JOURNAL OF COMBINATORIAL THEORY, Series B 39, 371–375 (1985)

Note

Vertex-transitive Triangulations of Compact Orientable 2-Manifolds

DAVID B. SUROWSKI

Department of Mathematics, Kansas State University, Manhattan, Kansas 66506

Communicated by P. A. Catlin

Received December 15, 1984

In this note we construct two infinite families of vertex-transitive triangulations of compact orientable 2-manifolds. Included in these families are two of the best known "classical" examples, viz., the triangulation of the genus 3 surface admitting the group $PSL(2, 7)$ and the triangulation of the genus 7 surface admitting $SL(2, 8)$. © 1985 Academic Press, Inc.

1. INTRODUCTION

Let Γ be a graph with vertex set V and edge set E. If $\{x, y\}$ is in E we sometimes write $x \sim y$. Denote by $\mathrm{Aut}(\Gamma)$ the group of automorphisms of Γ. Of particular interest to us are the graphs $\Gamma = (V, E)$ which satisfy the following four conditions:

(1.1) Γ is connected.

(1.2) $|V| < \infty$.

(1.3) $\mathrm{Aut}(\Gamma)$ *is transitive on* V.

(1.4) If $x \in V$ *and if* $\Gamma(x) = \{y \in V \mid y \neq x$ *and* $y \sim x\}$, *then the induced graph on* $\Gamma(x)$ *is an ordinary n-gon,* $n \geqslant 4$.

(1.5) *Let S be the set of* 3-*element cliques in* Γ. *Then each* $s \in S$ *has a cyclic ordering such that if* $s_1, s_2 \in S$ *have orderings* $<_1$, $<_2$, *respectively, and if* $s_1 \cap s_2 = \{x, y\}$ *with* $x <_1 y$, *then* $y <_2 x$.

The import of conditions (1.1)-(1.5) is that if Δ is the simplicial complex with vertices V and simplexes equal to the finite cliques in V, then Δ triangulates a compact orientable 2-manifold of genus g, where

(1.6) $2 - 2g = v - nv/2 + nv/3$, $v = |V|$, and n is the valence of Γ.

If Γ is a graph satisfying (1.1)–(1.5), and if g is the genus as in (1.6), we shall call Γ a *vertex-transitive triangulation* (or simply a VTT) of *genus g*.

0095-8956/85 \$3.00

582b/39/3-13

For a discussion of a rather more general aspect of this problem, see [1, Chap. 5].

Our two families of VTT's are constructed as follows:

First of all, let $G = PSL(2, p)$, where p is a rational prime satisfying $16 \mid p^2 - 1$. Let $\mathscr{C}$ be one of the (two) conjugacy classes of elements of order p. Agree that $x \sim y$ if xy is an involution. Note that this is a symmetric relation since $yx = x^{-1}(xy)\, x$. We shall prove in Section 2 that the graph Γ so obtained is a VTT and has genus $g = (p+2)(p-3)(p-5)/24$. In particular, if $p = 7$ then $g = 3$ and $G = PSL(2, 7)$, which is "extremal" in that $|G| = 84(g-1)$. (We always have $|G| \leqslant 84(g-1)$; see [1, (5.5.2)].)

For the second family, let $G = SL(2, q)$, where $q = 2^r \geqslant 8$. Let $\mathscr{C}$ be a conjugacy class of elements of order $q-1$. If $x, y \in \mathscr{C}$ then define $x \sim y$ if xy^2 is an involution. In Section 3 we prove that the relation $\sim$ so defined is symmetric and that the resulting graph is a VTT of genus $g = (q-1)(q^2 - 5q - 12)/12$. If $q = 8$ then $g = 7$ and $|G| = SL(2, 8)$ which satisfies $|G| = 84(g-1)$.

We have made no attempt to give exhaustive lists of examples of graphs which the methods described herein will produce. Instead, we expect that the basic idea of the constructions will lead to further similar constructions and thereby shed light on the problem of which genera can actually occur.

2. $G = PSL(2, p)$, $16 \mid p^2 - 1$

Let $\mathscr{C}$ be a fixed class of elements of order p and let $\mathscr{I}$ be the class of involutions. Let Γ be the graph with vertices $\mathscr{C}$ and adjacency $x \sim y$ if $xy \in \mathscr{I}$. From [2, (19.2)] we have that for any $\tau \in \mathscr{I}$

$$|\{(x, y) \in \mathscr{C} \times \mathscr{C} \mid xy = \tau\}| = \frac{|\mathscr{C}|^2}{|G|} \sum \frac{\chi(g)^2\, \chi(\tau)}{\chi(1)}$$

where the summation is over the irreducible characters of G, and where $g \in \mathscr{C}$. From the character table of G [2, Sect. 38], one calculates that the right-hand side above is Σ, where

$$\begin{aligned} \Sigma &= p - 1 \qquad \text{if } p \equiv 1 \pmod 4, \\ &= p + 1 \qquad \text{if } p \equiv 3 \pmod 4. \end{aligned}$$

We have $|\mathscr{C}| = \frac{1}{2}(p^2 - 1)$ and

$$\begin{aligned} \mathscr{I}| &= \tfrac{1}{2}\, p(p+1) \qquad \text{if } p \equiv 1 \pmod 4, \\ &= \tfrac{1}{2}\, p(p-1) \qquad \text{if } p \equiv 3 \pmod 4. \end{aligned}$$

In either case we conclude that if $x \in \mathscr{C}$, then $|\Gamma(x)| = p$, i.e., the valence of Γ is p. The connectivity of Γ follows from the fact that G is generated by any pair of non-commuting subgroups of order p.

Next, let x, y, and z be elements of G, represented by the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & u^{-1} \\ -u & 0 \end{bmatrix},$$

respectively, where u is a square root of 2. (Since $16 \mid p^2 - 1$, the quadratic reciprocity theorem guarantees the existence of u.) We assume that $\mathscr{C}$ is the class containing x. Easy calculations show that $zxz^{-1} = y$ and that $(xy)^2 = 1$. Therefore $y \in \Gamma(x)$. Since we already know that the graph Γ has valence p, we infer that

$$\Gamma(x) = \{x^i y x^{-i} \mid 0 \leqslant i \leqslant p-1\}.$$

2.1. LEMMA. *$x^i y x^{-i} \sim y$ if and only if $2i^2 \equiv 1 \pmod p$.*

Proof. Calculation reveals that $(yx^i y x^{-i})^2$ is represented by the matrix

$$\begin{bmatrix} (-2i+1)^2 + 8i^2(i-1) & 2i^2(-4i^2+2) \\ 4(i-1)(-4i^2+2) & 8i^2(i-1) + (4i^2-2i-1)^2 \end{bmatrix}.$$

This matrix represents the identity in G if and only if $2i^2 \equiv 1 \pmod p$, as required.

The above lemma, together with the fact that the cyclic group generated by x acts transitively on $\Gamma(x)$, shows that $\Gamma(x)$ inherits the structure of an ordinary p-gon. Thus, we have already shown that Γ satisfies (1.1) through (1.4).

2.2. LEMMA. *Γ satisfies* (1.5).

Proof. Since G acts transitively on vertices and since the cyclic group $\langle g \rangle$ acts transitively on $\Gamma(g)$ for any $g \in \mathscr{C}$, we infer that G acts transitively on 3-cliques. Let $s = \{x_1, x_2, x_3\}$ be a fixed 3-clique; let the cyclic ordering be (x_1, x_2, x_3). If $g \in G$ and if $s' = gs$, define the cyclic ordering of s' to be (gx_1, gx_2, gx_3). This is well defined provided that in the stabilizer of s, no element fixes one vertex and interchanges the remaining two. Say that $z \in G$ fixes x_1 and interchanges x_2 and x_3. Then z has even order and centralizes x_1. Since the centralizer of x_1 in G is $\langle x_1 \rangle$, z cannot exist as above. Finally, assume that s', s'' are 3-cliques with $s' \cap s'' = \{x'_1, x'_2\}$, and assume that the cyclic ordering of s' is (x'_1, x'_2, x'_3). Let H be the stabilizer in G of the edge $\{x'_1, x'_2\}$. Since G acts transitively on the $\frac{1}{2}|G|$ edges, we infer that H has order 2. As above, the nonidentity element τ of H cannot fix either x_1' or x_2'. It follows that $\tau x_3' \in s''$ and so the cyclic ordering of s'' is (x_2', x_1', x_3').

2.3. THEOREM. *Γ is a VTT of genus* $g = (p+2)(p-3)(p-5)/24$.

Proof. Only the genus needs to be computed. To this end, apply (1.6) with $v = \frac{1}{2}(p^2-1)$, $n = p$.

3. $G = SL(2, q)$, $q = 2^r \geqslant 8$

Let a be a generator of the multiplicative group $\mathbb{F}_q^x$. Define elements x, y of G by setting

$$x = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \qquad y = \frac{1}{(a^2+1)} \begin{bmatrix} 1 & a^3 \\ (a^4+a^2+1)/a & a^4 \end{bmatrix}$$

and let $\mathscr{C}$ be the conjugacy class in G containing x. Let $\mathscr{I}$ be the class of involutions in G. Routine calculation shows that xy^2 is an involution, as is yx^2. Further calculation shows that y has eigenvalues a and a^{-1}, which already shows that x and y are conjugate in $GL(2, q)$. Thus, assume that $zxz^{-1} = y$, where $\det z = d$. Since $\mathbb{F}_q$ is a perfect field of characteristic 2 we infer that d has a square root, say $c^2 = d$. Thus we may write $z = z_1 z_2$ where $z_1 \in G$ and $z_2 = \mathrm{diag}(c, c)$, and so $z_1 x z_1^{-1} = y$. This proves that $y \in \mathscr{C}$.

Next, a character theoretic calculation as in Section 2 reveals that if τ is an involution in G,

$$|\{(x, y) \in \mathscr{C} \times \mathscr{C} \mid xy^2 = \tau\}| = q.$$

Since there are $q^2 - 1$ involutions, we conclude that

$$|\{z \in \mathscr{C} \mid xz^2 \in \mathscr{I}\}| = q - 1.$$

Thus, we infer that the cyclic group $\langle x \rangle$ acts transitively of the set $\{z \in \mathscr{C} \mid xz^2 \in \mathscr{I}\}$. Since y is in this set, and since $yx^2 \in \mathscr{I}$ as well, we conclude that $xz^2 \in \mathscr{I}$ if and only if $zx^2 \in \mathscr{I}$. Since G acts transitively on $\mathscr{C}$, we infer that the relation $z \sim w$ if $zw^2 \in \mathscr{I}$ is actually a symmetric relation on $\mathscr{C}$. We let Γ be the corresponding graph, which by the above, has valence $q-1$.

If Γ_0 is a connected component of Γ, then Γ_0 must contain at least $q+1$ vertices. But then the stabilizer in G of Γ_0 must contain at least $q+1$ conjugates of the cyclic group $\langle x \rangle$ and hence must be all of G. By transitivity we get $\Gamma_0 = \Gamma$, i.e., Γ is connected.

Calculations similar to those in (2.1) give us the following:

3.1. LEMMA. *Let x, y be as above. Then $x^i y x^{-i} \sim y$ if and only if $i \equiv \pm 1$ (mod $q-1$).*

Therefore, we already have that (1.1) through (1.4) are satisfied by Γ. Since the verification of (1.5) is identical with the proof of (2.2) we have the following:

3.2. THEOREM. *The graph Γ is a VTT of genus $g=(q-1)(q^2-5q-12)/12$.*

Proof. Of course, the genus is computed using (1.6).

REFERENCES

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