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Square nearly nonpositive sign pattern matrices[☆]

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Abstract

A sign pattern matrix A is called square nearly nonpositive if all entries but one of A^2 are nonpositive. We characterize the irreducible sign pattern matrices that are square nearly nonpositive. Further we determine the maximum (resp. minimum) number of negative entries that can occur in A^2 when A is irreducible square nearly nonpositive (SNNP), and then we characterize the sign patterns that achieve this maximum (resp. minimum) number. Finally, we discuss some spectral properties of the sign patterns which are square nonpositive or square nearly nonpositive. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

A *sign pattern matrix* is a matrix whose entries are in the set $\{+, -, 0\}$. A *generalized sign pattern matrix* is a matrix whose entries are in the set $\{+, -, 0, \#\}$, where $\# = (+) + (-)$ is qualitatively ambiguous. Associated with each *n*-by-*n* sign pattern matrix $A = (a_{ij})$ is a class of real matrices, called the sign pattern class of A, defined by

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$$Q(A) = \{B \in M_n(R)\colon \operatorname{sign} b_{ij} = a_{ij} \text{ for all } i \text{ and } j\}.$$

Given a sign pattern matrix A, A^2 is in general a generalized sign pattern. If only the signs of a real matrix A are given, then generally the sign pattern of A^2 will be unpredictable, although not entirely arbitrary. Relatively little is known about the possible sign correlations between the ambiguously signed entries of A^2, or about the locations of the ambiguous entries. In qualitative matrix analysis, the knowledge of the sign pattern of A^2 would be useful (see [2–4,7]).

The motivation of this paper is from [7]. In [7], the authors obtained $N_-(A^2) \leqslant (n-1)^2 + 1$ when $A^2 \leqslant 0$ and $N_-(A^2) \leqslant n^2 - 2$ for arbitrary sign pattern matrix and the maximum number can be achieved, where $N_-(A^2)$ denotes the number of negative entries in A^2 and n is the order of A. A natural question now arises, what properties must A have if all the entries but one of A^2 are nonpositive? A sign pattern A is called *square nearly nonpositive* (SNNP) if all entries but one of A^2 are nonpositive. In Section 2 of this paper, we characterize the irreducible sign patterns that are SNNP, and consider the number of negative entries in A^2 when A is irreducible SNNP. In Section 3, we discuss some spectral properties of the sign patterns which are square nonpositive or SNNP.

Let Q_n denote the set of all n-by-n sign pattern matrices. If $A \in Q_n$ is entrywise nonpositive (resp. nonnegative), we write $A \leqslant 0$ (resp. $A \geqslant 0$). Similarly, $A < 0$ $(A > 0)$ is used to represent an entrywise negative (positive) pattern.

To obtain our results, we need some graph theoretic concepts. For $A = (a_{ij}) \in Q_n$, let SD(A) be the *signed directed graph* on vertex set $N = \{1, 2, \ldots, n\}$, where (i, j) is an arc in SD(A) if and only if $a_{ij} \neq 0$, and the sign of the arc (i, j) is a_{ij}, denoted $i \overset{a_{ij}}{\to} j$. A path of length k in SD(A) (or, say in A), called a k-path, is a sequence of k arcs $(i_1, i_2), (i_2, i_2), \ldots, (i_k, i_{k+1})$, corresponding to a path in A, consisting of the nonzero product of entries $a_{i_1 i_2} \cdots a_{i_k i_{k+1}} \neq 0$. A path in SD$(A)$ is said to be positive (negative) if the number of negative entries in it is even (odd). If $i_{k+1} = i_1$ and $i_1, i_2, \ldots, i_k$ are all distinct, then the above k-path of SD(A) is called a k-cycle.

2. Irreducible SNNP sign patterns

In this section we give a characterization of irreducible SNNP patterns.

Lemma 2.1. *Let $A \in Q_n$. Then A is SNNP if and only if there exists an ordered pair (i_1, j_1) such that there is a positive 2-path from i_1 to j_1 in SD(A) and for all other ordered pair $(i, j) \neq (i_1, j_1)$, all 2-paths from i to j in SD(A) are negative.*

Proof. Let $A = (a_{ij}) \in Q_n$. Then $(A^2)_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj} = -(\text{or } 0)$ if and only if each term $a_{ik} a_{kj} = -(\text{or } 0)$ for all $k = 1, 2, \ldots, n$. Hence the result follows. □

Corollary 2.2. *Let $A \in Q_n$ $(n \geqslant 2)$ be irreducible and SNNP. Then $a_{ii} = 0$ for $i = 1, 2, \ldots, n$.*

Proof. If $a_{ii} \neq 0$ for some i, without loss of generality, we may assume that $i = 1$ and $a_{11} > 0$. Thus $a_{11}a_{11} > 0$ and $(A^2)_{11} \nleqslant 0$ and all other entries of A^2 are nonpositive. Since A is irreducible, there exist vertices, say 2 and 3 such that $1 \to 2$ and $3 \to 1$ are arcs of SD(A). Since A is SNNP and $(A^2)_{11} \nleqslant 0$, $(A^2)_{12} \leqslant 0$, $(A^2)_{31} \leqslant 0$. Hence $1 \xrightarrow{-} 2$, $3 \xrightarrow{-} 1$. Thus $3 \xrightarrow{-} 1 \xrightarrow{-} 2$ is a positive 2-path from 3 to 2, a contradiction. Therefore $a_{ii} = 0$ for $i = 1, 2, \ldots, n$. $\square$

Corollary 2.3. *If $A \in Q_n$ $(n \geqslant 2)$ is irreducible and SNNP, then $(A^2)_{ii} \leqslant 0$ for $i = 1, 2, \ldots, n$.*

Proof. If $(A^2)_{ii} \nleqslant 0$ for some i, there exists a vertex j, $j \neq i$, such that $i \to j \to i$ is a positive 2-path of A. Thus $j \to i \to j$ is also a positive 2-path of A. Hence $(A^2)_{jj} \nleqslant 0$, a contradiction. $\square$

From the following proposition, we know that every irreducible SNNP sign pattern has exactly one positive entry in its square, hence its square does not contain any qualitatively ambiguous entries.

Proposition 2.4. *Let $A \in Q_n$ be an irreducible and SNNP pattern. Then there exists exactly one positive entry in A^2.*

Proof. If $n = 1$, then the result is obvious. Let $n \geqslant 2$. As A is an irreducible and SNNP pattern, there exist vertices i and j such that $i \neq j$ and $(A^2)_{ij} \nleqslant 0$. If $(A^2)_{ij} = \#$, then there exist vertices k and l such that $i \to k \to j$ and $i \to l \to j$ are positive and negative paths, respectively. Hence the signs of $i \to k$ and $i \to l$ or $k \to j$ and $l \to j$ are different. Without loss of generality, we may assume that $i \xrightarrow{+} k$ and $i \xrightarrow{-} l$. Since A is irreducible, there exists a vertex $s \neq i$ such that $s \to i$ is an arc of SD(A). If $s \xrightarrow{-} i$, then $s \xrightarrow{-} i \xrightarrow{-} l$ is a positive 2-path from s to l, this contradicts Lemma 2.1. Similarly, if $s \xrightarrow{+} i$, we also reach a contradiction. Therefore $(A^2)_{ij} = +$. $\square$

If A^2 has two entries that are not nonpositive, then A^2 is not necessarily unambiguous. For example, let

$$A = \begin{pmatrix} 0 & + & + & 0 & 0 \\ 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & + \\ - & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & \# & 0 \\ 0 & 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 & + \\ - & 0 & 0 & 0 & 0 \\ 0 & - & - & 0 & 0 \end{pmatrix},$$

and there are entries # and $+$ in A^2. $\square$

From Corollary 2.2 and Proposition 2.4, it can be seen that there is no irreducible and SNNP pattern of order 2. In what follows we always assume that $n \geqslant 3$.

Theorem 2.5. *Let $A \in Q_n$ $(n \geqslant 3)$ be an irreducible pattern. Then A is SNNP if and only if A is permutation similar to a partitioned matrix with the form*

$$\pm \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & 0 & B_{23} \\ B_{31} & B_{32} & 0 \end{pmatrix}, \tag{1}$$

where the diagonal blocks are square, $B_{11} \leqslant 0,\ B_{12} \leqslant 0,\ B_{23} \geqslant 0,\ B_{32} \leqslant 0,\ B_{31} \leqslant 0,\ B_{11}^2 = 0,\ B_{11}B_{12} = 0,\ B_{31}B_{11} = 0,\ B_{31}B_{12} = 0,$

$$B_{13} = \begin{pmatrix} + & 0 & \cdots & 0 \\ + & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ + & 0 & \cdots & 0 \end{pmatrix},$$

and

$$B_{21} = \begin{pmatrix} + & + & \cdots & + \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Proof. If A is an irreducible SNNP pattern, then by Corollary 2.3 and Proposition 2.4, we may assume that $(A^2)_{rs} = +$, where $r \neq s$, and there exists a vertex k such that $r \to k \to s$ is a positive path, and $(A^2)_{ij} \leqslant 0$ for all $(i, j) \neq (r, s),\ i, j = 1, \ldots, n$.

Set $V_1 = \{j :\ r \to j \to s$ is a positive 2-path in SD(A)$\}$. Then $V_1 \neq \emptyset$. Performing a permutation similarity on A if necessary, we may assume $V_1 = \{1, \ldots, k_1\}$. Clearly $1 \leqslant k_1 < n$. If there exist two vertices $j_1, j_2 \in V_1$ such that $r \overset{+}{\to} j_1 \overset{+}{\to} s$ but $r \overset{-}{\to} j_2 \overset{-}{\to} s$, since A is irreducible, there exists a vertex x such that $x \to r$ is an arc of SD(A). Thus one of the paths $x \to r \to j_1$ and $x \to r \to j_2$ is positive, which is a contradiction. Hence all arcs from r to the vertices in V_1 have the same sign. Similarly, all arcs from the vertices in V_1 to s also have the same sign. Without loss of generality, we may assume that $r \overset{+}{\to} j \overset{+}{\to} s$ for all $j \in V_1$.

Claim. *For every $x \notin V_1$, the lengths of all simple paths from s to x have the same parity.*

Otherwise, there are two simple paths P_1 and P_2 from s to x such that P_1 is odd but P_2 is even, let P_1 be $s \stackrel{-}{\to} x_1 \to \cdots \to x_{2k} \to x$, and P_2 be $s \stackrel{-}{\to} y_1 \to \cdots \to y_{2m-1} \to x$. Since A is irreducible and SNNP, and no 2-path contained in P_1 or P_2 has the form $r \to j \to s$, we have $x_{2k} \stackrel{-}{\to} x$, $y_{2m-1} \stackrel{+}{\to} x$, and there exists a vertex y such that $x \to y$. If $x \stackrel{+}{\to} y$, then $y_{2m-1} = r$, and $y = s$. Hence $x \in V_1$, this is a contradiction. If $x \stackrel{-}{\to} y$, then $x_{2k} \stackrel{-}{\to} x \stackrel{-}{\to} y$. Hence $(A^2)_{x_{2k},y} \nleqslant 0$, a contradiction. Therefore, the above claim holds. $\square$

Set $V_2 = \{i \notin V_1;\ \text{there exists an odd simple path from } s \text{ to } i\}$, $|V_2| = k_2$, and $V_3 = \{i \notin V_1;\ \text{there exists an even simple path from } s \text{ to } i\}$, it is easy to see that $r \in V_2$, $s \in V_3$, and V_1, V_2, V_3 form a partition of V.

By the claim, any two vertices in V_2 (V_3) are not adjacent, the arcs to V_2 are negative and the arcs to V_3 are positive. Specially, the arcs from V_3 to V_2 are negative and the arcs from V_2 to V_3 are positive. Also, since any 2-path whose middle vertex is not in V_1 is negative, it follows that any arc from V_2 is positive and any arc from V_3 is negative. Performing a permutation similarity on A if necessary, we may assume $V_2 = \{k_1+1, \ldots, k_1+k_2\}$, $V_3 = \{k_1+k_2+1, \ldots, n\}$ and further, $r = k_1+1$, $s = k_1+k_2+1$. Thus A can be partitioned into the following form:

$$\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & 0 & B_{23} \\ B_{31} & B_{32} & 0 \end{pmatrix},$$

where $B_{12} \leqslant 0$, $B_{13} \geqslant 0$, $B_{21} \geqslant 0$, $B_{23} \geqslant 0$, $B_{31} \leqslant 0$, $B_{32} \leqslant 0$.

Suppose there is a vertex j in V_1 and a vertex x in V_3 such that $x \neq s$ and $j \to x$ is an arc of SD(A). From the above, $j \to x$ is positive, and hence, $r \to j \to x$ is a positive 2-path, which is a contradiction. Thus

$$B_{13} = \begin{pmatrix} + & 0 & \cdots & 0 \\ + & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ + & 0 & \cdots & 0 \end{pmatrix}.$$

Similarly,

$$B_{21} = \begin{pmatrix} + & \cdots & + \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Since all 2-paths not from r to s in $\mathrm{SD}(A)$ are negative, we obtain $B_{11} \leqslant 0,\ B_{11}^2 = 0,\ B_{11}B_{12} = 0,\ B_{31}B_{11} = 0,\ B_{31}B_{12} = 0.$

The converse is obvious. □

If A is an irreducible SNNP pattern, then A is permutation similar to form (1). Since there is no 2-path in $\mathrm{SD}(B_{11})$, we can partition the vertex set V_1 of $\mathrm{SD}(B_{11})$ into three parts as follows:

$$X_1 = \{x \in V_1,\ x \text{ is neither an initial vertex nor a terminal vertex of any arcs in } \mathrm{SD}(B_{11})\};$$

$$X_2 = \{x \in V_1,\ x \text{ is an initial vertex of some arc in } \mathrm{SD}(B_{11})\};$$

$$X_3 = \{x \in V_1,\ x \text{ is a terminal vertex of some arc in } \mathrm{SD}(B_{11})\}.$$

By the proof of Theorem 2.5, we can prove:

Theorem 2.6. *Let $A \in Q_n$ $(n \geqslant 3)$ be an irreducible pattern. Then A is SNNP if and only if A is permutation similar to a partitioned matrix with the form*

$$\pm \begin{pmatrix} 0 & 0 & 0 & C_{14} & C_{15} \\ 0 & 0 & C_{23} & C_{24} & C_{25} \\ 0 & 0 & 0 & 0 & C_{35} \\ C_{41} & C_{42} & C_{43} & 0 & C_{45} \\ C_{51} & 0 & C_{53} & C_{54} & 0 \end{pmatrix},$$

where $C_{14} \leqslant 0,\ C_{23} \leqslant 0,\ C_{24} \leqslant 0,\ C_{45} \geqslant 0,\ C_{51} \leqslant 0,\ C_{53} \leqslant 0,\ C_{54} \leqslant 0,$ and $C_{51}C_{14} = 0$,

$$\begin{pmatrix} C_{15} \\ C_{25} \\ C_{35} \end{pmatrix} = \begin{pmatrix} + & 0 & \cdots & 0 \\ + & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ + & 0 & \cdots & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} C_{41} & C_{42} & C_{43} \end{pmatrix} = \begin{pmatrix} + & \cdots & + \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

We now turn our attention to finding the maximum and minimum number of negative entries in A^2 when A is square nearly nonpositive. Let $N_-(A^2)$ denote the number of negative entries in A^2. Note that if

$$A = \begin{array}{c} \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & 0 & B_{23} \\ B_{31} & B_{32} & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ k_1 \\ k_2 \\ k_3 \end{array}, \tag{2}$$

where $B_{23} \geqslant 0$, $B_{32} \leqslant 0$, $B_{11} \leqslant 0$, $B_{12} \leqslant 0$, $B_{31} \leqslant 0$, $B_{11}^2 = 0$, $B_{11}B_{12} = 0$, $B_{31}B_{11} = 0$, $B_{31}B_{12} = 0$,

$$B_{13} = \begin{pmatrix} + & 0 & \cdots & 0 \\ + & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ + & 0 & \cdots & 0 \end{pmatrix},$$

and

$$B_{21} = \begin{pmatrix} + & \cdots & + \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix},$$

then

$$A^2 = \begin{pmatrix} B_{12}B_{21} + B_{13}B_{31} & B_{13}B_{32} & B_{11}B_{13} + B_{12}B_{23} \\ B_{21}B_{11} + B_{23}B_{31} & B_{21}B_{12} + B_{23}B_{32} & B_{21}B_{13} \\ B_{32}B_{21} & 0 & B_{32}B_{23} + B_{31}B_{13} \end{pmatrix},$$

where

$$B_{21}B_{13} = \begin{pmatrix} + & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If the row i of $B_{11}B_{13} + B_{12}B_{23}$ contains a nonzero entry, that is, there exists a negative 2-path $i \to x \to y$ from vertex $i \in V_1$ to a vertex $y \in V_3$, then $i \xrightarrow{-} x$. Thus there is no negative arc to vertex i and hence, the column i of $B_{21}B_{11} + B_{23}B_{31}$ must be 0. Similarly if column j of $B_{21}B_{11} + B_{23}B_{31}$ contains a nonzero entry, then the row j of $B_{11}B_{13} + B_{12}B_{23}$ must be 0. Therefore, there are at least $\min(k_1k_3, k_1k_2)$ zero entries in $B_{11}B_{13} + B_{12}B_{23}$ and $B_{21}B_{11} + B_{23}B_{31}$. Thus we have proven:

Lemma 2.7. *Let A be of the form* (2). *Then*

$$N_-(A^2) \leqslant k_1^2 + k_2^2 + k_3^2 + 2(k_2 + k_3)k_1 - \min(k_1k_2, k_1k_3).$$

Theorem 2.8. *Let $A \in Q_n (n \geqslant 3)$ be an irreducible SNNP pattern. Then $N_-(A^2) \leqslant n^2 - n$, with equality if and only if A* (*or* A^{T}) *is permutation similar to*

$$\pm\begin{pmatrix} 0 & \cdots & 0 & 0 & + \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & + \\ + & \cdots & + & 0 & + \\ - & \cdots & - & - & 0 \end{pmatrix}.$$

Proof. It is obvious for $n=3$. Let $n \geqslant 4$. From Theorem 2.5 and Lemma 2.7, it follows that

$$N_-(A^2) \leqslant k_1^2 + k_2^2 + k_3^2 + 2(k_2+k_3)k_1 - \min(k_1k_2, k_1k_3),$$

where $k_1+k_2+k_3=n$. Let $k_4=\min\{k_2,k_3\}$, and $f(k_1,k_4)=k_1^2+k_4^2+(n-k_1-k_4)^2+2(n-k_1)k_1-k_1k_4$. Then $f(k_1,k_4)$ achieves its maximum value at the boundary points on the domain $1 \leqslant k_1 \leqslant n-2,\ 1 \leqslant k_4 \leqslant n-2$. If $k_1=1$, then $f(k_1,k_4) \leqslant (n-1)^2+2 < n^2-n$ for $n \geqslant 4$. If $k_1=n-2$, then $f(k_1,k_4) \leqslant (n-2)^2+1+1+2(n-(n-2))(n-2)-(n-2)=n^2-n$.

Now assume that A (or A^{T}) is of the form (3). Then

$$A^2=\begin{pmatrix} - & \cdots & - & - & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ - & \cdots & - & - & 0 \\ - & \cdots & - & - & + \\ - & \cdots & - & 0 & - \end{pmatrix},$$

and $N_-(A^2)=n^2-n$.

Conversely, assume that A is an irreducible SNNP pattern and $N_-(A^2)=n^2-n$. By Theorem 2.5 we may assume that A is of the form (2). From the above proof, it follows $k_1=n-2$. From the proof of Lemma 2.7, there are exactly $1+\min\{k_1k_2,k_1k_3\}=1+k_1=n-1$ zero entries in A^2. Thus

$$A^2=\begin{pmatrix} - & \cdots & - & - & \\ \vdots & \ddots & \vdots & \vdots & u^{\mathrm{T}} \\ - & \cdots & - & - & \\ & v & & - & + \\ - & \cdots & - & 0 & - \end{pmatrix},$$

where $u^{\mathrm{T}}=B_{11}B_{13}+B_{12}B_{23},\ v=B_{21}B_{11}+B_{23}B_{31}$. If $B_{11}\neq 0$, then there exist $j_1, j_2 \in V_1$ such that $j_1 \overset{-}{\rightarrow} j_2$, then there is no negative 2-path from j_2 to j_1. This is a contradiction with $(A^2)_{j_1j_2}<0$. Hence $B_{11}=0$. Therefore $u^{\mathrm{T}}=B_{12}B_{23}=B_{12},\ v=B_{23}B_{31}=B_{31}$. If $B_{12}\neq 0$ and $B_{31}\neq 0$, and let $B_{12}^{\mathrm{T}}=(a_1,a_2,\ldots,a_{n-2})$, $B_{31}=(b_1,b_2,\ldots,b_{n-2})$, then there exist i,j such that $1\leqslant i,\ \ j\leqslant n-2,\ \ i\neq j$ and $a_i=-,\ b_j=-$, thus $b_i=0,\ a_j=0$ by $u^{\mathrm{T}}=B_{12}$, and $v=B_{31}$. Hence $(B_{12}B_{31}+B_{13}B_{31})_{ji}=a_j+b_i=0$, this is a contradiction with $B_{12}B_{21}+B_{13}B_{31}<0$. Therefore at least one of B_{12} and B_{31} is a zero block. Since there are exactly $n-2$ zero entries in $u^{\mathrm{T}}=B_{12}$ and $v=B_{21}$, we have $B_{12}=0$, $B_{31}=(-,-,\cdots,-)$, or

$B_{31} = 0$, $B_{12}^{\mathrm{T}} = (-, -, \cdots, -)$. Therefore A (or A^{T}) is permutation similar to form (3). $\square$

We conclude this section by considering the minimum number of $N_-(A^2)$ when A is an irreducible and SNNP pattern.

Theorem 2.9. *Let $A \in Q_n$ $(n \geqslant 3)$ be an irreducible SNNP pattern. Then $N_-(A^2) \geqslant n-1$, with equality if and only if SD(A) is an odd cycle in which certain two incident arcs have the same sign and the other arcs have alternating signs.*

Proof. Since A is irreducible and SNNP, SD(A) is strongly connected. Thus there exists at least one 2-path from each vertex of SD(A), hence $N_-(A^2) \geqslant n-1$. The equality holds if and only if there exists exactly one 2-path from each vertex of SD(A). Therefore the outdegree of every vertex of SD(A) is one, so SD(A) is a cycle. By Theorem 2.5, $k_1 = 1$, $n = 2l+1$ for some l and the cycle contains only two incident arcs that have the same sign and all other arcs have alternating signs. $\square$

3. Some spectral properties

In [7], the authors have given some interesting properties for the eigenvalues of sign patterns, specially, they obtained that the square maximally nonpositive patterns that require exactly one pure imaginary eigenvalue pair, and they asked if the square maximally nonpositive patterns are the only patterns that require exactly one pure imaginary eigenvalue pair. The following example gives a negative answer.

Example 3.1. Let

$$A = \begin{pmatrix} 0 & + & 0 & 0 \\ + & 0 & - & 0 \\ 0 & + & 0 & - \\ 0 & 0 & + & 0 \end{pmatrix}.$$

Then A is irreducible, and for each $B \in Q(A)$, B is similar to

$$\tilde{B} = \begin{pmatrix} 0 & a & 0 & 0 \\ 1 & 0 & -b & 0 \\ 0 & 1 & 0 & -c \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where $a, b, c > 0$. Then the characteristic polynomial of $\tilde{B}$ is given by $P_{\tilde{B}}(x) = x^4 + (b+c-a)x^2 - ac$. By Descartes's rule of signs (see [8]), $t^2 + (b+c-a)t - ac = 0$ has one positive root and one negative root. Thus A requires exactly one pure imaginary eigenvalue pair, but A is not a square maximally nonpositive pattern. But we have:

Proposition 3.2. *Let $A \in Q_n (n \geqslant 2)$ be irreducible. If A requires $n-2$ eigenvalues equal to* 0, *and a pure imaginary eigenvalue pair, then up to permutation similarity and signature similarity (that is, similarity via a diagonal matrix, each of its diagonal entry is $+$ or $-$), A is equivalent to the square maximally nonpositive patterns*

$$\pm \begin{pmatrix} 0 & + & \cdots & + \\ - & 0 & \cdots & 0 \\ \vdots & & & \\ - & 0 & \cdots & 0 \end{pmatrix}.$$

Proof. If sign pattern A satisfies the condition of this proposition, then A must be a sign skew-symmetric matrix whose digraph is a doubly directed tree (see [1,6]). Since A requires $n-2$ eigenvalues equal to 0, term rank of A is 2 (see [5]). Thus $D(A)$ is a doubly directed star. Hence, up to permutation similarity and signature similarity, A is equivalent to the square maximally nonpositive patterns. □

For square nearly nonpositive patterns, we have:

Proposition 3.3. *Let $A \in Q_n$ $(n \geqslant 3)$ be an irreducible and SNNP pattern, and $N_-(A^2) = n^2 - n$. Then A requires $n-3$ eigenvalues equal to* 0, *a nonzero real eigenvalue, and a pair of nonreal eigenvalues.*

Proof. From Theorem 2.8, $N_-(A^2) = n^2 - n$ if and only if A or A^{T} is permutation similar to

$$\pm \begin{pmatrix} 0 & \cdots & 0 & 0 & + \\ \vdots & & \vdots & \vdots & \\ 0 & \cdots & 0 & 0 & + \\ + & \cdots & + & 0 & + \\ - & \cdots & - & - & 0 \end{pmatrix}.$$

Note that the characteristic polynomial of a matrix $B \in Q(A)$ has the form $P_B(x) = x^{n-3}(x^3 + ax + b)$, for some $a > 0$ and $b \neq 0$. By Descartes's rule of signs, $x^3 + ax + b = 0$ (where $a > 0,\ b > 0$ or $b < 0$) has one nonzero real root and a pair of nonreal roots. □

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