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# On the regularity of weak solutions to the magnetohydrodynamic equations

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**Abstract**

In this paper, we study the regularity of weak solution to the incompressible magnetohydrodynamic equations. We obtain some sufficient conditions for regularity of weak solutions to the magnetohydrodynamic equations, which is similar to that of incompressible Navier–Stokes equations. Moreover, our results demonstrate that the velocity field of the fluid plays a more dominant role than the magnetic field does on the regularity of solution to the magneto-hydrodynamic equations.


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## 1. Introduction

We consider the viscous incompressible magneto-hydrodynamics (MHD) equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{Re}\,\Delta u + (u\cdot\nabla)u - S(B\cdot\nabla)B + \nabla\left(p + \dfrac{S}{2}|B|^2\right) = f, \\ \dfrac{\partial B}{\partial t} - \dfrac{1}{Rm}\,\Delta B + (u\cdot\nabla)B - (B\cdot\nabla)u = 0, \\ \operatorname{div} u = 0, \quad \operatorname{div} B = 0. \end{cases} \tag{1.1}$$

Here $u, p, B$ are non-dimensional qualities corresponding to the velocity of the fluid, its pressure and the magnetic field, respectively. $f(x,t)$ represents a non-dimensional volume density force. The non-dimensional number $Re$ is the Reynolds number, $Rm$ is the magnetic Reynolds and $S = M^2/(Re\,Rm)$ with $M$ being the Hartman number.

In this paper, we will discuss two classes of sufficient conditions which guarantee the weak solutions are regular. At first, we will present some results about the relationship between the regularity for weak solution to the MHD equations and the smoothness of the direction of vorticity for large vorticity field. Secondly, we will show the smoothness of solutions to the MHD equations when the velocity field $u$ belongs to $L^p(0,T;L^q(\mathbb{R}^3))$ with $1/p + 3/2q \leqslant 1/2$ for $q \geqslant 3$. In general, it is not known whether the smooth solution of the Cauchy problem exists for all time, for given sufficient smooth, divergence free initial data. Duvaut and Lions [8] constructed a class of global weak solutions, similar to the Leray–Hopf weak solutions to the three-dimensional Navier–Stokes equations. But the strong solution is only local, in general. For the two-dimensional case, the smoothness of solutions have been shown. And same results hold in the case of three-dimensional case under the assumption that $(u, B)$ belongs to $L^\infty(0,T;H^1(\mathbb{R}^3))$. For details, see Sermange and Teman [12]. As pointed out by Constantin and Fefferman [4] in the case of incompressible Navier–Stokes equations, the main difference between the two-dimensional and three-dimensional cases can be well understood by considering the dynamics of the fluid vorticity. In the two-dimensional case, the vorticity field is perpendicular to the plane of motion and its magnitude is uniformly bounded, while in the three-dimensional case, there exists a stretching mechanism for the vorticity magnitude which is non-linear and potentially capable of producing finite time singularities. At the same time, Constantin and Fefferman showed that the solution is smooth, if the direction of vorticity is sufficiently well behaved in the region of high vorticity magnitude, i.e., they obtained the smoothness of solution if the direction field $\xi$ of the vorticity $w(x,t)$ satisfies that

$$|\xi(x) - \xi(y)| \leqslant |x-y|/\rho \tag{1.2}$$

for some positive constant $\rho$ when $|w(x,t)| \geqslant \Omega$ and $|w(y,t)| \geqslant \Omega$ for some $\Omega > 0$. Similar results were given in [5] for three-dimensional incompressible Euler equations and for quasi-geostrophic active scalar equation in [6,7].

The first purpose of this paper is to discuss the important role of the smoothness of direction of vorticity in the region of the large vorticity in the regularity theory for the incompressible magnetohydrodynamics equations. Comparing with the incompressible Navier–Stokes equations, a important characteristic of the magneto-hydrodynamics is the induction effect. This effect brings about the coupling of the magnetic field and the velocity field. As a result of the inclusion of the magnetic field, the equation of magneto-hydrodynamics are considerably more complicated than those of ordinary hydrodynamics. This first resulting difficulty is that, there is no global $L^1$-estimate about the vorticities; The second, that is the key new difficulties come from the appearance of the strong coupled terms of vorticities of the velocity $u$ and magnetic field $B$. We use the fine estimate of singular integral to overcome the difficulties resulting from the lack of the $L^1$-estimate of the vorticity. Furthermore, some integral formulas are deduced for the coupling terms. By a careful treatment of the kernels of these integrals, we show that the solution is smooth, if the vorticity field $w^+(x,t)$ of velocity field $u$ satisfies the following estimate: there exist three positive constants $K, \rho, \Omega$, such that

$$|w^+(x+y,t)/|w^+(x+y,t)| - w^+(x,t)/|w^+(x+y,t)|| \leqslant K|y|^{1/2} \tag{1.3}$$

when $|y| \leqslant \rho$ and $|w^+(x,t)| \geqslant \Omega$ for some positive constants $K, \rho$ and $\Omega$.

The second purpose of this paper is to show the smoothness of weak solutions in $L^p(0,T;L^q(\mathbb{R}^3))$. Many authors have studied the regularity for weak solutions of the Navier–Stokes equations as long as any one of following three conditions hold:

(1) $u \in L^p(0,T;L^q(\mathbb{R}^3))$ for $1/p + 3/2q \leqslant 1/2$ and $q > 3$,
(2) $u \in C([0,T];L^3(\mathbb{R}^3))$,
(3) $\nabla u \in L^\alpha(0,T;L^\beta(\mathbb{R}^3))$ for $1/\alpha + 3/2\beta = 1$ with $1 < \alpha \leqslant 2$.

See [2,9,13,14]. It must be noticed that case (3) cannot be included into cases (1) and (2). Moreover, the borderline case $\alpha = 2$ is significant. It shows that $L^2(0,T;W^{1,3}(\mathbb{R}^3))$ is a regularity class. This cannot be deduced from (1) and (2), since $W^{1,3}(\mathbb{R}^3)$ cannot be imbedded into $L^\infty(\mathbb{R}^3)$. In this paper, we also show that one of (1)–(3) is sufficient condition for regularity of the solution to MHD equations. As pointed out by Beirão da Veiga, (3) shows that the loss of regularity in time turns out to be balanced by some additional regularity in space.

It should be noted that condition (1.3) is somewhat stronger than condition (1.2), while the conditions for our second results are the same as those for the Navier–Stokes equations. However, it is worthy to emphasize that there are no assumptions on the magnetic field $B$. In other word, our results demonstrate that the magnetic field plays less dominant role than the velocity field does in the regularity theory of solutions to the magneto-hydrodynamics equations. In a certain sense, our results are consistent with the recent numerical simulations of Politano et al. in [11]. Furthermore, observations of space and laboratory plasmas alike reveal that the magnetic field of the plasma tends to self-organize through a turbulent phase of relaxation into a simple spiral configuration [10]. Thus, an incompressible three-dimensional magneto-hydrodynamics equations should exhibits a greater degree of regularity than does an ordinary incompressible three-dimensional Navier–Stokes equation, in some sense. However, we cannot show this here.

Finally, it will follow easily from our proofs that for the three-dimensional incompressible Navier–Stokes equations, the Hölder continuity of the direction of the vorticity with exponent $1/2$, in the high vorticity magnitude region and a ball of every point with fixed radius, is sufficient to ensure the regularity of the solution.

## 2. Mathematical preliminaries

Let $w^+(x,t) = \operatorname{curl} u(x,t)$, $w^-(x,t) = \operatorname{curl} B(x,t)$. Then the vorticity equations for the three-dimensional incompressible magneto-hydrodynamics equation can be written as

$$\begin{cases} \dfrac{\partial w^+}{\partial t} - \dfrac{1}{Re}\Delta w^+ + (u\cdot\nabla)w^+ - (w^+\cdot\nabla)u - S(B\cdot\nabla)w^- + S(w^-\cdot\nabla)B = F, \\ \dfrac{\partial w^-}{\partial t} - \dfrac{1}{Rm}\Delta w^- + (u\cdot\nabla)w^- - (w^-\cdot\nabla)u - (B\cdot\nabla)w^+ + (w^+\cdot\nabla)B = 2T(B,u) \end{cases} \tag{2.1}$$

with

$$T(B,u) = \begin{pmatrix} \partial_2 B\cdot\partial_3 u - \partial_3 B\cdot\partial_2 u \\ \partial_3 B\cdot\partial_1 u - \partial_1 B\cdot\partial_3 u \\ \partial_1 B\cdot\partial_2 u - \partial_2 B\cdot\partial_1 u \end{pmatrix} \quad \text{and} \quad F = \operatorname{curl} f.$$

Here $\partial_i$ denote $\partial/\partial x_i$ for $i = 1,2,3$.

Let $C^\infty_{0,\sigma}(\mathbb{R}^3)$ denote the set of all $C^\infty$ real vector-valued functions $\phi = (\phi_1,\phi_2,\phi_3)$ with compact support in $\mathbb{R}^3$, such that $\operatorname{div}\phi = 0$. Let $H$ and $V$ be the closure of $C^\infty_{0,\sigma}(\mathbb{R}^3)$ in $L^2(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$, respectively. And let $\|\cdot\|_p$ denote the norm in $L^p(\mathbb{R}^3)$ for $1\leqslant p\leqslant\infty$. If the initial data $(u_0, B_0)$ belong to $H$ and $f\in L^2(0,\infty;V')$, it is well known that there exists a global weak solution $(u,B)$ in $L^\infty(0,\infty;H)\cap L^2_{\mathrm{loc}}(0,\infty;V)$, which satisfies the energy inequality

$$\begin{aligned} &\|u(t)\|_2^2 + S\|B(t)\|_2^2 + 2\int_0^t \left(\frac{1}{Re}\|\nabla u(s)\|_2^2 + \frac{S}{Rm}\|\nabla B(s)\|_2^2\right) ds \\ &\quad\leqslant \|u_0\|_2^2 + S\|B_0\|_2^2 + 2\int_0^t (u(s), f(s))\, ds \end{aligned} \tag{2.2}$$

for any $t\geqslant 0$ (cf. [8]). If $(u_0,B_0)\in V$ and $f\in L^2(0,\infty;L^2(\mathbb{R}^3))$, then there exists a unique solution $(u,B)$, such that

$$u, B \in L^\infty(0,T^*;V)\cap L^2(0,T^*;H^2(\mathbb{R}^3)) \tag{2.3}$$

for some $T^* > 0$. According to the regularity result obtained in [12], $u$ and $B$ are sufficient smooth, if $(u_0, B_0)$ and $f$ are sufficient smooth.

By the Biot–Savart law, the velocity field and the magnetic field can be expressed in terms of their vorticities, respectively, as follows:

$$u(x,t) = -\frac{1}{4\pi}\int \nabla\left(\frac{1}{|y|}\right) \times w^+(x+y)\,dy,$$
$$B(x,t) = -\frac{1}{4\pi}\int \nabla\left(\frac{1}{|z|}\right) \times w^-(x+z)\,dz. \tag{2.4}$$

As in [4], the gradient matrix can be decomposed as the strain matrix and the anti-symmetric parts

$$\begin{cases} \nabla u(x,t) = S^+(x,t) + \frac{1}{2}\, w^+(x,t) \times \cdot, \\ \nabla B(x,t) = S^-(x,t) + \frac{1}{2}\, w^-(x,t) \times \cdot \end{cases} \tag{2.5}$$

with

$$S^+(x,t) = \tfrac{1}{2}(\nabla u(x,t) + (\nabla u(x,t))^T), \quad S^-(x,t) = \tfrac{1}{2}(\nabla B(x,t) + (\nabla B(x,t))^T). \tag{2.6}$$

The following two integral equations were obtained in [4]:

$$\begin{cases} w^+(x,t) = \dfrac{1}{4\pi}\,\text{P.V.}\int \sigma(\hat{y})w^+(x+y,t)\dfrac{dy}{|y|^3}, \\ w^-(x,t) = \dfrac{1}{4\pi}\,\text{P.V.}\int \sigma(\hat{z})w^-(x+z,t)\dfrac{dz}{|z|^3}, \\ S^+(x,t) = \dfrac{3}{4\pi}\,\text{P.V.}\int M(\hat{y}, w^+(x+y,t))\dfrac{dy}{|y|^3}, \\ S^-(x,t) = \dfrac{3}{4\pi}\,\text{P.V.}\int M(\hat{z}, w^-(x+z,t))\dfrac{dz}{|z|^3}, \end{cases} \tag{2.7}$$

where the matrixes

$$\begin{cases} \sigma(\hat{y}) = 3\hat{y}\otimes\hat{y} - I, \\ M(\hat{y}, w) = \frac{1}{2}(\hat{y}\otimes(\hat{y}\times w) + (\hat{y}\times w)\otimes\hat{y}) \end{cases} \tag{2.8}$$

with $\hat{y} = y/|y|$, $I$ is the identity matrix and the tensor product simply denotes the matrix $(\hat{y}\otimes\hat{y})_{ij} = \hat{y}_i\hat{y}_j$. Moreover, the matrix $\sigma$ is symmetric, traceless and has zero mean on the unit sphere. The matrix $M$ is also traceless and symmetric; Its mean on the unit sphere is zero when the second variable $w$ is held fixed and $M$ is viewed as a function of $\hat{y}$ alone. The property with zero mean on the unit sphere is very important to deduce the necessary estimates about the coupling terms.

In the following, we deduce the integral representations for coupling terms. For this purpose, let $T(B,u) = (T_1(B,u), T_2(B,u), T_3(B,u))$. Differentiating the Biot–Savart law (2.4), one can obtain that

$$\frac{\partial B}{\partial x_i} = \frac{1}{4\pi}\text{P.V.}\int \nabla\partial_i\left(\frac{1}{|z|}\right) \times w^-(x+z)\,dz$$

here the property of zero mean of $\sigma$ on unit sphere has been used. Note that

$$\nabla\frac{\partial}{\partial z_i}\left(\frac{1}{|z|}\right) = (-e_i + 3\hat{z}_i\hat{z})/|z|^3 \overset{\Delta}{=} v_i$$

with $e_i$ the unit vector along the $z_i$-axis. Then

$$\begin{aligned}
T_1(B,u) &= \partial_2 B\cdot\partial_3 u - \partial_3 B\cdot\partial_2 u \\
&= \frac{1}{4\pi}\,\text{P.V.}\int (v_2\times w^-(x+z)\cdot\partial_3 u - v_3\times w^-(x+z)\cdot\partial_2 u)\,dz \\
&= \frac{1}{4\pi}\,\text{P.V.}\int (\partial_3 u\times v_2 - \partial_2 u\times v_3)\cdot w^-(x+z)\,dz \\
&= \frac{1}{4\pi}\,\text{P.V.}\int (-\nabla u_1\cdot w^-(x+z) + 3(\hat{z}_2\partial_3 u\times\hat{z} - \hat{z}_3\partial_2 u\times\hat{z}) \\
&\quad \times w^-(x+z))\frac{dz}{|z|^3} \\
&= \frac{1}{4\pi}\,\text{P.V.}\int (-\nabla u_1\cdot w^-(x+z) + 3(\hat{z}_2\partial_3 u - \hat{z}_3\partial_2 u)\cdot(\hat{z}\times w^-(x+z))\frac{dz}{|z|^3}.
\end{aligned}$$

Similarly,

$$T_2(B,u) = \frac{1}{4\pi}\,\text{P.V.}\int (-\nabla u_2\cdot w^-(x+z) + 3(\hat{z}_3\partial_1 u - \hat{z}_1\partial_3 u)\cdot(\hat{z}\times w^-(x+z))\frac{dz}{|z|^3},$$

$$T_3(B,u) = \frac{1}{4\pi}\,\text{P.V.}\int (-\nabla u_3\cdot w^-(x+z) + 3(\hat{z}_1\partial_2 u - \hat{z}_2\partial_1 u)\cdot(\hat{z}\times w^-(x+z))\frac{dz}{|z|^3}.$$

Thus,

$$\begin{aligned}
T(B,u)\cdot w^-(x,t) = &\ \frac{1}{4\pi}\,\text{P.V.}\int (-w^-(x+z,t)\cdot\nabla u\cdot w^-(x,t) \\
&+3(w^-(x,t)\times\hat{z})\cdot\nabla u\cdot(\hat{z}\times w^-(x+z,t))\frac{dz}{|z|^3}. \qquad (2.9)
\end{aligned}$$

In (2.9), the vectors at the right-hand side of matrix $\nabla u$ are viewed as column vector. Applying representations (2.5)–(2.7), it follows that

$$
\begin{aligned}
&T(B,u)\cdot w^-(x,t)\\
&\quad= -\frac{1}{4\pi}\,\text{P.V.}\int (w^-(x+z,t)\cdot S^+(x,t)\cdot w^-(x,t))\frac{dz}{|z|^3}\\
&\qquad -\frac{1}{8\pi}\,\text{P.V.}\int \text{Det}(w^-(x,t), w^-(x+z,t), w^+(x,t))\frac{dz}{|z|^3}\\
&\qquad +\frac{3}{4\pi}\,\text{P.V.}\int ((w^-(x,t)\times\hat z)\cdot S^+(x,t)\cdot(\hat z\times w^-(x+z,t)))\frac{dz}{|z|^3}\\
&\qquad +\frac{3}{8\pi}\,\text{P.V.}\int (\hat z, w^+(x,t))\,\text{Det}(\hat z, w^-(x,t), w^-(x+z,t))\frac{dz}{|z|^3},
\end{aligned}
\tag{2.10}
$$

where Det denotes the determinant of the matrix whose columns are the three column vectors in the bracket.

## 3. The main result

In this section, we intend to present our main results. Our major assumption about the vorticity $w^+(x,t)$ of the velocity field $u(x,t)$ is

**Assumption A.** There exist three positive constants $K$, $\rho$ and $\Omega$ such that

$$
|w^+(x+y,t)-w^+(x,t)|\leqslant K|w^+(x+y,t)||y|^{1/2} \tag{3.1}
$$

holds if both $|y|\leqslant\rho$ and $|w^+(x,t)|\geqslant\Omega$ for any $t\in[0,T]$.

Under this assumption on $w^+(x,t)$, one can show the following a priori estimate.

**Theorem 1.** *Let* $u_0, B_0 \in V$ *and* $f\in L^2(0,T;L^2(\mathbb{R}^3))$. *Assume that* $(u,B)$ *is a smooth solution of MHD equations* (1.1) *on some interval* $[0,T]$ *with* $0<T\leqslant\infty$. *Then if the Assumption A holds on* $[0,T]$, *one has*

$$
w^+, w^- \in L^\infty(0,T;L^2(\mathbb{R}^3)), \tag{3.2}
$$

$$
\nabla w^+, \nabla w^- \in L^2(0,T;L^2(\mathbb{R}^3)). \tag{3.3}
$$

*Moreover, for* $t\in[0,T]$,

$$
\begin{aligned}
&\|w^+(t)\|_2^2 + S\|w^-(t)\|_2^2 + \int_0^t ((1/Re)\|\nabla w^+(s)\|_2^2 + (S/Rm)\|\nabla w^-(s)\|_2^2)\,ds\\
&\quad\leqslant C\left(\|w^+(0)\|_2^2 + S\|w^-(0)\|_2^2 + \int_0^T \|f(s)\|_2^2\,ds\right)e^{CA_0}
\end{aligned}
\tag{3.4}
$$

*with* $A_0 = (\Omega^{1/2} + (K + \rho^{-1/2})^2)(\|u_0\|_2^2 + S\|B_0\|_2^2 + \int_0^T \|f(s)\|_{V'}ds)$ *and* $C$ *is an absolute constant.*

**Theorem 2.** (a) *Let* $u_0, B_0 \in V$ *and* $f \in L^2(0,T;L^2(\mathbb{R}^3))$. *Assume that* $(u,B)$ *is a smooth solution of MHD equations* (1.1) *on some interval* $[0,T]$ *with* $0 < T \leqslant \infty$. *Assume that one of the following two conditions holds*:

(1) $u \in L^p(0,T;L^q(\mathbb{R}^3))$ *for* $1/p + 3/2q = 1/2$ *and* $q > 3$,
(2) $u \in C([0,T];L^3(\mathbb{R}^3))$,

*then*

$$u \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)). \tag{3.5}$$

*Moreover,*

$$\|\nabla u(t)\|_2^2 + S\|\nabla B(t)\|_2^2 + \int_0^t \left(\frac{1}{Re}\|D^2u(s)\|_2^2 + \frac{S}{Rm}\|D^2B(s)\|_2^2\right) ds \leqslant C_0(\|\nabla u_0\|_2 + \|\nabla B_0\|_2^2) + C\int_0^T \|f\|_2^2\,ds \tag{3.6}$$

*holds for any* $t \in [0,T]$. *Here* $D^2 = \sum_{i,j=1}^3 \partial_i\partial_j$ *and* $C_0$ *is a constant depending on* $\int_0^T \|u(s)\|_q^p\,ds$ *in case* (1) *and* $\|u\|^2_{C([0,T];L^3(\mathbb{R}^3))}T$ *in the case* (2), *respectively*.

(b) *Let* $u_0, B_0 \in L^\beta(\mathbb{R}^3)$ *for some* $\beta \geqslant 3$. *If* $\nabla u \in L^\alpha(0,T;L^\beta(\mathbb{R}^3))$ *for* $1/\alpha + 3/2\beta = 1$ *with* $1 < \alpha \leqslant 2$, *then*

$$u, \quad B \in L^\infty(0,T;L^\beta(\mathbb{R}^3)); \quad |u|^{\frac{\beta-2}{2}}\nabla u, \quad |B|^{\frac{\beta-2}{2}}\nabla B \in L^2(0,T;L^2(\mathbb{R}^3)).$$

*Moreover,*

$$\|u(t)\|_\beta^\beta + \|B(t)\|_\beta^\beta + \frac{\beta(\beta-1)}{2Re}\int_0^t \||u|^{\frac{\beta-2}{2}}\nabla u(s)\|_2^2\,ds + \frac{\beta(\beta-1)}{2Rm}\int_0^t \||B|^{\frac{\beta-2}{2}}\nabla B(s)\|_2^2\,ds \leqslant C\left(\|B_0\|_\beta^\beta,\ \|u_0\|_\beta^\beta,\ e^{\int_0^T \|\nabla u\|_\beta^\alpha d\tau}\right) \tag{3.7}$$

*for any* $t \in [0,T]$.

Employing the above a priori estimates, one can show that

**Theorem 3.** *Let $u_0, B_0 \in V$ and $f \in L^2(0,T; L^2(\mathbb{R}^3))$. Suppose that $(u, B)$ is the weak solution of MHD equations (1.1) on $[0, T)$. If $w^+(x,t)$ satisfies the assumption A on $[0,T]$, then*

$$w^+, w^- \in L^\infty(0,T; L^2(\mathbb{R}^3)), \quad \nabla w^+, \nabla w^- \in L^2(0,T; L^2(\mathbb{R}^3)). \tag{3.8}$$

*Moreover,*

$$\begin{aligned} &\|w^+(t)\|_2^2 + S\|w^-(t)\|_2^2 + \int_0^t ((1/Re)\|\nabla w^+(s)\|_2^2 + (S/Rm)\|\nabla w^-(s)\|_2^2)\, ds \\ &\leqslant C\left(\|w^+(0)\|_2^2 + S\|w^-(0)\|_2^2 + \int_0^T \|f(s)\|_2^2\, ds\right) e^{CA_0} \end{aligned} \tag{3.9}$$

*with $A_0 = (\Omega^{1/2} + (K + \rho^{-1/2})^2)(\|u_0\|_2^2 + S\|B_0\|_2^2 + \int_0^T \|f(s)\|_{V'} ds)$ and $C$ is an absolute constant. Therefore $(u, B)$ is the unique strong solution to the MHD equations on $[0, T]$.*

**Theorem 4.** *Let $u_0, B_0 \in V$ and $f \in L^2(0,T; L^2(\mathbb{R}^3))$. Assume that $(u, B)$ is a weak solution of MHD equations (1.1) on some interval $[0, T]$ with $0 < T \leqslant \infty$. Assume that one of the following two conditions holds:*

(1) *$u \in L^p(0,T; L^q(\mathbb{R}^3))$ for $1/p + 3/2q = 1/2$ and $q > 3$,*
(2) *$u \in C([0,T]; L^3(\mathbb{R}^3))$,*

*then*

$$u \in L^\infty(0,T; H^1(\mathbb{R}^3)) \cap L^2(0,T; H^2(\mathbb{R}^3)). \tag{3.10}$$

*Moreover,*

$$\begin{aligned} &\|\nabla u(t)\|_2^2 + S\|\nabla B(t)\|_2^2 + \int_0^t \left(\frac{1}{Re}\|D^2 u(s)\|_2^2 + \frac{S}{Rm}\|D^2 B(s)\|_2^2\right) ds \\ &\leqslant C_0(\|\nabla u_0\|_2 + \|\nabla B_0\|_2^2) + C\int_0^T \|f\|_2^2\, ds \end{aligned} \tag{3.11}$$

*hold for any $t \in [0,T]$. Here $C_0$ is a constant depending on $\int_0^T \|u(s)\|_q^p ds$ in case (1) and $\|u\|^2_{C([0,T];L^3(\mathbb{R}^3))} T$ in case (2) respectively.*

(b) *Let $u_0, B_0 \in L^\beta(\mathbb{R}^3)$ for some $\beta \geqslant 3$. If $\nabla u \in L^\alpha(0,T; L^\beta(\mathbb{R}^3))$ for $1/\alpha + 3/2\beta = 1$ with $1 < \alpha \leqslant 2$, then*

$$u, \quad B \in L^\infty(0,T; L^\beta(\mathbb{R}^3)); \quad |u|^{\frac{\beta-2}{2}}\nabla u, \quad |B|^{\frac{\beta-2}{2}}\nabla B \in L^2(0,T; L^2(\mathbb{R}^3)).$$

*Moreover,*

$$
\begin{aligned}
&\|u(t)\|_\beta^\beta + \|B(t)\|_\beta^\beta + \frac{\beta(\beta-1)}{2Re} \int_0^t \||u|^{\frac{\beta-2}{2}} \nabla u(s)\|_2^2 \, ds \\
&\quad + \frac{\beta(\beta-1)}{2Rm} \int_0^t \||B|^{\frac{\beta-2}{2}} \nabla B(s)\|_2^2 \, ds \\
&\quad \leqslant C \left( \|B_0\|_\beta^\beta, \ \|u_0\|_\beta^\beta, \ e^{\int_0^T \|\nabla u\|_\beta^\alpha d\tau} \right)
\end{aligned} \tag{3.12}
$$

*for any* $t \in [0, T]$.

**Remarks.** 1. If $u_0, B_0$ and $f$ are sufficiently smooth, then the strong solution $(u, B)$ are sufficient smooth, by the regularity results in [12].

2. Constantin and Fefferman showed the smoothness of solutions to the three-dimensional incompressible Navier–Stokes equations under the assumption that

$$|w(x,t)/|w(x,t)| - w(y,t)/|w(y,t)|| \leqslant |x-y|/\rho$$

if both $|w(x,t)| \geqslant \Omega$ and $|w(y,t)| \geqslant \Omega$ for some positive constants $\Omega$ and $\rho$. In view of our estimate below, it is obvious that, in order to obtain their regularity result, it is sufficient to assume that

$$|w(x,t)/|w(x,t)| - w(y,t)/|w(y,t)|| \leqslant |x-y|^{1/2}/\rho$$

holds if both $|w(x,t)| \geqslant \Omega$, $|w(y,t)| \geqslant \Omega$ and $|x-y| \leqslant \delta$ for some positive constants $\Omega$, $\rho$ and $\delta$.

3. A regularity result was obtained in [3,15] for inviscid MHD equation in $\mathbb{R}^3$ under the assumption that

$$\int_0^T [\|w^+(s)\|_\infty + \|w^-(s)\|_\infty] \, ds < \infty.$$

But here, in order to obtain the regularity for viscous MHD equations, we only need to assume that $w^+(x,t)$ satisfy Assumption A.

4. Serrin [13], Giga [9], von Wahl [14], etc. obtained the regularity of solutions to the Navier–Stokes equations in the case of (1)–(3) in Theorem 3. It is worthy to point out that case (3) is a natural extension of (1)–(2). Moreover, the borderline case $\alpha = 2$ is significant. It follows from Theorem 3 that $u$ and $B$ are regular if $\nabla u \in L^2(0, T; L^3(\mathbb{R}^3))$. This cannot be implied by cases (1) and (2), since $W^{1,3}(\mathbb{R}^3)$ cannot be imbedded into $L^\infty(\mathbb{R}^3)$.

The proof of Theorem 3. Since $u_0, B_0 \in V$ and $f \in L^2(0, T; L^2(\mathbb{R}^3))$, then the weak solution $(u, B)$ is strong and unique on $[0, T_1]$ for some $T_1 < T$. By the a priori estimate in Theorem 1, together with the Assumption A, it follows that estimate (3.9) is valid on $[0, T_1]$, which is independent of $T_1$. Thus, the strong solution $(u, B)$ satisfies estimate (3.9) as long as $w^+(x,t)$ satisfies the Assumption A. By the standard continuation argument, the strong solution can be extended to $[0, T]$. $\square$

The proof of Theorem 4 is completely same with that of Theorem 3. We omit it here.

## 4. A priori estimate I

In this section, we will give the a priori estimate and complete the proof of Theorem 1. So we assume that the solution $(u, B)$ is sufficient smooth on $[0, T]$.

For this purpose, we multiply the first equation of (2.1) by $w^+$, the second equation of (2.1) by $Sw^-$, then add the resulting equations to obtain that

$$\begin{aligned}
&\frac{d}{dt}(\|w^+(t)\|_2^2 + S\|w^-(t)\|_2^2) + \frac{2}{Re}\|\nabla w^+(t)\|_2^2 + \frac{2}{Rm}\|\nabla w^-(t)\|_2^2 \\
&\quad = 2\int (w^+(x,t)\cdot\nabla u(x,t)\cdot w^+(x,t) + Sw^-(x,t)\cdot\nabla u(x,t)\cdot w^-(x,t) \\
&\qquad -Sw^-(x,t)\cdot\nabla B(x,t)\cdot w^+(x,t) - Sw^+(x,t)\cdot\nabla B(x,t)\cdot w^-(x,t) \\
&\qquad +2ST(B,u)\cdot w^-(x,t) + F\cdot w^+(x,t))\,dx. \qquad (4.1)
\end{aligned}$$

Let $\phi(r)$ be a smooth cut-off function such that $1 \leqslant \phi(r) \leqslant 1$, $\phi(r) = 1$ for $0 \leqslant r \leqslant 1$, and $\phi(r) = 0$ for $r \geqslant 2$. In the following, we will estimate all the terms on the right-hand side of (4.1). By (2.5), the first term at the right-hand side can be written as

$$\begin{aligned}
I_1 &= 2\int w^+(x,t)\cdot\nabla u(x,t)\cdot w^+(x,t)\,dx \\
&= 2\int w^+(x,t)\cdot S^+(x,t)\cdot w^+(x,t)\,dx \\
&= 2\int \phi\left(\frac{|w^+(x,t)|}{\Omega}\right)(w^+(x,t)\cdot S^+(x,t)\cdot w^+(x,t))\,dx \\
&\quad +2\int \left(1-\phi\left(\frac{|w^+(x,t)|}{\Omega}\right)\right)(w^+(x,t)\cdot S^+(x,t)\cdot w^+(x,t))\,dx.
\end{aligned}$$

By the Calderon–Zygmund estimate and formula (2.7), we have that

$$\begin{cases} \|S^+\|_p \leqslant C(p)\|w^+\|_p, \\ \|S^-\|_p \leqslant C(p)\|w^-\|_p \end{cases} \qquad (4.2)$$

for any $1 < p < \infty$. Then by the Hölder and Sobolev inequalities,

$$
\begin{aligned}
I_{11} &= 2\left|\int \phi\left(\frac{|w^+(x,t)|}{\Omega}\right)(w^+(x,t)\cdot S^+(x,t)\cdot w^+(x,t))\,dx\right| \\
&\leqslant C\Omega^{1/5}\|w^+\|_2\|w^+\|_{24/5}^{4/5}\|S^+\|_3 \\
&\leqslant C\Omega^{1/5}\|w^+\|_2^{8/5}\|\nabla w^+\|_2^{6/5} \\
&\leqslant \delta_1\|\nabla w^+\|_2^2 + C(\delta_1)\Omega^{1/2}\|w^+\|_2^4
\end{aligned}
\tag{4.3}
$$

for any $\delta_1 > 0$, where the Young inequality has been used.

Since the mean of $M$ on the unit sphere is zero when the second variable $w^+$ is fixed and $M$ is viewed as a function of $\hat{y}$ alone, then its integral vanishes in any ball in this case, i.e.,

$$
\text{P.V.}\int_{|y|\leqslant\rho} M(\hat{y}, w^+(x,t))\frac{dy}{|y|^3} = 0. \tag{4.4}
$$

Thus, by formula (2.7) and Assumption A,

$$
\begin{aligned}
\left|\left(1-\phi\left(\frac{|w^+(x,t)|}{\Omega}\right)\right)S^+(x,t)\right| &\leqslant \text{CP.V.}\int_{|y|\geqslant\rho} M(\hat{y}, w^+(x+y,t))\frac{dy}{|y|^3} \\
&\quad +\text{P.V.}\int_{|y|\leqslant\rho}\left(1-\phi\left(\frac{|w^+(x,t)|}{\Omega}\right)\right) \\
&\quad \times M(\hat{y}, w^+(x+y,t)-w^+(x,t))\frac{dy}{|y|^3} \\
&\leqslant C(\rho^{-1/2}+K)\text{P.V.}\int |w^+(x+y,t)|\frac{dy}{|y|^{5/2}}.
\end{aligned}
$$

Therefore, by Calderon–Zygmund estimate, the Hölder and Sobolev inequalities, we have

$$
\begin{aligned}
I_{12} &= 2\left|\int\left(1-\phi\left(\frac{|w^+(x,t)|}{\Omega}\right)\right)(w^+(x,t)\cdot S^+(x,t)\cdot w^+(x,t))\,dx\right| \\
&\leqslant C\|w^+\|_2\|w^+\|_6\left\|\left(1-\phi\left(\frac{|w^+(\cdot,t)|}{\Omega}\right)\right)S^+(\cdot,t)\right\|_3 \\
&\leqslant C(\rho^{-1/2}+K)\|w^+\|_2^2\|\nabla w^+\|_2 \\
&\leqslant \delta_1\|\nabla w^+\|_2^2 + C(\delta_1)(\rho^{-1/2}+K)^2\|w^+\|_2^4.
\end{aligned}
\tag{4.5}
$$

Therefore,

$$I_1 \leqslant 2\delta_1 \|\nabla w^+\|_2^2 + C(\delta_1)[\Omega^{1/2} + (\rho^{-1/2} + K)^2]\|w^+\|_2^4. \tag{4.6}$$

Applying formulas (2.5) and (2.7), one may rewrite the second term at the right-hand side of (4.1) as

$$\begin{aligned} I_2 &= 2S \int w^-(x,t) \cdot \nabla u(x,t) \cdot w^-(x,t)\, dx \\ &= 2S \int w^-(x,t) \cdot S^+(x,t) \cdot w^-(x,t)\, dx \\ &= \frac{3S}{2\pi} \text{P.V.} \int \int (\hat{y}, w^-(x,t)) \,\text{Det}(\hat{y}, w^+(x+y,t), w^-(x,t)) \frac{dy}{|y|^3}\, dx. \end{aligned}$$

Using the cut-off function $\phi(|w^+(x+y,t)|/\Omega)$, one decomposes the last integral into two parts,

$$\begin{aligned} I_{21} = \frac{3S}{2\pi} \text{P.V.} \int \int (\hat{y}, w^-(x,t)) \,\text{Det}\Big(\hat{y}, \phi\left(\frac{|w^+(x+y,t)|}{\Omega}\right) \\ \times w^+(x+y,t), w^-(x,t)\Big) \frac{dy}{|y|^3}\, dx, \end{aligned}$$

$$\begin{aligned} I_{22} = \frac{3S}{2\pi} \text{P.V.} \int \int (\hat{y}, w^-(x,t)) \,\text{Det}\left(\hat{y}, \left(1 - \phi\left(\frac{|w^+(x+y,t)|}{\Omega}\right)\right)\right. \\ \left. \times w^+(x+y,t), w^-(x,t)\right) \frac{dy}{|y|^3}\, dx. \end{aligned}$$

Similar to the treatment of $I_{11}$, $I_{21}$ can be estimated as

$$I_{21} \leqslant C\Omega^{1/5} \|w^-\|_2 \|w^-\|_6 \|w^+\|_{12/5}^{4/5}.$$

By Gagliardo–Nirenberg, Sobolev and the Young inequalities, one has

$$I_{21} \leqslant \delta_1 \|\nabla w^+\|_2^2 + \delta_2 \|\nabla w^-\|_2^2 + C(\delta_1, \delta_2)\Omega^{1/2}(\|w^+\|_2^4 + \|w^-\|_2^4)$$

for any $\delta_2 > 0$. In order to estimate $I_{22}$, we need to use the property of zero mean of $\sigma$ on unit sphere, i.e.,

$$\int_{|y| \leqslant \rho} \sigma(\hat{y}) w^+(x,t) \frac{dy}{|y|^3} = 0 \tag{4.7}$$

By representation (2.7), one has that

$$\left(1-\phi\left(\frac{|w^+(x+y,t)|}{\Omega}\right)\right)w^+(x+y,t)$$
$$=\frac{1}{4\pi}\text{P.V.}\int_{|z|\leqslant\rho}\left(1-\phi\left(\frac{|w^+(x+y,t)|}{\Omega}\right)\right)\sigma(\hat{z})w^+(x+y+z,t)\frac{dz}{|z|^3}$$
$$+\frac{1}{4\pi}\text{P.V.}\int_{|z|\geqslant\rho}\left(1-\phi\left(\frac{|w^+(x+y,t)|}{\Omega}\right)\right)\sigma(\hat{z})w^+(x+y+z,t)\frac{dz}{|z|^3}.$$

Using (4.7) and Calderon–Zygmund estimate, one has that

$$\left\|\left(1-\phi\left(\frac{|w^+(\cdot,t)|}{\Omega}\right)\right)w^+(\cdot,t)\right\|_3\leqslant C(K+\rho^{-1/2})\|w^+\|_2.$$

Thus we obtain the estimate of $I_{22}$

$$\begin{aligned}I_{22}&\leqslant C\|w^-\|_2\|w^-\|_6\|(1-\phi(|w^+(\cdot,t)|\Omega))w^+(\cdot,t)\|_3\\&\leqslant \delta_2\|\nabla w^-\|_2^2+C(\delta_2)(K+\rho^{-1/2})^2(\|w^+\|_2^4+\|w^-\|_2^4).\end{aligned}$$

Consequently,

$$\begin{aligned}&I_2\leqslant\delta_1\|\nabla w^+\|_2^2+2\delta_2\|w^-\|_2^2+C(\delta_1,\delta_2)\\&[\Omega^{1/2}+(K+\rho^{-1/2})^2](\|w^+\|_2^4+\|w^-\|_2^4).\end{aligned}\tag{4.8}$$

It follows from (2.5) and (2.7) that the third term at the right-hand side of (4.1) can be written as

$$\begin{aligned}I_3&=-2S\int[w^-(x,t)\cdot(\nabla B(x,t)\cdot w^+(x,t)+w^+(x,t)\cdot\nabla B(x,t)\cdot w^-(x,t)]\,dx\\&=-2S\int w^-(x,t)\cdot(\nabla B(x,t)+(\nabla B(x,t))^T)\cdot w^+(x,t)\,dx\\&=-4S\int w^-(x,t)\cdot S^-(x,t)\cdot w^+(x,t)\,dx.\end{aligned}$$

By (4.2) and (4.7), one has that

$$I_3\leqslant C\|w^-\|_2\|S^-\|_6\left\|\phi\left(\frac{|w^+(\cdot,t)|}{\Omega}\right)w^+(\cdot,t)\right\|_3$$

$$+\left\|\left(1-\phi\left(\frac{|w^{+}(\cdot,t)|}{\Omega}\right)\right)w^{+}(\cdot,t)\right\|_{3}$$
$$\leqslant C\Omega^{1/5}\|w^{-}\|_{2}\|\nabla w^{-}\|_{2}\|w^{+}\|_{12/5}^{4/5}+C(K+\rho^{-1/2})\|w^{-}\|_{2}\|\nabla w^{-}\|_{2}\|w^{+}\|_{2}$$
$$\leqslant \delta_1\|\nabla w^{+}\|_2^2+\delta_2\|\nabla w^{-}\|_2^2+C(\delta_1,\delta_2)[\Omega^{1/2}$$
$$+(K+\rho^{-1/2})^2](\|w^{+}\|_2^4+\|w^{-}\|_2^4). \tag{4.9}$$

Let

$$I_4=4S\int T(B,u)\cdot w^{-}(x,t)\,dx.$$

Applying representation (2.10) and taking into account of facts (4.4) and (4.7), we can deduce in a similar way as for $I_3$ that

$$I_4 \leqslant \delta_1\|\nabla w^{+}\|_2^2+\delta_2\|\nabla w^{-}\|_2^2+C(\delta_1,\delta_2)[\Omega^{1/2}+(K+\rho^{-1/2})^2]$$
$$\times(\|w^{+}\|_2^4+\|w^{-}\|_2^4). \tag{4.10}$$

Integrating by part, we get, with the help of the Hölder and Young inequalities, that

$$\begin{aligned} I_5 &= 2\left|\int F\cdot w^{+}(x,t)\,dx\right| \\ &\leqslant 2\int |f||\nabla w^{+}(x,t)|\,dx \\ &\leqslant \delta_1\|\nabla w^{+}\|_2^2+C\|f\|_2^2. \end{aligned} \tag{4.11}$$

Substituting above estimates into (3.9) and integrating from 0 to $t$ show that

$$(\|w^{+}(t)\|_2^2+S\|w^{-}(t)\|_2^2)+\int_0^t\left(\frac{1}{Re}\|\nabla w^{+}(s)\|_2^2+\frac{S}{Rm}\|\nabla w^{-}(s)\|_2^2\right)ds$$
$$\leqslant \|w^{+}(0)\|_2^2+S\|w^{-}(0)\|_2^2+\int_0^T\|f(s)\|_2^2\,ds$$
$$+C(\Omega^{1/2}+(K+\rho^{-1/2})^2)\int_0^t(\|w^{+}(s)\|_2^4+\|w^{-}(s)\|_2^4)\,ds \tag{4.12}$$

with $\delta_1=1/(6Re)$ and $\delta_2=S/(5Rm)$.

Combining the energy inequality with the fact

$$\|w^{+}(t)\|_2=\|\nabla u(t)\|_2,\quad \|w^{-}(t)\|_2=\|\nabla B(t)\|_2,$$

the one deduces estimate (3.9) from (4.12) by Gronwall inequality. □

## 5. A priori estimate II

In this section, we will deduce another kind of a priori estimate and prove Theorem 2. Here we also assume that $(u, B)$ is sufficient smooth on $[0, T]$.

First, we differentiate the first equations of (1.1) about $x_i$, then multiply the resulting equations by $\partial_i u$ to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_i u\|_2^2 + \frac{1}{Re}\|\nabla\partial_i u\|_2^2\\ &\quad = -\int(\partial_i u\cdot\nabla)u\cdot\partial_i u\,dx + S\int(\partial_i B\cdot\nabla)B\cdot\partial_i u\,dx\\ &\qquad + S\int(B\cdot\nabla)\partial_i B\cdot\partial_i u\,dx + \int\partial_i f\cdot\partial_i u\,dx.\end{aligned} \tag{5.1}$$

Similarly,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_i B\|_2^2 + \frac{1}{Rm}\|\nabla\partial_i B\|_2^2\\ &\quad = -\int(\partial_i u\cdot\nabla)B\cdot\partial_i B\,dx\\ &\qquad + \int(\partial_i B\cdot\nabla)u\cdot\partial_i B\,dx + \int(B\cdot\nabla)\partial_i u\cdot\partial_i B\,dx.\end{aligned} \tag{5.2}$$

Adding (5.1) and $S\times$ (5.2), we obtain, by integration by part, that

$$\begin{aligned}&\frac{d}{dt}(\|\partial_i u\|_2^2 + S\|\partial_i B\|_2^2) + 2\left(\frac{1}{Re}\|\nabla\partial_i u\|_2^2 + \frac{S}{Rm}\|\nabla\partial_i B\|_2^2\right)\\ &\quad = -2\int(\partial_i u\cdot\nabla)u\cdot\partial_i u\,dx + 2S\int(\partial_i B\cdot\nabla)B\cdot\partial_i u\,dx - 2S\int(\partial_i u\cdot\nabla)B\cdot\partial_i B\,dx\\ &\qquad + 2S\int(\partial_i B\cdot\nabla)u\cdot\partial_i B\,dx + \int\partial_i f\cdot\partial_i u\,dx.\end{aligned} \tag{5.3}$$

If $u \in L^p(0, T; L^q(\mathbb{R}^3))$ with $1/p + 3/2q = 1/2$ and $q > 3$, then $p = 2q/(q-3)$. We get, by the integration by part and the Hölder inequality, that

$$\begin{aligned}I_1 &= \left|-2\int(\partial_i u\cdot\nabla)u\cdot\partial_i u\,dx\right|\\ &\leqslant 2\left|\int(u\cdot\nabla)\partial_i u\cdot\partial_i u\,dx\right| + 2\left|\int(u\cdot\nabla)u\cdot\partial_i\partial_i u\,dx\right|\\ &\leqslant 4\|u\|_q\|\nabla u\|_{\frac{2q}{q-2}}\|D^2 u\|_2.\end{aligned}$$

By the Gagliardo–Nirenberg inequality and Young inequality, $I_1$ can be estimated as

$$I_1 \leqslant \frac{1}{10Re}\|D^2u\|_2^2 + C\|u\|_q^p\|\nabla u\|_2^2. \tag{5.4}$$

Similarly, we can estimate the other terms in (5.3) and obtain that

$$\begin{aligned}
I_2 &= \left|2S\int(\partial_i B\cdot\nabla)B\cdot\partial_i u\,dx\right| \\
&\leqslant \frac{S}{10Rm}\|D^2B\|_2^2 + C\|u\|_q^p\|\nabla B\|_2^2, \\
I_3 &= \left|-2S\int(\partial_i u\cdot\nabla B)\cdot\partial_i B\,dx\right| \\
&\leqslant \frac{S}{10Rm}\|D^2B\|_2^2 + C\|u\|_q^p\|\nabla B\|_2^2, \\
I_4 &= \left|2S\int(\partial_i B\cdot\nabla u)\cdot\partial_i B\,dx\right| \\
&\leqslant \frac{S}{10Rm}\|D^2B\|_2^2 + C\|u\|_q^p\|\nabla B\|_2^2, \\
I_5 &= \left|2\int(\partial_i f\cdot\partial_i u)\,dx\right| \\
&\leqslant \frac{1}{10Re}\|D^2u\|_2^2 + C\|f\|_2^2.
\end{aligned} \tag{5.5}$$

Substituting above estimates into (5.3) and summing $i$ from 0 to 3, one gets that

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_2^2 + S\|\nabla B\|_2^2) + \frac{1}{Re}\|D^2u\|_2^2 + \frac{S}{Rm}\|D^2B\|_2^2 \\
&\leqslant C\|u\|_q^p(\|\nabla u\|_2^2 + S\|\nabla B\|_2^2) + C\|f\|_2^2
\end{aligned} \tag{5.6}$$

which implies that

$$\begin{aligned}
\|\nabla u(t)\|_2^2 + S\|\nabla B(t)\|_2^2 \;&\leqslant\; (\|\nabla u_0\|_2^2 + S\|\nabla B_0\|_2^2)\exp\left\{C\int_0^t \|u(\tau)\|_q^p\,d\tau\right\} \\
&\quad + C\int_0^t \|f(s)\|_2^2 \exp\left\{C\int_s^t \|u(\tau)\|_q^p\,d\tau\right\} ds.
\end{aligned} \tag{5.7}$$

Thus, we obtain estimate (3.6) in the case of (1).

If $u \in C([0,T]; L^3(\mathbb{R}^3))$, then we can decompose $u = u^1 + u^2$ with $\|u^1\|_{C([0,T];L^3(\mathbb{R}^3))} \leqslant \varepsilon$ and $\|u^2\|_{L^\infty((0,T)\times\mathbb{R}^3)} \leqslant C(\varepsilon, \|u\|_{C([0,T];L^3(\mathbb{R}^3))})$ for any $\varepsilon > 0$. Then $I_1$ can be estimated as

$$\begin{aligned} I_1 &\leqslant C \int |u||D^2u||\nabla u|\,dx \\ &\leqslant C\|u^1\|_3 \|\nabla u\|_6 \|D^2 u\|_2 + C\|u^2\|_\infty \|\nabla u\|_2 \|D^2u\|_2 \\ &\leqslant C\varepsilon \|D^2u\|_2^2 + C\|u^2\|_\infty^2 \|\nabla u\|_2^2. \end{aligned} \tag{5.8}$$

Here we have used the Sobolev inequality and Cauchy inequality. The other terms in (5.3) can be treated as before, so we can obtain an inequality similar to (5.6). Then we deduce the result by same procedure as before.

Now we consider the case that $\nabla u \in L^\alpha(0,T; L^\beta(\mathbb{R}^3))$ for $1/\alpha + 3/2\beta = 1$ with $1 < \alpha \leqslant 2$. We multiply the both sides of the second equation in (1.1), integrate over $\mathbb{R}^3$ and get by integration by parts,

$$\frac{1}{\beta}\frac{d}{dt}\|B\|_\beta^\beta + \frac{\beta-1}{Rm}\||B|^{\frac{\beta-2}{2}}\nabla B\|_2^2 \leqslant \int |B|^\beta |\nabla u|\,dx \leqslant \|\nabla u\|_\beta \|B\|_{\beta^2/(\beta-1)}^\beta. \tag{5.9}$$

By the Gagliardo–Nirenberg inequality,

$$\|B\|_{\beta^2/(\beta-1)}^\beta \leqslant C\|B\|_\beta^{\frac{2\beta-3}{2}} \||B|^{\frac{\beta-2}{2}}\nabla B\|_2^{\frac{3}{\beta}}. \tag{5.10}$$

By the Young's inequality, we get

$$\frac{1}{\beta}\frac{d}{dt}\|B\|_\beta^\beta + \frac{\beta-1}{2Rm}\||B|^{\frac{\beta-2}{2}}\nabla B\|_2^2 \leqslant C\|\nabla u\|_\beta^\alpha \|B\|_\beta^\beta. \tag{5.11}$$

Therefore

$$\|B(t)\|_\beta^\beta + \frac{\beta(\beta-1)}{2Rm}\int_0^t \||B|^{\frac{\beta-2}{2}}\nabla B(s)\|_2^2\,ds \leqslant C\|B_0\|_\beta^\beta e^{\int_0^T \|\nabla u\|_\beta^\alpha d\tau} \tag{5.12}$$

for any $t \in [0,T]$.

Noting that the projector $P$ commutes with $\nabla$, we have

$$\begin{aligned} &\frac{1}{\beta}\frac{d}{dt}\|u\|_\beta^\beta + \frac{\beta-1}{Re}\||u|^{\frac{\beta-2}{2}}\nabla u\|_2^2 \\ &\quad \leqslant \int P(u\cdot\nabla)u\cdot|u|^{\beta-2}u\,dx - \int P(B\cdot\nabla)B\cdot|u|^{\beta-2}u\,dx \end{aligned}$$

$$\leqslant \|\nabla u\|_\beta \left( \|u\|^{\beta}_{\beta^2/(\beta-1)} + \|B\|^2_{\beta^2/(\beta-1)} \|u\|^{\beta-2}_{\beta^2/(\beta-2)} \right)$$

$$\leqslant C\|\nabla u\|_\beta \left( \|u\|_\beta^{\frac{2\beta-3}{2}} \||u|^{\frac{\beta-2}{2}} \nabla u\|_2^{\frac{3}{\beta}} \right.$$

$$\left. + \|B\|_\beta^{\frac{2\beta-3}{\beta}} \||B|^{\frac{\beta-2}{2}} \nabla B\|_2^{\frac{6}{\beta^2}} \|u\|_\beta^{\frac{(\beta-2)(2\beta-3)}{2\beta}} \||u|^{\frac{\beta-2}{2}} \nabla u\|_2^{\frac{3(\beta-2)}{\beta^2}} \right). \quad (5.13)$$

Thus

$$\frac{1}{\beta}\frac{d}{dt}\|u\|^\beta_\beta + \frac{\beta-1}{2Re}\||u|^{\frac{\beta-2}{2}}\nabla u\|_2^2 \leqslant \||B|^{\frac{\beta-2}{2}}\nabla B\|_2^2 + C\|\nabla u\|^\alpha_\beta \|u\|^\beta_\beta$$
$$+ C\|\nabla u\|^\alpha_\beta \|B\|^2_\beta \|u\|^{\beta-2}_\beta. \quad (5.14)$$

By Gronwall's inequality and (5.12), we obtain that

$$\|u(t)\|^\beta_\beta + \frac{\beta(\beta-1)}{2Re}\int_0^t \||u|^{\frac{\beta-2}{2}}\nabla u(s)\|_2^2\, ds \leqslant C\left( \|B_0\|^\beta_\beta, \|u_0\|^\beta_\beta, e^{\int_0^T \|\nabla u\|^\alpha_\beta d\tau} \right) \quad (5.15)$$

for any $t \in [0, T]$. This completes the proof of Theorem 2. □

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