Discrete Mathematics 309 (2009) 4355–4361

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Vertex domination of generalized Petersen graphs

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ARTICLE INFO

Article history:
Received 9 August 2007
Received in revised form 20 January 2009
Accepted 22 January 2009
Available online 23 February 2009

Keywords:
Generalized Petersen graph
Vertex domination
Efficient domination
Perfect domination

ABSTRACT

In a graph G a vertex v dominates all its neighbors and itself. A set D of vertices of G is (vertex) dominating set if each vertex of G is dominated by at least one vertex in D. The (vertex) domination number of G, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set of G. A set D of vertices in G is efficient dominating set if every vertex of G is dominated by exactly one vertex of D. For natural numbers n and k, where $n > 2k$, a generalized Petersen graph $P(n, k)$ is obtained by letting its vertex set be $\{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ and its edge set be the union of $\{u_iu_{i+1}, u_iv_i, v_iv_{i+l}\}$ over $1 \le i \le n$, where subscripts are reduced modulo n. We prove a necessary and sufficient condition for these graphs to have an efficient dominating set, and we determine exact values of $\gamma(P(n, k))$ for $k \in \{1, 2, 3\}$. Also we prove that for an odd number k, $\gamma(P(n, k)) = \frac{n}{2} + O(k)$ and for an even number $k > 2$, $\gamma(P(n, k)) \le \frac{5n}{9} + O(k)$.

1. Introduction

For the definition of basic concepts not given here we refer the reader to a textbook in graph theory, for example [8]. For surveys on the domination concept in graph theory we refer the reader to [5,6].

A set D of vertices of a graph G is a (vertex) dominating set if each vertex in $V - D$ is adjacent to at least one vertex in D. The (vertex) domination number of G, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set of G. A minimum dominating set of G is a γ-set. A set D of vertices is efficient dominating set or a perfect dominating set if each vertex of G is dominated by exactly one vertex in D. Note that every efficient dominating set is necessarily independent. Also, any efficient dominating set in a graph must be of size $\gamma(G)$.

In a generalized Petersen graph $P(n, k)$ we let its vertex set be the union of $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$, and its edge set be $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k}\}$, $1 \le i \le n$. The first set of vertices is u-vertices and the second ones v-vertices. By a u-path in $P(n, k)$ we mean a path whose vertices consist of just u-vertices. A v-path is defined similarly. The edge of the form u_iv_i is spoke. Fig. 1 shows the generalized Petersen graph $P(16, 5)$ and an efficient dominating set.

Georges et al. [4] and Zelinka [9] studied other domination parameters on generalized Petersen graphs. Here we study their vertex domination. In Section 2 we characterize generalized Petersen graphs that have efficient dominating sets. By applying this result, in Section 3 we find the exact values of $\gamma(P(n, k))$ for $1 \le k \le 3$. In Section 4 we discuss $\gamma(P(n, k))$ for any k.

2. Efficient vertex domination

In the following lemma a useful necessary condition is given for $P(n, k)$ to have an efficient dominating set.

Lemma 1. *If $P(n, k)$ has an efficient dominating set, then $\gamma(P(n, k)) = \frac{n}{2}$ and $4|n$.*

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2009.01.018

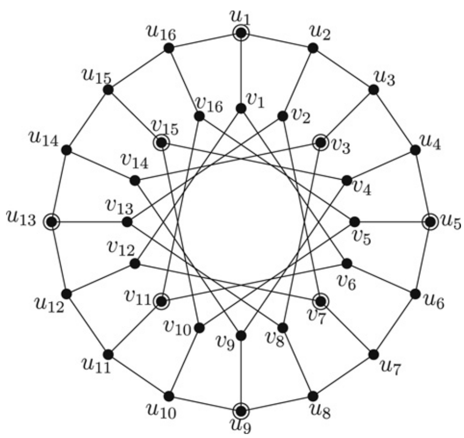


Fig. 1. An efficient dominating set in $P(16, 5)$.

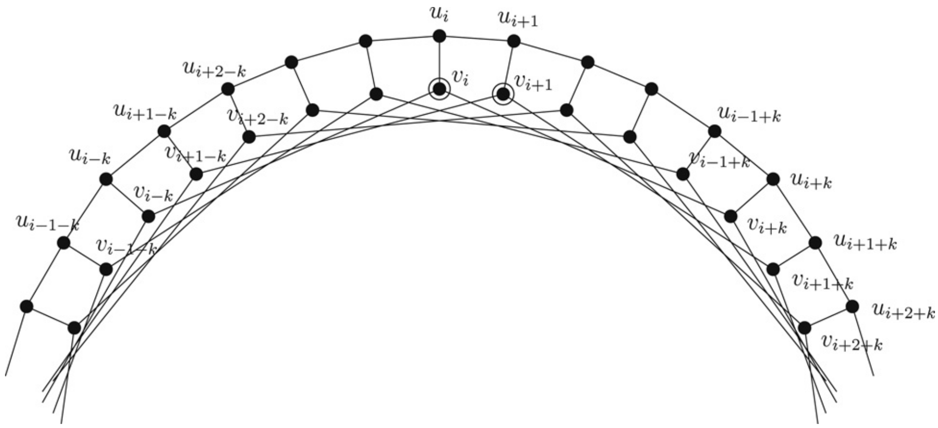


Fig. 2. If v_i and v_{i+1} belong to a dominating set in $P(n, k)$.

Proof. Since $P(n, k)$ is a 3-regular graph with $2n$ vertices, if $P(n, k)$ has an efficient dominating set, then $\gamma(P(n, k)) = \frac{2n}{3+1} = \frac{n}{2}$.

Now if we let $\gamma(P(n, k)) = m$, then $n = 2m$. Assume that S is a dominating set of size m and suppose that l of its elements are u-vertices and $m - l$ of them v-vertices. Each u-vertex dominates three of u-vertices and each v-vertex dominates one u-vertex. Since $P(n, k)$ has an efficient dominating set, $3l + (m - l) = n = 2m$, and hence $m = 2l$. As $n = 2m$, we have $n = 4l$ and so $4|n$. ■

Lemma 2. *If k is an odd number and $4|n$, then $\gamma(P(n, k)) = \frac{n}{2}$, and therefore $P(n, k)$ has an efficient dominating set.*

Proof. Let $n = 4l$. We construct an efficient dominating set $S = A \cup B$, where

$$A = \{u_{4i+1} \mid 0 \le i \le l - 1\} \quad \text{and} \quad B = \{v_{4i+3} \mid 0 \le i \le l - 1\}.$$

Here A dominates vertices u_{4i}, u_{4i+1}, and u_{4i+2}, and B dominates u_{4i+3}. Also the vertices v_{4i+3}, v_{4i+3+k}, and v_{4i+3-k} are dominated by B, while for each i the vertex v_{4i+1} is dominated by A. Since k is odd, any v-vertex v_j with $j = 4i + r$, for each $r \in \{1, 2, 3, 4\}$ is dominated. But $|S| = \frac{n}{2}$, so $\gamma(P(n, k)) = |S| = \frac{n}{2}$ and therefore S is an efficient dominating set. See Fig. 1 for an example. ■

Lemma 3. *Suppose that S is an efficient dominating set for $P(n, k)$. If a v-vertex $v_i \in S$, then $v_{i+1} \notin S$, where subscripts are taken modulo n.*

Proof. Suppose to the contrary that $v_i, v_{i+1} \subseteq S$ for some i, as in Fig. 2.

To dominate u_{i+k} and u_{i+k+1}, we must have $u_{i-1+k} \in S$ and $u_{i+2+k} \in S$. To dominate u_{i-k} and u_{i+1-k} we must have $u_{i-1-k} \in S$ and $u_{i+2-k} \in S$. Now, neither v_{i-1} nor its neighbors can be used for dominating v_{i-1}, as there will be some overlaps in dominating. ■

Theorem 1. *A generalized Petersen graph $P(n, k)$ has an efficient dominating set if and only if $n \equiv 0 \pmod 4$ and k is odd.*

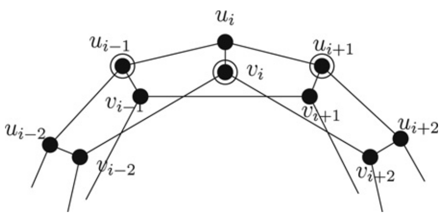


Fig. 3. A building block of a γ-set in $P(n, 2)$.

Proof. Sufficiency of the statement follows from Lemma 2. For necessity, suppose that S is an efficient dominating set in $P(n, k)$. As in Lemma 1, we have $|S| = \frac{n}{2} = 2l$, where l is the number of u-vertices in S, which is equal to the number of v-vertices in S. Each u-vertex dominates three u-vertices (including itself) and one v-vertex. So there are $3l$ u-vertices dominated by u-vertices, and l of them dominated by v-vertices. Let u_i and u_j be two u-vertices in S, such that on one of the u-paths from u_i to u_j there is no other u-vertex in S. Now there are exactly five u-vertices on the u-path from u_i to u_j, including u_i and u_j. For, since S is an efficient dominating set and by Lemma 3 the number of vertices on that path dominated by a v-vertex is at most 1, and also since there are l v-vertices in S, there must be at least one vertex of that path dominated by a v-vertex. So there is a unique pattern for the u-vertices in S, say $\{u_i, u_{i+4}\} \subseteq S$, and similarly $\{v_{i-2}, v_{i+2}\} \subseteq S$, see Fig. 1 for the pattern. By this unique pattern, it is clear that $P(n, k)$ does not have an efficient dominating set for even values of k. ■

3. Some exact values for $\gamma(P(n, k))$

In this section we establish some formulas for the vertex domination number of three classes of generalized Petersen graphs.

3.1. The Case $k = 1$

Theorem 2. *If $n \geq 3$, then we have*

$$\gamma(P(n, 1)) = \begin{cases} \dfrac{n}{2} + 1 & \text{if } n \equiv 2 \pmod 4 \\ \left\lceil \dfrac{n}{2} \right\rceil & \text{otherwise.} \end{cases}$$

Where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Proof. Obviously $\gamma(P(n, 1)) \geq \lceil \frac{n}{2} \rceil$. For the case $n \equiv 2 \pmod 4$, by Lemma 1, $P(n, 1)$ is not efficient, so in this case $\gamma(P(n, 1)) \geq \frac{n}{2} + 1$. For the construction of γ-sets with desired sizes, the same pattern as of Lemma 2 works, except in the case of $n = 4l + 2$, we need an extra vertex v_1. ■

3.2. The Case $k = 2$

Behzad and Behzad [1] have shown that $\gamma(P(2k + 1, k)) \leq \lceil \frac{3(2k+1)}{5} \rceil$. Since for each odd number n, the graph $P(n, 2)$ is isomorphic to $P(2k + 1, k)$ (see [7]), we generalize their result and show that equality holds for any n.

Theorem 3. *For $n \geq 5$ we have $\gamma(P(n, 2)) = \lceil \frac{3n}{5} \rceil$.*

Proof. For sufficiency, to show that $\gamma(P(n, 2)) \leq \lceil \frac{3n}{5} \rceil$, all we need is to construct a set that uses $\lceil \frac{3n}{5} \rceil$ vertices to dominate $P(n, 2)$. We cover $P(n, 2)$ by blocks of 10 vertices each, as shown in Fig. 3.

We dominate vertices of each block with 3 vertices as shown in Fig. 3. For $n = 5l$, vertices of $P(n, 2)$ can be partitioned by these blocks, therefore, $\gamma(P(5l, 2)) \leq 3l$. If $n \equiv 1(\mod 5)$, then we can cover all vertices by these blocks, except two adjacent vertices which can be dominated just with one more vertex. Hence $\gamma(P(5l + 1, 2)) \leq 3l + 1$. If $n \equiv 2$ or 3 $(\mod 5)$ then we can dominate remaining vertices with two more vertices. So, $\gamma(P(n, 2)) \leq \lceil \frac{3n}{5} \rceil$ for $n = 5l+2$ or 3. If $n \equiv 4$ $(\mod 5)$, then we dominate eight remaining vertices with three more vertices, and we still have $\gamma(P(n, 2)) \leq \lceil \frac{3n}{5} \rceil$.

For necessity, we need to show that $\gamma(P(n, 2)) \geq \lceil \frac{3n}{5} \rceil$. By Theorem 1, $P(n, 2)$ never has an efficient dominating set. So $\gamma(P(n, 2)) > \frac{n}{2}$, which implies that $\gamma(P(n, 2)) = \lceil \frac{3n}{5} \rceil$, for $n = 5, 6, 8, 10$. Also note that $P(7, 2)$ is isomorphic to $P(7, 3)$ and we will see in Theorem 4 that $\gamma(P(7, 3)) = 5$, so $\gamma(P(7, 2)) = 5 = \lceil \frac{3\cdot 7}{5} \rceil$. For $n = 9$, we need some more work. $\gamma(P(9, 2)) \geq \lceil \frac{9}{2} \rceil = 5$. If $\gamma(P(9, 2)) = 5$, then in any γ-set S, either the number of u-vertices or the number of v-vertices must be at most 2. Obviously none of them can contain just one vertex of S. If there are just two u-vertices in S, then without loss of generality, we may assume that $u_1 \in S$. The non-trivial cases are $\{u_1, u_4\} \subset S$ or $\{u_1, u_5\} \subset S$. In either case we are forced to include a set of three v-vertices in S. In both cases, the resulting set of 5 vertices does not form a dominating set. The case that S only contains two v-vertices is similar. Thus $\gamma(P(9, 2)) \geq 6$.

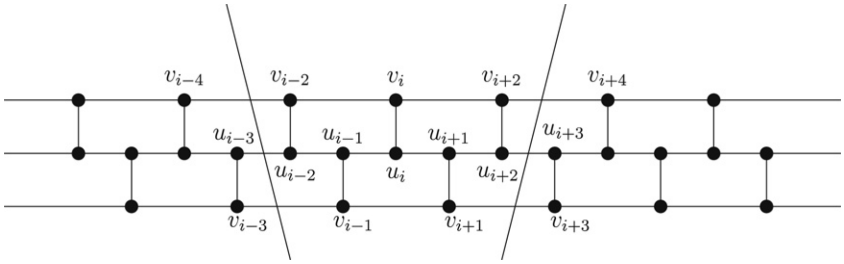


Fig. 4. A $\mathcal{P}_i$-block.

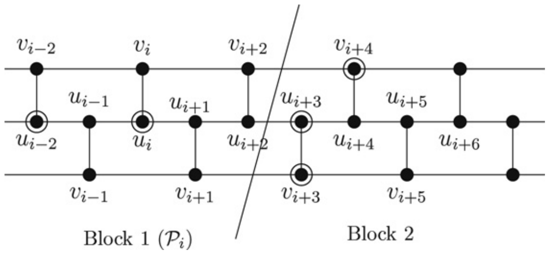


Fig. 5. Block $\mathcal{P}_i$ and its neighbor block.

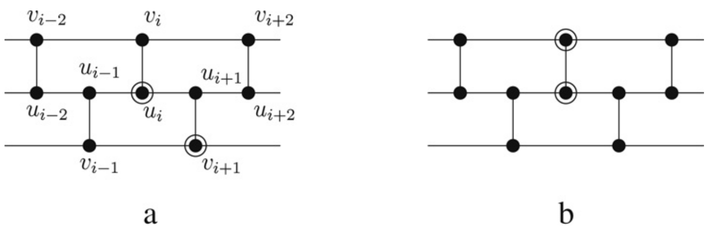


Fig. 6. Two possible forms of blocks with $\gamma_i = 2$.

Now assume that $n > 10$ and let S be a γ-set for $P(n, 2)$. For each $i = 1, \ldots, n$ we define a $\mathcal{P}_i$-block to be induced subgraph of $P(n, 2)$ on the set of vertices $\{u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}, v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$, where the subscripts are taken modulo n. See Fig. 4.

Let $\gamma_i = |S \cap V(\mathcal{P}_i)|$. We proceed as follows.

First, we show that there exists a γ-set in which all $\mathcal{P}_i$-blocks have $\gamma_i > 1$. Note that each of the vertices $\{u_{i-1}, u_i, u_{i+1}, v_i\}$ can be dominated *only* with some vertex of $\mathcal{P}_i$. So for all i, $\gamma_i \geq 1$. Now suppose that S is a γ-set for which the cardinality of the set $\{i | \gamma_i = 1\}$ is minimum. We show that this cardinality is zero. Indeed, if for some i, $|S \cap V(\mathcal{P}_i)| = 1$ then obviously the only vertex in $V(\mathcal{P}_i)$ belonging to S, must be u_i. To dominate vertices v_{i+1}, u_{i+2}, and v_{i+2} we need to have $\{v_{i+3}, u_{i+3}, v_{i+4}\} \subseteq S$. Now, the set $T = (S - \{u_{i+3}\}) \cup \{u_{i+2}\}$ is a γ-set and it has less blocks with $\{i | \gamma_i = 1\}$ than S. A contradiction.

Next, let S be a γ-set for which $\gamma_i > 1$ for all i, and the cardinality of the set $\{i | \gamma_i = 2\}$ for S is minimum. We show that in any $\mathcal{P}_i$-block with $\gamma_i = 2$, we have

(a) $u_i \in \mathcal{P}_i$.
(b) $\gamma_{i\pm 2}, \gamma_{i\pm 4} \geq 3$ and γ_{i+2} or $\gamma_{i-2} \geq 4$.

Let $\mathcal{P}_i$ be a block with $\gamma_i = 2$ (see Fig. 4). To show (a), note that, as we noticed earlier, the vertices $\{u_{i-1}, u_i, u_{i+1}, v_i\}$ can be dominated only with some vertices of $\mathcal{P}_i$. So either $u_i \in \mathcal{P}_i$ or $|\mathcal{P}_i \cap S| \geq 3$.

To show (b) we prove that in $\mathcal{P}_i$, neither of the vertices u_{i-2}, v_{i-2}, u_{i+2}, nor v_{i+2} can belong to S. For, if one of these vertices, say u_{i-2}, belongs to S then v_{i+1}, u_{i+2}, and v_{i+2} must be dominated by other vertices than those of $\mathcal{P}_i$. So we must have $\{v_{i+3}, u_{i+3}, v_{i+4}\} \subset S$. See Fig. 5.

On the other hand, to dominate u_{i+5}, one of the vertices u_{i+6}, v_{i+5}, u_{i+4}, or u_{i+5} must belong to S. So, Block 2 has at least 4 vertices in S. Now, $(S - \{u_{i+3}\}) \cup \{u_{i+2}\}$ is another γ-set which has fewer blocks with $\gamma_i = 2$, and this is a contradiction to the way that S is chosen. So, without loss of generality, we may assume that each block with $\gamma_i = 2$, up to symmetry, is one of the forms given in Fig. 6. Assume $\mathcal{P}_i$ is of the form in Fig. 6(a). Since u_{i+2} and v_{i+2} cannot be dominated by the vertices of $\mathcal{P}_i \cap S$, we have $v_{i+4}, u_{i+3} \in S$. Also, at least one of the vertices u_{i+4}, u_{i+5}, u_{i+6}, or v_{i+5} must belong to S. Similarly, u_{i-3}, v_{i-4} and at least one of the vertices u_{i-4}, u_{i-5}, u_{i-6} or v_{i-5} must belong to S. Now, assertion (b) is clear. Proof of the second case, Fig. 6(b), is similar.

Now, we count the elements of S. From the above, we know that $\gamma_i \geq 2$, and also that if $\gamma_i = 2$ then $\gamma_{i-2} + \gamma_i + \gamma_{i+2} \geq 9$. Let L be a set defined as $L = \{i-2, i, i+2 | \gamma_i = 2\}$. Obviously $|L|$ is of multiple 3, and we have

$$\sum_{i=1}^{n} \gamma_i = \sum_{\gamma_i=2} (\gamma_{i-2} + \gamma_i + \gamma_{i+2}) + \sum_{i \notin L} \gamma_i.$$

$$\geq \sum_{\gamma_i=2} 9 + \sum_{i \notin L} 3 = 9\frac{|L|}{3} + 3(n - |L|) = 3n.$$

Therefore, $\sum_{i=1}^{n} \gamma_i \geq 3n$. Note that each vertex of $P(n, 2)$ belongs to exactly 5 $\mathcal{P}_i$-blocks. So, $5|S| = \sum_{i=1}^{n} \gamma_i$. Hence, $5|S| \geq 3n$ and $|S| \geq \lceil \frac{3n}{5} \rceil$. ■

3.3. The Case $k = 3$

Xueliang Fu, Yuansheng Yang and Baoqi Jiang have proved the following theorem in [3]. Here we give a short and different proof.

Theorem 4. *For $n \geq 7$ we have*

$$\gamma(P(n, 3)) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod 4 \\ \left\lceil \frac{n}{2} \right\rceil & \text{if } n \equiv 1, 0 \pmod 4 \text{ or } n = 11 \\ \left\lceil \frac{n}{2} \right\rceil + 1 & \text{if } n \equiv 3 \pmod 4,\ n \neq 11. \end{cases}$$

Proof. First, we construct an efficient dominating set for each case. For a given number l, let A and B be two sets defined as in Lemma 2, i.e.

$$A = \{u_{4i+1} \mid 0 \leq i \leq l-1\} \quad \text{and} \quad B = \{v_{4i+3} \mid 0 \leq i \leq l-1\}.$$

Now it can be easily checked that each of the following sets is a dominating set of the appropriate size in each case:

1. $n = 4l,\ S = A \cup B$;
2. $n = 4l + 1,\ S = A \cup B \cup \{v_{n-1}\}$;
3. $n = 4l + 2,\ S = A \cup B \cup \{u_{n-2}, v_{n-1}\}$;
4. $n = 4l + 3\ (n \neq 11),\ S = A \cup B \cup \{u_{n-2}, v_{n-3}, v_2\}$;
5. $n = 11,\ S = \{u_1, u_5, u_8, v_1, v_3, v_{10}\}$.

Next, we prove that each of the given sets is indeed a γ-set. As we noted in the proof of Lemma 1, we have $\gamma(P(n, 3)) \geq \lceil \frac{n}{2} \rceil$. So, this takes care of cases 1, 2, and 5. Case 3 follows from Lemma 1. To see Case 4, if $\gamma(P(4l+3, 3)) = \lceil \frac{n}{2} \rceil = 2l+2$, and if S is a γ-set, then we have *exactly* two double dominations, i.e. there are two vertices of $P(4l+3, 3)$ each of which is dominated twice, or one vertex is dominated three times. Suppose that we have s of u-vertices and t of v-vertices in S. So,

$$3s + t \geq 4l + 3, \qquad 3t + s \geq 4l + 3, \quad \text{and} \quad s + t = 2l + 2.$$

These imply $s = t = l + 1$.

Therefore there are $3(l+1) + (l+1)$ many u-vertices dominated and the same number for v-vertices. Thus, there is no vertex dominated three times, and one of the two doubly dominated vertices is a u-vertex and the other one is a v-vertex. Two adjacent u-vertices or v-vertices cannot be in S since then we have a vertex dominated by two vertices. Therefore, there are three cases to be discussed:

(a) Two vertices of a spoke belong to S. Let u_1 and v_1 be such vertices. For $n \geq 15$, to dominate u_3 we need to have $v_3 \in S$. Also similar argument may be used to show that the vertices $u_5, u_8, v_{n-1}, v_{10}, v_{12}$, and v_{14}, orderly are forced to be in S. Now we have no choice to dominate u_{11}.

Note that if $n = 11$ we do not face this situation, because $v_{n-1} = v_{10}$. Indeed the γ-set given in the above has two such double dominated vertices. But if $n = 7$, then v_3 is forced to be in S, and there is no choice for u_4 to be dominated without having another double domination vertex.

(b) Let the double dominated u-vertex be u_2, which is dominated by u_1 and u_3. If $n = 7$, then to dominate u_5 and u_6 we must have $v_6, v_5 \in S$, also to dominate v_4 we need either v_4 or v_7 to be in S. Therefore, $S = \{u_1, u_3, v_5, v_6, v_7\}$ or $\{u_1, u_3, v_4, v_5, v_6\}$. If $n > 7$, then to dominate u_5 we must have u_6 or v_5 in S. If $u_6 \in S$, then to dominate v_4 there will be another double domination in u-vertices, namely u_4. So, $v_5 \in S$. Now to dominate v_4 we need $v_7 \in S$, and for u_6 we need $v_6 \in S$. Now to dominate u_8 we must have $u_9 \in S$. But then there will be two double dominated v-vertices, namely v_3 and v_9.

(c) Let the double dominated u-vertex be u_1, which is dominated by v_1 and u_2. If $n > 7$, then vertices u_5, v_n, and v_9 are forced to be in S. Now that we have $\{u_5, v_9, v_n\} \subset S$, vertices u_6 and u_9 are dominated, but in order u_7 to be dominated we must have $v_7 \in S$. Then we do not have choice to dominate u_8 unless we have another double domination in v-vertices. For the case $n = 7$ since u_1 is the only double dominated u-vertex by v_1 and u_2, so u_5 is the only candidate to dominate u_4. Now to dominate u_7, we must choose v_7. Thus v_6 is left undominated. Therefore $\gamma(P(7, 3)) > 4$. ■

4. Final notes

In this section we introduce some bounds for domination number of generalized Petersen graphs.

Proposition 1. *If k is an odd number and $n > 2k$ is any integer, then $\gamma(P(n, k)) = \frac{n}{2} + O(k)$.*

Proof. Let A and B be two sets defined as in the proof of Lemma 2, and let $S = A \cup B$. The vertices u_j, for $j \equiv 0, 1, 2 \pmod 4$ are dominated by u_{j+1}, u_j, or u_{j-1} and if $j \equiv 3 \pmod 4$, then u_j is dominated by v_j. So, all u-vertices are dominated. For the v-vertices, if $k \le j \le n - k$, then v_j is dominated by u_j, $v_{j\pm k}$, v_j, or $v_{j\mp k}$ according to the value of j modulo 4. The number of possible remaining v-vertices that are not dominated by the vertices in S is at most $2k$. We may add all of them to S to have a dominating set. So, we have

$$\frac{n}{2} \le \gamma(P(n, k)) \le \frac{n}{2} + 2k \implies \gamma(P(n, k)) = \frac{n}{2} + O(k). \quad ■$$

Note that when k is a fixed integer the upper and lower bounds given in the proof of Proposition 1 are close to each other, but for large values of k (for example close to $\frac{n}{2}$) the gap between them is significant.

In the following we find an upper bound by introducing appropriate blocks of vertices in each case.

Proposition 2. *If k is an even number greater than 2 and $n > 2k$, then $\gamma(P(n, k)) \le \frac{5n}{9} + O(k)$. Indeed, this upper bound can be improved:*

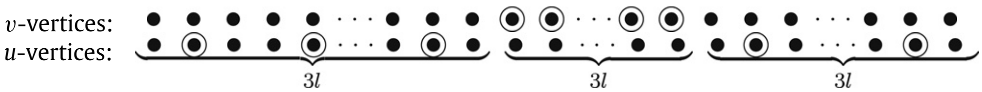
(a) $\gamma(P(n, k)) \le (5l)\lceil \frac{n}{9l} \rceil$ $(k = 3l)$;
(b) $\gamma(P(n, k)) \le (5l + 2)\lceil \frac{n}{9l+4} \rceil$ $(k = 3l + 1)$;
(c) $\gamma(P(n, k)) \le (5l + 4)\lceil \frac{n}{9l+6} \rceil$ $(k = 3l + 2)$.

Proof. To show the inequality, we choose blocks, described in each case in the following, and cover $P(n, k)$ by these blocks.

(a) $k = 3l$

In this case consider a block $\mathcal{B}_i$ of size $9l$ having $5l$ vertices $S(\mathcal{B}_i)$, in the dominating set as follows:

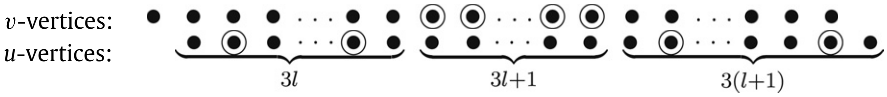
$$\begin{aligned} S(\mathcal{B}_i) \ = \ &\{u_{i+1}, u_{i+4}, \ldots, u_{i+3l-2}, u_{i+6l+1}, u_{i+6l+4}, \ldots, u_{i+9l-2}\} \cup \\ &\{v_{i+3l}, v_{i+3l+1}, \ldots, v_{i+6l-1}\} \end{aligned}$$



(b) $k = 3l + 1$

In this case consider a block $\mathcal{C}_i$ of size $9l + 4$, having $5l + 2$ vertices $S(\mathcal{C}_i)$, in the dominating set as follows:

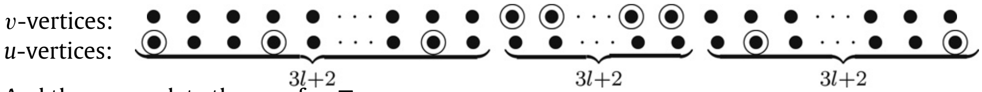
$$\begin{aligned} S(\mathcal{C}_i) \ = \ &\{u_{i+2}, u_{i+5}, \ldots, u_{i+3l-1}, u_{i+6l+3}, u_{i+6l+6}, \ldots, u_{i+9l+3}\} \cup \\ &\{v_{i+3l+1}, v_{i+3l+2}, \ldots, v_{i+6l+1}\} \end{aligned}$$



(c) $k = 3l + 2$

In this case consider a block $\mathcal{D}_i$ of size $9l + 6$ having $5l + 4$ vertices $S(\mathcal{D}_i)$, in the dominating set as follows:

$$\begin{aligned} S(\mathcal{D}_i) \ = \ &\{u_i, u_{i+3}, \ldots, u_{i+3l}, u_{i+6l+5}, u_{i+6l+8}, \ldots, u_{i+9l+5}\} \cup \\ &\{v_{i+3l+2}, v_{i+3l+3}, \ldots, v_{i+6l+3}\} \end{aligned}$$



And these complete the proof. ■

Note. The Generalized Petersen graphs are particular cases of the I-graphs (see for example [2]). The I-graph $I(n, j, k)$ is a graph with vertex and edge set

$$V(I(n, j, k)) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$$
$$E(I(n, j, k)) = \{u_i u_{i+j}, u_i v_i, v_i v_{i+k} \mid i = 1, 2, \ldots, n\},$$

where subscripts are reduced modulo n.

Clearly, $P(n, k) = I(n, 1, k)$. It could be an interesting project to investigate the domination number for this class of graphs as well, and we propose this research problem to the interested reader.

Acknowledgements

We thank anonymous referees for their useful comments which improved the paper. One of the authors (E.S.M.) completed part of this work while visiting IRMACS, Simon Fraser University. He would like to thank the Department of Mathematics and Computer Science for their warm and generous hospitality and financial support.

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