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Boolean Representation of Manifolds and Functions

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1. INTRODUCTION

Let U be a set, let X be a subset of U, and let $\mathscr{X} = \{X_i\}_{i\in I}$ be a family of subsets of U. We say that X admits a *Boolean representation in terms of* $\mathscr{X}$ if

$$X = \bigcup_{j\in J} \bigcap_{i\in S_j} X_i \tag{1}$$

for some family $\{S_j\}_{j\in J}$ of subsets of I.

Similarly, let f be a real valued function on a set Ω and let $\mathscr{G} = \{g_i\}_{i\in I}$ be a family of real valued functions on Ω. We say that f admits a *Boolean representation in terms of* $\mathscr{G}$ if

$$f(x) = \sup_{j\in J} \inf_{i\in S_j} g_i(x), \qquad \forall\, x \in \Omega \tag{2}$$

for some family $\{S_j\}_{j\in J}$ of subsets of I.

The goal of this paper is to establish Boolean representations for smooth domains in $\mathbb{R}^n$ and smooth real valued functions on closed convex subsets of $\mathbb{R}^n$.

The paper is organized as follows.

In Section 2, we show that a smooth n-dimensional manifold in $\mathbb{R}^n$ with a boundary admits Boolean representation (1) in terms of a family of closed half subspaces of $\mathbb{R}^n$.

In Section 3, we establish Boolean representation (2) for smooth real valued functions on closed convex domains in $\mathbb{R}^n$. Although this result can be established using the method developed in Section 2, we prefer to present an alternative proof that gives an explicit formula for the family $\mathscr{G}$ in (2).

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In Section 4, we investigate the relation between the Boolean representation established in Section 3 and the Legendre transform.

Finally, in Section 5, we establish an integral representation of smooth functions in terms of the Choquet integral. This representation is equivalent to one obtained in Section 3.

2. SMOOTH MANIFOLDS

Let $\Gamma \subset \mathbb{R}^n$ be a smooth n-dimensional manifold with a boundary; i.e., each $x \in \Gamma$ has a neighborhood diffeomorphic to an open subset of a closed half space in $\mathbb{R}^n$. For each point x in the boundary $\partial\Gamma$ we denote by Q_x the closed half space consisting of all tangent and "inward" vectors at x.

Let a be a point in Γ and let R be a closed ray with the origin at a. The connected component of $\Gamma \cap R$ in R is a closed interval $[a, a_R]$ in R (it is possible that $a_R = a$). In other words, $[a, a_R]$ is the set of all points in Γ that are "visible" from a in the "direction" R. Clearly, $a_R \in \partial\Gamma$ and $[a, a_R] \subset Q_{a_R}$.

Let $\mathcal{R}_a$ be the set of all closed rays with the origin at a. Then $\bigcap_{R \in \mathcal{R}_a} Q_{a_R}$ is a closed convex subset of Γ containing a. This we have

$$\Gamma = \bigcup_{a \in \Gamma} \bigcap_{R \in \mathcal{R}_a} Q_{a_R}.$$

The above argument proves the following theorem.

THEOREM 1. *Let $\Gamma \subset \mathbb{R}^n$ be a smooth n-dimensional manifold with the boundary $\partial\Gamma$ and let Q be the family of all closed half spaces Q_x, $x \in \partial\Gamma$. There is a family $\{Q_i\}_{i \in I}$ of subsets of Q such that Γ admits a Boolean representation*

$$\Gamma = \bigcup_{i \in I} \bigcap_{Q_x \in Q_i} Q_x. \tag{3}$$

Remark 1. Suppose Γ is convex. Then $\Gamma = \bigcap_{x \in \partial\Gamma} Q_x$ which is clearly a special case of (3). If Γ is concave, then $\Gamma = \bigcup_{x \in \partial\Gamma} Q_x$ which is again a special case of (3).

Remark 2. Representations similar to (3) are well known in the area of constructive solid geometry. Namely, any simple polytope can be represented by a Boolean formula based on the half spaces supporting the faces of the polytope [1].

3. SMOOTH FUNCTIONS

We use notation $x = (x_1, \ldots, x_n)$ for points in $\mathbb{R}^n$. A closed domain in $\mathbb{R}^n$ is the closure of an open set.

Let f be a smooth function on a closed domain Ω in $\mathbb{R}^n$. For a point $t \in \Omega$ we define

$$g_t(x) = \langle \nabla f(t), x - t \rangle + f(t), \qquad x \in \Omega, \tag{4}$$

where $\nabla f(t)$ is the gradient vector of f at t and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. Geometrically, the graphs of these affine linear functions are tangent hyperplanes to the graph of f.

In this section we prove the following theorem.

THEOREM 2. *Given closed convex domain Ω and $f \in C^1(\Omega)$, there exists a family $\{S_j\}_{j \in J}$ of subsets of Ω such that*

$$f(x) = \sup_{j \in J} \inf_{t \in S_j} g_t(x), \qquad \forall\, x \in \Omega. \tag{5}$$

First we prove two technical lemmas.

LEMMA 1. *Let h be a differentiable function on* $[0, 1]$. *There exists $\lambda_0 \in$* $[0, 1]$ *such that*

$$h'(\lambda_0)(-\lambda_0) + h(\lambda_0) \le h(0) \quad \textit{and} \quad h'(\lambda_0)(1 - \lambda_0) + h(\lambda_0) \ge h(1). \tag{6}$$

Proof. Let $m = h(1) - h(0)$. If $h'(0) \ge m$, then $\lambda_0 = 0$ satisfies both inequalities. Thus we may assume that $h'(0) < m$. Similarly, if $h'(1) \ge m$, then $\lambda_0 = 1$ satisfies both inequalities and we may assume that $h'(1) < m$.

Consider function $H(\lambda) = h(\lambda) - m\lambda - h(0)$. We have

$$H(0) = H(1) = 0 \quad \text{and} \quad H'(0) < 0, H'(1) < 0.$$

It follows that H is negative in some neighborhood of 0 and positive in some neighborhood of 1. Hence the set $U = \{\lambda \in (0, 1) : H(\lambda) = 0\}$ is a nonempty closed subset of (0, 1). Let $\lambda_0 = \inf U$. Then $H(\lambda_0) = 0$ and $H'(\lambda_0) \ge 0$; i.e.,

$$h(\lambda_0) = m\lambda_0 + h(0) = m(\lambda_0 - 1) + h(1)$$

and $h'(\lambda_0) \ge m$. We have

$$h'(\lambda_0)(-\lambda_0) + h(\lambda_0) = h'(\lambda_0)(-\lambda_0) + m\lambda_0 + h(0) \le h(0)$$

and

$$h'(\lambda_0)(1 - \lambda_0) + h(\lambda_0) = h'(\lambda_0)(1 - \lambda_0) + m(\lambda_0 - 1) + h(1) \ge h(1). \qquad \blacksquare$$

LEMMA 2. *Let* $f \in C^1(\Omega)$. *For any given* $a, b \in \Omega$ *there exists* $c \in \Omega$ *such that*

$$g_c(a) \leq f(a) \quad \textit{and} \quad g_c(b) \geq f(b).$$

Proof. Let $h(\lambda) = f((1-\lambda)a + \lambda b)$ for $\lambda \in [0,1]$. By Lemma 1, there is $\lambda_0 \in [0,1]$ satisfying inequalities (6). Let $c = (1-\lambda_0)a + \lambda_0 b$. We have

$$\begin{aligned} g_c(a) &= \langle \nabla f(c), a - c\rangle + f(c) = (-\lambda_0)\langle \nabla f(c), b - a\rangle + f(c) \\ &= (-\lambda_0)h'(\lambda_0) + h(\lambda_0) \leq h(0) = f(a) \end{aligned}$$

and

$$\begin{aligned} g_c(b) &= \langle \nabla f(c), b - c\rangle + f(c) = (1-\lambda_0)\langle \nabla f(c), b - a\rangle + f(c) \\ &= (1-\lambda_0)h'(\lambda_0) + h(\lambda_0) \geq h(1) = f(b). \end{aligned}$$ ∎

Now we proceed with the proof of Theorem 2. For a given $u \in \Omega$, we define $S_u = \{t \in \Omega : g_t(u) \geq f(u)\}$ and

$$f_u(x) = \inf_{t \in S_u} g_t(x), \quad x \in \Omega.$$

which is well defined, since $f \in C^1(\Omega)$.

By Lemma 2, for given $x, u \in \Omega, x \neq u$, there exists $v \in \Omega$ such that $g_v(u) \geq f(u)$ and $g_v(x) \leq f(x)$. Hence, $v \in S_u$ and

$$f_u(x) = \inf_{t \in S_u} g_t(x) \leq g_v(x) \leq f(x).$$

In addition,

$$f_u(u) = \inf_{t \in S_u} g_t(u) = g_u(u) = f(u),$$

since $u \in S_u$ and $g_t(u) \geq f(u)$ for $t \in S_u$. Therefore we have

$$\sup_{u \in \Omega} \inf_{t \in S_u} g_t(x) = \sup_{u \in \Omega} f_u(x) = f(x),$$

which completes the proof.

Remark 3. Let f be a strictly convex function. Then

$$f(x) = \sup_{u \in \Omega} g_u(x), \qquad x \in \Omega,$$

since $S_u = \{u\}$ for all $u \in \Omega$. Similarly, for a strictly concave f, $S_u = \Omega$ and we have

$$f(x) = \inf_{u \in \Omega} g_u(x), \qquad x \in \Omega.$$

Both facts are well known properties of convex (concave) functions.

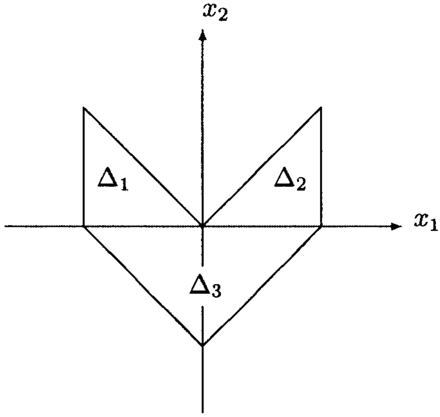

FIGURE 1

Remark 4. Convexity of Ω is an essential assumption in Theorem 2. Consider, for instance, the domain Ω in $\mathbb{R}^2$ which is a union of three triangles defined by the sets of their vertices as follows (see Fig. 1):

$$\Delta_1 = \{(-1,0),(-1,-1),(0,0)\}, \qquad \Delta_2 = \{(0,0),(1,1),(1,0)\},$$
$$\text{and} \qquad \Delta_3 = \{(-1,0),(1,0),(0,-1)\}.$$

Let us define

$$f(x) = \begin{cases} x_2^2, & \text{for } x \in \Delta_2, \\ 0, & \text{for } x \in \Delta_1 \cup \Delta_3. \end{cases}$$

Clearly, $f \in C^1(\Omega)$. Suppose f has a Boolean representation

$$f(x) = \sup_{j \in J} \inf_{t \in S_j} g_t(x), \qquad \forall\, x \in \Omega,$$

for some family $\{S_j\}_{j\in J}$ of subsets of Ω. Let a be a point in the interior of Δ_2. Since $f(a) > 0$, there is S_j such that $\inf\{g_t(a) : t \in S_j\} > 0$. Since $g_t(a) = 0$ for $t \in \Delta_1 \cup \Delta_3$, we have $S_j \subseteq \Delta_1$. Let $b = (-a_1, a_2)$. Then $g_t(b) = g_t(a)$, since $g_t(x) = 2t_2x_2 - t_2^2$. Thus $\inf\{g_t(b) : t \in S_j\} = \inf\{g_t(a) : t \in S_j\} > 0$ which contradicts $f(b) = 0$.

Remark 5. A Boolean representation of piecewise linear functions in terms of their linear components similar to (5) is obtained in [6].

4. THE LEGENDRE TRANSFORM

Let f be a strictly convex smooth function on a compact convex domain $\Omega \subset \mathbb{R}^n$. Then (cf. Remark 3)

$$\begin{aligned} f(x) &= \sup_{t\in\Omega} g_t(x) \\ &= \sup_{t\in\Omega}\{\langle \nabla f(t), x\rangle - [\langle \nabla f(t), t\rangle - f(t)]\}, \qquad \forall x \in \Omega. \end{aligned} \tag{7}$$

Let us introduce variables

$$p = \nabla f(t), \tag{8}$$

$$H = \langle \nabla f(t), t\rangle - f(t). \tag{9}$$

Since f is strictly convex, Eq. (8) defines a one-to-one mapping of Ω onto $\Omega' = \nabla f(\Omega)$ and we can express t in terms of p in (7) to obtain the following representation of $f(x)$ in terms of its *Legendre transform* $H(p)$ (cf. [2]):

$$f(x) = \sup_{p\in\Omega'}\{\langle p, x\rangle - H(p)\}.$$

In general, let Γ denote the graph of a smooth function $f\colon \Omega \to \mathbb{R}$. Then Γ is the envelope of the set of its tangent hyperplanes. The *Legendre transform* [7] of Γ is the surface defined parametrically by (8) and (9) in the $(n+1)$-dimensional (p, H)-space. One can view our Boolean representation (5) as a representation of an arbitrary smooth function f in terms of its Legendre transform.

5. INTEGRAL REPRESENTATION

It was shown in Section 3 that a smooth function f on a closed convex domain Ω admits the Boolean representation

$$f(x) = \sup_{u\in\Omega}\inf_{t\in S_u} g_t(x), \tag{10}$$

where S_u is a closed subset of Ω defined by $S_u = \{t \in \Omega : g_t(u) \geq f(u)\}$.

Let μ be a monotonic set function on Ω defined by

$$\mu(S) = \begin{cases} 1 & \text{if } S \supseteq S_u \text{ for some } u \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then (10) can be written equivalently in the form (Proposition 2.2 in [4]; see also [5])

$$f(x) = \int g_t(x)d\mu,$$

where the integral on the right side is the Choquet integral (alternatively, the Sugeno integral) with respect to the non-additive measure μ. For the definitions and properties of the Choquet and Sugeno integrals the reader is referred to [3]. The book also has a comprehensive bibliography on the subject.

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