functions defined on an arbitrary totally ordered set Γ (see [9]). The main problems which arise are related to the continuity of α and the connectedness of Γ. For example, for any continuous function α defined on $[a,b]$ and having only finitely many zeros among which at least one is of the first kind, we have $M(\alpha) > 1$. Nevertheless, it is shown in [9] that both the Proposition of Sec. 2 and our Theorem may be generalized for an arbitrary totally ordered set. These generalizations however are not yet satisfactory, and it will be a matter of further study to get more exact results for infinite Γ.

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