MATHEMATICS

ON ADO'S THEOREM

BY

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Dedicated to Professor Freudenthal in gratitude and esteem on the occasion of his 60th birthday

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Introduction

In this introduction „Lie algebra (group)" will mean „real finite dimensional Lie algebra (group)".

For the theorem that a Lie algebra L belongs to a global Lie group two different methods of proof are known in the literature. The first one consists in proving that L is isomorphic with a linear Lie algebra (Ado's theorem) and therefore with the Lie algebra of a linear Lie group. The second one proceeds by direct construction and succeeds owing to the fact that the second betti number of a simply connected Lie group is zero, or more precisely that the second betti number of the universal covering of the adjoint group of L is zero. This topological circumstance is essential to the proof as has been pointed out several times in the literature ([1], [3], [4]). Hence one would expect this topological fact to turn up in some disguise in the first method of proof. There is indeed some connection with topology albeit a little remote. The usual proofs of Ado's theorem rely on the Levi theorem. Into the proof of the latter two properties of semi-simple Lie algebras seem to go in inevitably, notably first the complete reducibility of their linear representations and secondly the vanishing of their second betti number or, in other words, the triviality of their central extensions. Both properties can be seen as extensions via the unitary trick of the corresponding properties of compact semi-simple Lie algebras. In this case the vanishing of the second betti number of the Lie algebra or the corresponding global group is the same thing.

This note is an attempt to look at Ado's theorem from a global point of view in order to bring out its connection with topology more distinctly. It is shown how Ado's theorem ties up with the vanishing of the second betti number of a simply connected Lie group without the intervention of Levi's theorem. The ideas involved can be traced back to e.g. E. CARTAN [1, 2] Y. MATSUSHIMA [5], M. GOTÔ [6], and probably many others.

1. Heuristics.

We shall try to prove that any simply connected Lie group admits a locally faithful linear representation. It will be sufficient to construct a representation that is locally faithful on the centre.

Let us first look at the trivial case of a simply connected commutative group G with Lie algebra L (although G may be identified with L, we shall not use this fact). The identity map $L \to L$ is an L-valued linear form on L, and thus it determines a right invariant L-valued pfaffian form ω on G (of course ω is also left invariant). Because of the commutativity of G, ω is a closed 1-form. Since G is simply connected ω is exact, i.e. $\omega = df$, where f is an L-valued function on G. The invariance of ω means that f is invariant modulo the constant L-valued functions. Hence the finite dimensional space V of L-valued functions, spanned by f and the constant functions, is invariant under right translations. Since $df = \omega$ is the identity on L, G is locally faithfully represented in V.

This suggests the following construction for an arbitrary simply connected G with Lie algebra L. Assume that J is the centre of L. Extend the identity map $J \to J$ to a projection $L \to J$ (for the vector space structure). The projection $L \to J$ gives rise to a right invariant J-valued pfaffian form ω on G. If ω were a closed form, one could repeat the above argument and obtain a right invariant finite dimensional vectorspace of functions in which the linear representation of G induced by the right translations is locally faithful on $\exp J$. In general however $d\omega$ is non zero. But, denoting the natural map $G \to G/\exp J$ by p, we have $d\omega = p^*\tau_2$, where τ_2 is a right invariant closed 2-form on $G/\exp J$. Since $G/\exp J$ is simply connected, its second betti number vanishes. Hence τ_2 is exact. Suppose that we could find on $G/\exp J$ a J-valued 1-form τ_1 of finite span (i.e. its right translates span a finite dimensional vectorspace) such that $\tau_2 = d\tau_1$. Then $\omega - p^*\tau_1$ is a closed 1-form of finite span on G. Because of the simple connectedness of G, $\omega - p^*\tau_1 = df$, where f is a J-valued function on G of finite span modulo the constant functions. Hence f and the constant functions together with their right translates span a finite dimensional vector space V. Since $df = \omega - p^*\tau_1$ induces the identity map on J considered as a Lie algebra of invariant tangent fields on G, the representation of G in V is locally faithful on the centre of G.

So the question arises: Under what conditions does there exist a form τ_1 of finite span on $G/\exp J$ such that $\tau_2 = d\tau_1$? In case $G/\exp J$ is an algebraic linear group, τ_2, as an invariant form, is rational. Therefore in this case the most natural candidate for a τ_1 is likewise a rational form, which is then automatically of finite span. This would suggest to look for a rational version of the de Rham cohomology[1]); such a version has

[1]) For the case of affine non-singular curves one establishes the existence of such a version by employing the classical theory of differentials of the second kind (see e.g. Hermann Weyl: Die Idee der Riemannschen Fläche) along with some general facts.

been obtained recently by GROTHENDIECK [7]. However, since we are concerned with groups, the existence of such a τ_1 follows already from the Hopf theorem on "Hopf algebras" [8].

In the next sections we consider first an algebraic case of Ado's theorem and then reduce the general case to this one.

2. THE ALGEBRAIC CASE.

Before we proceed to discuss a special case of Ado's theorem in section 2.3 we make a few preliminary remarks in sections 2.1 and 2.2. In 2.1 we are concerned with the notion of differential form of finite span on a local Lie group; in 2.2 we recall the Hopf theorem in a version that we need.

2.1. Let W be a local Lie group, ω a differential form on W, U and V open local subgroups of W with $UV \subset W$. For any $v \in V$ define the map $\varrho_v: U \to W$ by $\varrho_v(u) = uv$. Let further $\Omega_{U,V}$ denote the linear space of differential forms on U spanned by the set $\{\varrho_v^* \omega; v \in V\}$. Note that for $V_1 \subset V$ we have $\Omega_{U,V_1} \subset \Omega_{U,V}$.

ω is said to be *of finite span* if there exists a pair U, V such that $\Omega_{U,V}$ is finite dimensional. We suppose from now on that ω has this property. Then, because of $\Omega_{U,V_1} \subset \Omega_{U,V}$ for $V_1 \subset V$, we may suppose V to be such that $\Omega_{U,V} = \Omega_{U,V_1}$ for any open local subgroup $V_1 \subset V$. Assume that V has this property, and that X is an open local subgroup with $X^2 \subset V$. Then any $\omega' \in \Omega_{U,V} = \Omega_{U,X}$ is a finite linear combination of forms $\varrho_{x'}^* \omega$, $x' \in X$. If $\omega' = \sum \lambda_{x'} \varrho_{x'}^* \omega \in \Omega_{U,V} = \Omega_{U,X}$ we put $\varrho_x^* \omega' = \sum \lambda_{x'} \varrho_{xx'}^* \omega$; since $xx' \in V$, $\varrho_{xx'}^* \omega \in \Omega_{U,V}$ and $\varrho_x^* \omega' \in \Omega_{U,V}$, $x \to \varrho_x^*$ is a linear representation of X in the space $\Omega_{U,V}$. If ω is an analytic form, the representation ϱ^* is analytic, and we obtain a representation of the Lie algebra of W in $\Omega_{U,V}$.

Further we note the following properties:

If ω is right invariant on W, then ω is of finite span. (A form ω is said to be right invariant on W if around every point of W there exists a neighbourhood U such that $\varrho_v^* \omega = \omega | U$, v sufficiently near the neutral element, ϱ_v being the map $U \to W$ defined by $u \to uv$.)

The exterior product of two forms of finite span is of finite span.

If f is a 0-form on W, i.e. a function, such that df is of finite span, then f is of finite span modulo the constant functions, and hence of finite span itself.

If W' is a local Lie group and $W' \xrightarrow{\varphi} W$ an analytic homomorphism, then ω of finite span on W implies that $\varphi^* \omega$ is of finite span on W'.

Finally in the above definitions and statements ω may be as well a differential form with values in a finite dimensional vector space.

2.2. Let $\mathfrak{G}$ be an irreducible algebraic linear group defined over the complex numbers. It is no loss in generality to consider $\mathfrak{G}$ as an irreducible algebraic subgroup of $SL(n, C)$, and therefore as the null set of a finite set of polynomials in $E(n, C)$ – the space of endomorphisms of complex n-space. Let z_i, $i = 1, \ldots, n^2$, be coordinates in $E(n, C)$.

A *rational differential form* in $E(n, C)$ is to be a differential form of type $\sum f(z_i)\, dz_{j_1} \wedge \ldots \wedge dz_{j_k}$, with polynomial coefficients f. The rational differential forms in $E(n, C)$ constitute a complex Ω_E. A *rational differential form on* $\mathfrak{G}$ is to be the restriction of a rational differential form in E. Among the rational forms on $\mathfrak{G}$ we find the right invariant differential forms. The rational forms on $\mathfrak{G}$ constitute a complex $\Omega_{\mathfrak{G}}$ [1]). From basic results in sheaf theory, or, if one prefers, in local algebra (see e.g. [9]), it follows that $\Omega_{\mathfrak{G}} = \Omega_E \bmod \Omega(\mathfrak{G})$, where $\Omega(\mathfrak{G})$ denotes the ideal of Ω_E which is differentially generated by the ideal of equations of $\mathfrak{G}$. Using this result it follows that $\Omega_{\mathfrak{G}\times\mathfrak{G}} \simeq \Omega_{\mathfrak{G}} \otimes \Omega_{\mathfrak{G}}$ and $H(\Omega_{\mathfrak{G}\times\mathfrak{G}}) = H(\Omega_{\mathfrak{G}}) \otimes H(\Omega_{\mathfrak{G}})$. Further the multiplication map $\mu: \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ defined by $(g_1, g_2) \to g_1 g_2$ is described by polynomial equations. This provides a homomorphism $\mu^*: \Omega_{\mathfrak{G}} \to \Omega_{\mathfrak{G}\times\mathfrak{G}} = \Omega_{\mathfrak{G}} \otimes \Omega_{\mathfrak{G}}$ which turns $H(\Omega_{\mathfrak{G}})$ into a Hopf algebra. Therefore $H(\Omega_{\mathfrak{G}})$ is a free Grassmann algebra over the complex numbers on generators of odd dimension.

Hence a closed rational form τ of dimension 2, and in particular a closed right invariant form τ of dimension 2, can be represented as $\tau = \sum \omega_{1i} \wedge \tau_{1i} + d\sigma$ where ω_{1i}, τ_{1i}, σ are rational forms of dimension 1 and ω_{1i}, τ_{1i} closed.

Finally we note that a right translation on $\mathfrak{G}$ induces a linear map in the coordinate functions z_i. Hence a rational form on $\mathfrak{G}$ is of finite span. Similar definitions and statements apply to the notion of rational form with values in a finite dimensional complex vectorspace.

2.3. Let $\mathfrak{G}$ be an irreducible algebraic complex linear group with Lie algebra G, and let

$$(*) \qquad 0 \to J \to H \to G \to 0$$

be a central extension of G in the category of complex (finite dimensional!) Lie algebras. We want to show that H admits a faithful linear representation.

Let W be a simply connected local Lie group with Lie algebra H. The map $H \xrightarrow{\varphi} G$ induces an analytic homomorphism $\varphi: W \to \mathfrak{G}$. We assume W to be so small that $\varphi(W)$ is contained in a simply connected open neighbourhood U of the neutral element of $\mathfrak{G}$. We proceed to construct on W a suitable function of finite span.

Let $\omega: H \to J$ be a projection onto J for the vector space structure. Then $d\omega = \varphi^* \tau_2$ ($d\omega$ is defined by $d\omega(h_1, h_2) = \omega([h_1, h_2])$, $h_{1,2} \in H$), where τ_2 is a J-valued 2-cocycle on G describing the extension (*).

The linear map $\omega: H \to J$ and the alternating bilinear map $\tau_2: G \times G \to J$

[1]) According to the theorem of Grothendieck, on a non-singular affine variety the cohomology based on rational forms coincides with the de Rham cohomology.

induce right invariant forms on W and $\mathfrak{G}$ respectively; these forms will also be denoted by ω and τ_2. For these forms and the map $\varphi\colon W \to \mathfrak{G}$ we still have the relation $d\omega = \varphi^*\tau_2$. τ_2 being a closed right invariant form on $\mathfrak{G}$, it may, by the Hopf theorem, be written as

$$\tau_2 = \sum \omega_{1i} \wedge \tau_{1i} + d\sigma,$$

where ω_{1i}, τ_{1i}, σ are rational forms, ω_{1i} complex valued, τ_{1i} and σ J-valued, ω_{1i} and τ_{1i} closed. In the simply connected local subgroup U there exist functions g_i with $dg_i = \omega_{1i}$ and hence in U we have $\tau_2 = d\tau_1$, where $\tau_1 = \sum g_i\tau_{1i} + \sigma$. Since the dg_i are of finite span, the g_i are of finite span, and therefore τ_1 is on U a form of finite span. Since $\varphi(W) \subset U$ we have $\varphi^*\tau_2 = d\varphi^*\tau_1$ or, since $d\omega = \varphi^*\tau_2$, $d(\omega - \varphi^*\tau_1) = 0$; $\omega - \varphi^*\tau_1$ is of finite span since both ω and τ_1 are of finite span. W being simply connected $\omega - \varphi^*\tau_1 = df$, where f is an analytic J-valued function on W, and of finite span since df is of finite span. Finally $\omega - \varphi^*\tau_1$ takes on the elements of J, considered as invariant tangent fields on W, the same values as ω. Since $\omega|J$ is the identity map $J \to J$, the finite dimensional linear representation of H associated with the function f is faithful on the subalgebra J.

Combining this representation with the linear representation associated with the homomorphism $H \to G$ (G may be considered as a linear Lie algebra) we obtain

Proposition 1. *Let* $0 \to J \to H \to G \to 0$ *be a central extension in the category of complex Lie algebras, and let* G *be the Lie algebra of an irreducible algebraic complex linear group. Then* H *is isomorphic with a linear Lie algebra.*

3. The general case.

In this section we continue to consider complex Lie algebras only. Later we shall examine the validity of the results for Lie algebras over an arbitrary field of characteristic zero.

With any Lie algebra L there is associated a central extension $0 \to J \subset L \xrightarrow{ad} ad\, L \to 0$, where J is the centre of L. We shall show in section 3.2 that there is an injection of this extension into a central extension $0 \to J \to H \to G \to 0$, where G is the Lie algebra of an algebraic linear group. Since, by proposition 1, H is faithfully representable, we obtain automatically that L is isomorphic with a linear Lie algebra. In section 3.1 we make some preliminary remarks.

3.1. Let $\mathfrak{G}$ be a connected linear Lie group acting in a complex vector space V. The Zariski closure $Z(\mathfrak{G})$ of $\mathfrak{G}$ in $GL(V)$ is an irreducible algebraic linear group. If G is the Lie algebra of $\mathfrak{G}$, we denote by $Z(G)$ the Lie algebra of $Z(\mathfrak{G})$. For any group $\mathfrak{G}'$ with $\mathfrak{G} \subset \mathfrak{G}' \subset Z(\mathfrak{G})$ we have $Z(\mathfrak{G}') = Z(\mathfrak{G})$ and consequently for any Lie algebra G' with $G \subset G' \subset Z(G)$ we have $Z(G') = Z(G)$.

Let L be a $\mathfrak{G}$-invariant (pointwise $\mathfrak{G}$-invariant) subspace of V, then

L is $Z(\mathfrak{G})$-invariant (pointwise $Z(\mathfrak{G})$-invariant); a similar statement holds if $\mathfrak{G}$ is replaced by G.

Let us denote by $\mathfrak{A}_L(A_L)$ the group (Lie algebra) induced in L for any group $\mathfrak{A}$ (Lie algebra A) leaving L invariant. The notion of Zariski closure has the following property

$$(Z(\mathfrak{G}))_L = Z(\mathfrak{G}_L), \qquad (Z(G))_L = Z(G_L).$$

Actually, with some extra care, we could do also with the weaker property $(Z(\mathfrak{G}))_L \subset Z(\mathfrak{G}_L)$ and $(Z(G))_L \subset Z(G_L)$.

If V is a Lie algebra and $\mathfrak{G}$ a group of automorphisms, $Z(\mathfrak{G})$ is likewise an automorphism group.

3.2. *Lemma* 1. *Let L be an ideal of the Lie algebra M and centre $L \subset$ centre M. If centralizer $L \subset L$, then centralizer $L =$ centre $L =$ centre M.*

Proof: centre $M \subset$ centralizer $L = L \cap$ centralizer L (because centralizer $L \subset L$) $=$ centre $L \subset$ centre M (assumption).
This yields the assertion.

If L is an ideal in M we denote by ad_L the representation defined by $ad_L x(y) = [x, y]$, $x \in M$, $y \in L$.

Lemma 2. *Let L be an ideal in M and centre $L \subset$ centre M. The following two conditions are equivalent*

(i) *centralizer* $L \subset L$

(ii) $M/L \xrightarrow{ad_L} (ad_L M)/(ad_L L)$ *is an isomorphism.*

Proof: (i) $\rightarrow$ (ii). $ad_L m \in ad_L L$, $m \in M$, implies the existence of $l \in L$ such that $ad_L(m-l) = 0$. Hence $m - l \in$ centralizer $L \subset L$, therefore $m \in L$.

(ii) $\rightarrow$ (i) $m \in$ centralizer L implies $ad_L m = 0$. Therefore $ad_L m \in ad_L L$ and hence, by (ii), $m \in L$.

A Lie algebra M is said to be a *close hull* of a Lie algebra L if the conditions (a), (b), (c) are fulfilled; (a) L is an ideal in M; (b) $ad_L L \subset \subset ad_L M \subset Z(ad_L L)$; ($c$) $L \supset$ centralizer L.

If M is a close hull of L, $ad_L M$ annihilates the center of L, since $Z(ad_L L)$ annihilates the centre of L, and therefore centre $L \subset$ centre M. Therefore by condition (c) and lemma 1, we find that centre $M =$ centre L if M is a close hull of L.

Further by lemma 2 we see that the condition (c) may be replaced by

(c') $\qquad M/L \xrightarrow{ad_L} (ad_L M)/(ad_L L)$ is an isomorphism.

Because of (c') we find

$$\begin{aligned}\dim M &= \dim (ad_L M/ad_L L) + \dim L \\ &\leqslant \dim Z(ad_L L) - \dim (ad_L L) + \dim L \\ &= \dim Z(ad_L L) + \dim \text{ centre } L.\end{aligned}$$

Hence there is among the close hulls of L at least one of maximal dimension.

Lemma 3. *Let M be a close hull of L and δ a derivation of M with $\delta(L) \subset L$, $\delta_L \in Z(ad_L L)$, $\delta_L \notin ad_L M$. Let M' be the semi-direct sum of M and a 1-dimensional Lie algebra with basis element d such that $ad_M d = \delta$. Then M' is a close hull of L.*

Proof: Since $\delta(L) \subset L$, L is an ideal in M' (condition (a)). Further $ad_L M' = ad_L M + \{\delta_L\} \subset Z(ad_L L)$ (condition (b)); $\{\delta_L\}$ denotes the set of all multiples of δ_L.

Finally because of the hypothesis, that $\delta_L \notin ad_L M$, $\dim (ad_L M'/ad_L L) = \dim (ad_L M + \{\delta_L\}/ad_L L) = \dim (ad_L M/ad_L L) + 1$. Further since M is a close hull of L, $ad_L M/ad_L L \simeq M/L$, and hence $\dim (ad_L M'/ad_L L) = \dim (M/L) + 1 = \dim (M'/L)$. This shows that the surjective map $M'/L \xrightarrow{ad_L} ad_L M'/ad_L L$ preserves dimension and is therefore an isomorphism (condition c').

Corollary: *Let M be a close hull of L of maximal dimension and δ a derivation of M with $\delta(L) \subset L$, $\delta_L \in Z(ad_L L)$. Then $\delta_L \in ad_L M$.*

Let M be a close hull of L of maximal dimension. The Lie algebra $Z(adM)$, consisting of derivations of M, leaves L invariant. Further, by section 3.2, $(Z(adM))_L = Z(ad_L M)$. Since $ad_L L \subset ad_L M \subset Z(ad_L L)$, we have $Z(ad_L M) = Z(ad_L L)$. Hence $(Z(adM))_L = Z(ad_L L)$. Therefore applying the corollary to the derivations $\delta \in (Z(adM))$ we find: $(Z(adM))_L = ad_L M$ and $ad_L M = Z(ad_L L)$. Finally the kernel of the homomorphism $M \xrightarrow{ad_L} ad_L M = Z(ad_L L)$ consists of the centralizer of L which is, by lemma 1, the centre of M = centre of L. Putting $Z(ad_L L) = ad_L M = G$, and substituting the letter H for M, we find

Proposition 2. *Let L be a Lie algebra with centre J and H a close hull of maximal dimension. Then the central extension $0 \to J \subset L \xrightarrow{ad} adL \to 0$ is contained in the central extension $0 \to J \subset H \xrightarrow{ad_L} G \to 0$, where $G = ad_L H = Z(ad_L L)$ is the Lie algebra of an irreducible algebraic linear group.*

Proposition 2 in conjunction with proposition 1 establishes the faithful representability of any complex Lie algebra.

4. Arbitrary fields of characteristics zero.

Throughout this section "field" will mean field of characteristic zero. C denotes the field of complex numbers.

The purpose of this section is to show the adequacy of the preceding constructions to yield also Ado's theorem for Lie algebras over arbitrary fields.

4.1. *On proposition* 2. The proof of proposition 2 rests on a few simple properties of the notion of the Zariski closure $Z(\ )$ of a linear Lie algebra. These properties remain valid for arbitrary fields. More in detail one has the following situation.

Let K be a given field, and let vectorspaces, algebras, Lie algebras to be considered, for the moment be over K. Let V be a finite dimensional vectorspace, $E(V)$ the Lie algebra of its endomorphisms. On the set of Lie subalgebras of $E(V)$ there exists a closure operation $Z(\ )$ with the following properties:

For any Lie algebra $L \subset E(V)$, $Z(L)$ is a Lie subalgebra of $E(V)$.

If L' is a Lie algebra with $L \subset L' \subset Z(L)$, then $Z(L') = Z(L)$.

If the subspace $V' \subset V$ is (pointwise)[1] L-invariant, then V' is also (pointwise) $Z(L)$-invariant, and the closure operations in $E(V')$ and $E(V)$ are connected by $Z(L_{V'}) = (Z(L))_{V'}$, the subscript V' indicates that the induced Lie algebra on V' is considered.

If V has an algebra structure, and L is a Lie algebra of derivations, then $Z(L)$ is also a Lie algebra of derivations.

Let K' be an extension of K, and denote by V', L', etc. the objects obtained from V, L etc. by extending the field of scalars to K'. Then $Z(L') = Z(L)'$.

If K' is algebraically closed $Z(L') = Z(L)'$ is the Lie algebra of an irreducible algebraic group whose field of definition is contained in K.

The above statements can be inferred from the MAURER–CHEVALLEY theory [10]. In the appendix we shall outline an elementary way to obtain them without using the deeper results from the Maurer–Chevalley theory.

In any case, the above properties are just those which permit to copy § 2.2. Hence we obtain

Proposition 2'. Let L be a Lie algebra defined over a field K. Let J denote the centre of L. Then the central extension of K-Lie algebras, $0 \to J \subset L \xrightarrow{ad} ad\, L \to 0$ is contained in a central extension $0 \to J \subset \subset H \to G \to 0$. If K is a subfield of C, the complexification of G is the Lie algebra of an irreducible algebraic complex linear group whose field of definition is contained in K.

Remarks. 1. The last sentence of the above proposition holds also mutatis mutandis if C is replaced by an algebraically closed field. The reason why we insist on C is that for the construction of § 3.2 we need functions g with $dg = \omega$ where ω is a rational form. These g are conveniently defined as *functions* in the complex case. In the general case perhaps one should imitate the "integration" procedure within the local ring of formal power series in the local parameters at the neutral element.

2. Since we are considering finite dimensional Lie algebras only, we

[1]) "Pointwise L – invariant" means "is annihilated by L".

may take them defined over a finitely generated field over the rationals, hence over a subfield of the complex numbers. Therefore it is sufficient to establish Ado's theorem for such fields.

4.2. *The construction of* § 3.2.

4.2.1. *Preliminaries.* Let $\mathfrak{G}'$ be an irreducible algebraic complex linear group acting in a complex vector space V, and suppose that $\mathfrak{G}'$ is defined over a field $K \subset C$. This means that there is a basis $e_1, \ldots, e_n$ of V such that with respect to the associated base $e_j^* \otimes e_i$ [1]) for $E(V)$ the prime ideal $\mathfrak{p}$ associated with $\mathfrak{G}'$ has a base in $K[z_{ij}]$; in other designation, putting $\mathfrak{p}_K = \mathfrak{p} \cap K[z_{ij}]$, $\mathfrak{p} = \mathfrak{p}_K \otimes_K C$.

We may as well suppose again that $\mathfrak{G}'$ is an algebraic subgroup of the unimodular group (otherwise we imbed $\mathfrak{G}'$ into the unimodular group of a space of one more dimension) so that $\mathfrak{G}'$ is an affine variety. The K-forms on $\mathfrak{G}'$ are defined to be differential forms of type $\sum f dz_{i_1 j_1} \wedge \ldots \wedge dz_{i_k j_k}$ with coefficients $f \in K[z_{ij}]$. The K-forms constitute a complex Ω_K. Since the ideal $\mathfrak{p}$ of $\mathfrak{G}'$ has a basis in K, the multiplication map $\mathfrak{G}' \times \mathfrak{G}' \to \mathfrak{G}'$ determines a map $\mu^*: \Omega_K \to \Omega_K \otimes_K \Omega_K$, which turns $H(\Omega_K)$ into a Hopf algebra over K.

Suppose J' is a complex vectorspace with a distinguished set $\mathfrak{B}$ of bases such that the transition of one base in $\mathfrak{B}$ to another (in $\mathfrak{B}$) is described by a matrix with coefficients in K. Then (with respect to $\mathfrak{B}$) a J'-valued K-form is to be a J'-valued rational differential form ω on $\mathfrak{G}'$ such that the components of ω with respect to a base from $\mathfrak{B}$ are K-forms.

A K-form ω on G' admits in general different expressions

$$\omega = \sum f dz_{i_1 j_1} \wedge \ldots \wedge dz_{i_k j_k}.$$

The *height* of such an expression is understood to be the maximal degree of its coefficients f. The height of ω is defined to be the minimal height of its expressions. Observe that *height* $(\omega_1 + \omega_2) \leqslant$ max (*height* ω_1, *height* ω_2). Hence the K-forms whose height does not exceed a given h constitute a finite dimensional vector-space over K.

For J'-valued K-forms one has a similar notion of height.

Since $\mathfrak{G}'$ is described by polynomial equations with coefficients in K, its space of tangent vectors at the neutral element is described by linear equations in z_{ij} with coefficients in K. The solutions in K constitute a Lie algebra G over K, and the Lie algebra G' of $\mathfrak{G}'$ is the complexification of G.

4.2.2. *The construction.* G, G', $\mathfrak{G}'$, V etc. have the same meaning as before.

Let (*) $0 \to J \subset H \to G \to 0$ be a central extension in the category of Lie algebras over K. Let $\omega: H \to J$ denote a projection for the vector space structure and let $\varphi^* \tau_2 = d\omega$. The pfaffian forms dz_{ij} induce in the

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[1]) $e_1^*, \ldots, e_n^*$ being the dual base in the dual of V.

neutral element K-linear functions on G. Hence τ_2 on G may be written as $\tau_2 = \sum \alpha_{i_1 j_1 i_2 j_2}(dz_{i_1 j_1})_1 \wedge (dz_{i_2 j_2})_1$, where $(dz_{ij})_1$ is the linear function induced by dz_{ij} on G considered as a K-linear subspace of the tangent space $E(V)$ to the variety $GL(V)$ in the neutral element; the $\alpha_{i_1 j_1 i_2 j_2}$ are elements of J.

We consider the K-bases of J as a distinguished set of bases of its complexification J'. Let further $0 \to J' \subset H' \to G' \to 0$ be the complexified extension (*). Then the form τ_2 may be considered to be a J'-valued cocycle on G', and right translation of τ_2 over $\mathfrak{G}'$ yields a right invariant K-form on $\mathfrak{G}'$. Adopting again the notation of § 2.3, we get

$$\tau_2 = \sum \omega_{1i} \wedge \tau_{1i} + d\sigma,$$

where all forms involved are pfaffian K-forms, ω_{1i}, τ_{1i} closed, τ_{1i} and σ J'-valued, ω_{1i} ordinary K-forms.

In the neighbourhood U of 1 we have $\tau_2 = d\tau_1$, where $\tau_1 = \sum g_i \tau_{1i} + \sigma$, g_i are ordinary functions (complex valued) with $dg_i = \omega_{1i}$ ($i = 1, \ldots, r$). We take the g_i to be such that they are zero in the neutral element.

Let W be again a simply connected local Lie group with Lie algebra H', and f a function on W determined by $df = \omega - \varphi^* \tau_1$. We take f to be zero in the neutral element. Recall that we obtained a local linear representation of W by making it act on f by *right* translations. Hence the associated Lie algebra representation of H' is obtained by identifying H' with the Lie algebra of *left-invariant* infinitesimal transformations on W and making them act as derivations.

We are to verify that thus on the Lie algebra H a finite dimensional linear representation over K is induced.

4.2.3. *The action of H.* To this end we examine first the action of H on $df = \omega - \varphi^* \tau_1$. Since ω is right invariant, the infinitesimal transformations on W belonging to H annihilate ω. Therefore there remains to examine the action of H on $\varphi^* \tau_1$, and this amounts to study the action of $\varphi(H) = G$ on τ_1.

On $GL(V)$ a left invariant infinitesimal transformation X belonging to G is represented by $\sum X_{ij} \partial / \partial z_{ij}$, where the X_{ij} are linear polynomials $\in K[z_{ij}]$. The action $\theta(X)$ of X on a differential form τ as a derivation of degree zero may be explained as follows (see [11] p. 18). Let

$$\tau = \sum f_{i_1 j_1; \ldots; i_k j_k} dz_{i_1 j_1} \wedge \ldots \wedge dz_{i_k j_k}$$

and let the summation on the right hand side be extended over a complete set of sequences $i_1 j_1; \ldots; i_k j_k$ of k pairs representing the *sets* of k pairs of indices, and further let us put for any sequence $\mathfrak{s}$: $t_1 s_1; \ldots; t_k s_k$,

$$f_{t_1 s_1; \ldots; t_k s_k} = 0 \text{ if repetition occurs in } \mathfrak{s}$$
$$= \operatorname{sgn} \pi.\ f_{i_1 j_1; \ldots; i_k j_k} \text{ if no repetition occurs in } \mathfrak{s},$$

$\mathfrak{i}$: $i_1 j_1; \ldots; i_k j_k$ represents the same set of k pairs as $\mathfrak{s}$, and π is the permutation carrying $\mathfrak{s}$ into $\mathfrak{i}$.

With this convention define $i(X)\tau$ to be

$$\sum X_{ij} f_{ij;\, i_2 j_2;\, \dots;\, i_k j_k} dz_{i_2 j_2} \wedge \dots \wedge dz_{i_k j_k}$$

where the sequence $i_2 j_2; \dots; i_k j_k$ is understood to run through a complete set of sequences representing the sets of $k-1$ pairs of indices, and where i, j runs through the complete set of pairs ij. $i(X)\tau$ is defined to be zero if τ has degree zero, i.e. if τ is a function.

Observe that if τ is a K-form and $X \in G$, then, because of $X_{ij} \in K[z_{ij}]$, $i(X)\tau$ is a K-form. Further, since for $X \in G$ the X_{ij} are linear polynomials, the height of $i(X)\tau$ on $\mathfrak{G}'$ (see 4.2.1) exceeds the height of τ at most by one unity. For $\theta(X)$ we have the equation ([11])

$$\theta(X) = i(X)d + di(X) \tag{1}$$

From (1) it is apparent that $\theta(X)d = d\theta(X)$. Further $\theta(X)$ is a derivation of degree zero ([11]). Finally for any k-form τ, and any $X \in G$, $\theta(X)\tau$ is a K-form and height $\theta(X)\tau \leqslant$ height τ.

Denoting the space of pfaffian J'-valued K-forms of height $\leqslant h$ by P_h, we see that θ induces a K-linear representation of G in P_h.

Let τ be a pfaffian J'-valued K-form of height $\leqslant h$, ω an ordinary closed pfaffian K-form of height $\leqslant l$, g a function on U with $dg = \omega$, and $X \in G$. Then from $\theta(X)(g\tau) = (i(X)\omega)\cdot\tau + g\cdot(\theta(X)\tau)$ it is apparent that $\theta(X)(g\tau) = P_{h+l+1} + gP_h$. Hence for $X \in G$, the space of pfaffian forms $P_{h+l+1} + gP_h$ is stable under $\theta(X)$. Returning to $\tau_1 = \sum g_i \tau_{1i} + \sigma$, let us assume that the maximum height of the forms τ_{1i} and σ is h and that the maximum height of the ω_{1i} is l. Then

$$\tau_1 \in P_h + g_1 P_h + g_2 P_h + \dots \subset P_{h+l+1} + \sum g_i P_h = P,$$

and θ provides a K-linear representation of G in the finite dimensional K-space P. Therefore θ induces a K-linear representation of H in the finite dimensional K-space P^* spanned by ω and $\varphi^*(P)$. Since $df = \omega - \varphi^*\tau_1 \in P^*$, we see that df and its successive derivatives

$$\theta(X_1)\theta(X_2) \dots \theta(X_p)df = \varphi^*\theta(\varphi(X_1)) \dots \theta(\varphi(X_p))\tau_1,$$

span a finite dimensional $\theta(H)$-invariant subspace S of P^*. Select among the successive derivatives of df a finite number $df_2, \dots, df_s$, such that $df = df_1, \dots, df_s$ constitute a base of S, and take the associated functions $f_2, \dots, f_s$ to be zero at the neutral element. Then for any $X \in H$ we obtain $\theta(X)df_i = \sum \xi_{ji}(X)df_j$, where $\xi_{ji}(X) \in K$. Hence we obtain

$$\theta(X)f_i = \sum \xi_{ji}(X)f_j + \gamma(X, i),$$

where $\gamma(X, i)$ is a constant. In order to determine $\gamma(X, i)$, we evaluate the function $\theta(X)f_i - \sum \xi_{ji}(X)f_j$ at the neutral element. Since the f_j are zero in the neutral element it remains to evaluate $\theta(X)f_i = df_i(X)$. We do this only for $\theta(X)f_1 = \theta(X)f = df(X)$, the argument in the other cases being similar. $df(X) = \omega(X) - (\varphi^*\tau_1)(X) = \omega(X) - \tau_1(\varphi(X))$. On H, ω is a projection $H \to J$, hence $\omega(X) \in J$. Further $\tau_1 \in P_{h+l+1} + g_1 P_h + g_2 P_h + \ldots$, the g_i are zero in the neutral element, and $\varphi(X) \in G$. Hence $\tau_1(\varphi(X))$ is the result one gets by substituting $\varphi(X) \in G$ into a J'-valued K-form $\in P_{h+l+1}$ and taking the value at the neutral element; this shows that $\tau_1(\varphi(X)) \in J$.

Consequently we see that the finite dimensional K-space spanned by the $f_1, \ldots, f_s$ and the constant functions with values in J is stable under $\theta(H)$.

The considerations of § 4.1 and 4.2 show that the constructions of § 2 and 3 actually suffice to obtain Ado's theorem for Lie algebras over arbitrary fields of characteristic zero.

Appendix.

As pointed out before, the existence of a Zariski closure operator $Z(\ )$ on the set of linear Lie algebras over a field K with the properties mentioned in § 4.1 follows from the Maurer–Chevalley theory[1]). For reasons of completeness we have tried to single out just that much from the elements of the theory as one needs in order to establish the existence of the closure operator with the properties mentioned.

A number of simple observations is collected in § 5; these are applied in § 6 to get the desired results.

All fields to be considered are assumed to be of characteristic zero.

§ 5. Let A be a commutative associative algebra with a unit element, over a field K. Let $\mathfrak{D}$ denote a set of derivations of A.

An ideal $\mathfrak{i} \subset A$ [2]) is said to *integrate* $\mathfrak{D}$ if $D\,\mathfrak{i} \subset \mathfrak{i}$ for every $D \in \mathfrak{D}$. (The terminology is inspired by the integration theory of vector fields.)

Proposition 3. *If* $\mathfrak{i}_1$, $\mathfrak{i}_2$ *integrate* $\mathfrak{D}$, *then both* $\mathfrak{i}_1 \cap \mathfrak{i}_2$ *and* $\mathfrak{i}_1 + \mathfrak{i}_2$ (*assuming that the latter is again a proper subset of* A) *integrate* $\mathfrak{D}$.

Proposition 4. (*false for non zero characteristic*) *If* $\mathfrak{i}$ *integrates* $\mathfrak{D}$, *then its radical* $R(\mathfrak{i})$ *also integrates* $\mathfrak{D}$.

Proof: Suppose that $f \in R(\mathfrak{i})$, i.e. $f^n \in \mathfrak{i}$ for some n. Applying D n times to f^n we obtain $n!(Df)^n + fg \in \mathfrak{i}$, where g is a sum of monomials of type $f^{i_1}(D^{i_2}f) \ldots (D^{i_k}f)$. The field being of characteristic zero we have $(Df)^n = fg'$ mod $\mathfrak{i}$, where $g' = -(1/n!)g$. Since $f^n \in \mathfrak{i}$ we find that $(Df)^{n^2} \in \mathfrak{i}$, or $Df \in R(\mathfrak{i})$.

Proposition 5. *Suppose that* $\mathfrak{i}$ *integrates* $\mathfrak{D}$, *and* $\mathfrak{i} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_k$ *is a minimal decomposition of* $\mathfrak{i}$ *into prime ideals* $\mathfrak{p}_i$. *Then every* $\mathfrak{p}_i$ *integrates* $\mathfrak{D}$.

Proof: Dividing out by $\mathfrak{i}$ shows that we may assume that $\mathfrak{i} = (0)$. Further it is sufficient to prove $D\mathfrak{p}_1 \subset \mathfrak{p}_1$ for $D \in \mathfrak{D}$.

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1) See e.g. [10], chap. II, §14, th. 13.

2) Ideals are always assumed to be proper subsets of A.

Take $\mathfrak{q}=\mathfrak{p}_2 \cap \dots \cap \mathfrak{p}_k$. Since $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ determine a minimal decomposition of (0) into primes, $\mathfrak{q}$ is necessarily $\neq (0)$. Let $q \in \mathfrak{q}$, $q \neq 0$. Then for every $x \in \mathfrak{p}_1$, $qx=0$. Hence $0=D(qx)=(Dq)x+q(Dx)$. Obviously $(Dq)x \in \mathfrak{p}_1$, and $q(Dx) \in \mathfrak{q}$. Since $\mathfrak{p}_1 \cap \mathfrak{q}=(0)$ we find $(Dq)x=q(Dx)=0$. Further from $q\neq 0$, $q \in \mathfrak{q}$, $\mathfrak{p}_1 \cap \mathfrak{q}=(0)$, it follows that $q \notin \mathfrak{p}_1$. This fact combined with the fact that $\mathfrak{p}_1$ is prime shows that the annihilator of q is $\subset \mathfrak{p}_1$. Hence $q(Dx)=0$ implies $Dx \in \mathfrak{p}_1$.

Let $\mathfrak{m}$ be an ideal. By $I(\mathfrak{m}, \mathfrak{D})$ we designate the maximal subideal of $\mathfrak{m}$ integrating $\mathfrak{D}$; such a maximal subideal exists by the second half of proposition 3 and observing that in any case (0) integrates $\mathfrak{D}$.

Proposition 6. *Let A be noetherian, and $\mathfrak{m}$ a prime ideal. Then $I(\mathfrak{m}, \mathfrak{D})$ is a prime ideal.*

Proof: Since $I(\mathfrak{m}, \mathfrak{D})$ integrates $\mathfrak{D}$, its radical integrates $\mathfrak{D}$ (prop. 4). Since $\mathfrak{m}$ is a prime ideal, the radical of $I(\mathfrak{m}, \mathfrak{D})$ is $\subset \mathfrak{m}$. $I(\mathfrak{m}, \mathfrak{D})$ being a maximal subideal of $\mathfrak{m}$ integrating $\mathfrak{D}$, we see that the radical of $I(\mathfrak{m}, \mathfrak{D})$ is contained in $I(\mathfrak{m}, \mathfrak{D})$, or $I(\mathfrak{m}, \mathfrak{D})$ is a radical ideal. Since A is noetherian, the radical ideal $I(\mathfrak{m}, \mathfrak{D})$ admits a minimal decomposition $I(\mathfrak{m}, \mathfrak{D})= =\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_k$ into prime ideals $\mathfrak{p}_i$. Since $\mathfrak{m}$ is prime and $I(\mathfrak{m}, \mathfrak{D}) \subset \mathfrak{m}$, at least one of the $\mathfrak{p}_i$ is $\subset \mathfrak{m}$, say $\mathfrak{p}_1 \subset \mathfrak{m}$. By proposition 5, $\mathfrak{p}_1$ integrates $\mathfrak{D}$. Hence again by the maximality of $I(\mathfrak{m}, \mathfrak{D})$ we must have $\mathfrak{p}_1 \subset I(\mathfrak{m}, \mathfrak{D})$. This fact combined with $I(\mathfrak{m}, \mathfrak{D}) \subset \mathfrak{p}_1$ yields $I(\mathfrak{m}, \mathfrak{D})=\mathfrak{p}_1$.

Assume from now on that $A=K[z_{ij}]$, where z_{ij} are n^2 indeterminates. Let further the derivations $D \in \mathfrak{D}$ be of type $\sum f_{ij}\partial/\partial z_{ij}$, where $f_{ij} \in K'[z_{ij}]$, K' a subfield of K. We shall say that K' contains the coefficients of $\mathfrak{D}$ or that $\mathfrak{D}$ has coefficients in K'.

Proposition 7. *Let the ideal $\mathfrak{m}$ of A have a basis in $K'[z_{ij}]$. Then with the above assumptions, $I(\mathfrak{m}, \mathfrak{D})$ has a basis in $K'[z_{ij}]$.*

Proof: Put $\mathfrak{m}'=\mathfrak{m} \cap K'[z_{ij}]$. The assumption on $\mathfrak{m}$ means that $\mathfrak{m}=K\cdot\mathfrak{m}'$, or any $f \in \mathfrak{m}$ is a linear combination $\sum c_i f_i'$, $f_i' \in \mathfrak{m}'$, $c_i \in K$. One infers from this by simple arguments that also the following holds: if $f=c_1f_1'+\dots+c_rf_r' \in \mathfrak{m}$, $f_i' \in K'[z_{ij}]$, $c_i \in K$, $c_1, \dots, c_r$ linearly independent over K', then $f_i' \in \mathfrak{m}'$.

Since $D \in \mathfrak{D}$ induces a derivation of $K'[z_{ij}]$, $I(\mathfrak{m}', \mathfrak{D}) \subset \mathfrak{m}'$ is a well defined ideal of $K'[z_{ij}]$. It is clear that the ideal $K\cdot I(\mathfrak{m}', \mathfrak{D})$ of $K[z_{ij}]$ integrates $\mathfrak{D}$. Hence $K\cdot I(\mathfrak{m}', \mathfrak{D}) \subset I(\mathfrak{m}, \mathfrak{D})$.

Let for any finite set $c_1, \dots, c_r \in K$, linearly independent over K', $I_{c_1,\dots,c_r}$ denote the $K'[z_{ij}]$-module $I(\mathfrak{m}, \mathfrak{D}) \cap \{c_1K'[z_{ij}]+\dots+c_rK'[z_{ij}]\}$. Hence any $f \in I_{c_1,\dots,c_r}$ is $\in I(\mathfrak{m}, \mathfrak{D})$ and admits a unique decomposition $c_1f_1'+\dots+c_rf_r'$, $f_i' \in K'[z_{ij}]$. Observe that for any $D \in \mathfrak{D}$, and $f \in I_{c_1,\dots,c_r}$, $Df \in I(\mathfrak{m}, \mathfrak{D})$ and $Df= \sum c_iDf_i'$, $Df_i' \in K'[z_{ij}]$. Hence the ideal $\mathfrak{f}_i' \in K'[z_{ij}]$ consisting of the f_i' (where f runs through $I_{c_1,\dots,c_r}$) integrates $\mathfrak{D}$.

Further for any $f \in \mathfrak{m} \cap \{c_1K'[z_{ij}]+\dots+c_rK'[z_{ij}]\}$ the f_i' in the representation $f=c_1f_1'+\dots+c_rf_r'$ are $\in \mathfrak{m}'$, see the remark above. Or the ideals $\mathfrak{f}_i'$ are integrating subideals of $\mathfrak{m}'$, and therefore $\subset I(\mathfrak{m}', \mathfrak{D})$. Hence $I_{c_1,\dots,c_r} \subset c_1I(\mathfrak{m}', \mathfrak{D})+\dots+c_rI(\mathfrak{m}', \mathfrak{D})$. Since any $f \in I(\mathfrak{m}, \mathfrak{D})$ is in some $I_{c_1,\dots,c_r}$ we obtain that $I(\mathfrak{m}, \mathfrak{D}) \subset K\cdot I(\mathfrak{m}', \mathfrak{D})$. Together with $K\cdot I(\mathfrak{m}', \mathfrak{D}) \subset \subset I(\mathfrak{m}, \mathfrak{D})$ we find $K\cdot I(\mathfrak{m}', \mathfrak{D})=I(\mathfrak{m}, \mathfrak{D})$.

Proposition 8. *Suppose that $\mathfrak{D}$ maps the subring $B=K[z_{ij}]$, $1\leqslant i\leqslant s$, $1\leqslant j\leqslant s$, into itself, and let $\tilde{\mathfrak{D}}$ be the set of induced derivations. Let $\mathfrak{m}$ be an ideal of $A=K[z_{ij}]$, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant n$, and $\tilde{\mathfrak{m}}=\mathfrak{m}\cap B$. Then $I(\tilde{\mathfrak{m}}, \tilde{\mathfrak{D}})= =I(\mathfrak{m}, \mathfrak{D})\cap B$.*

Proof: $I(\mathfrak{m}, \mathfrak{D})\cap B$ is a subideal of $\mathfrak{m}\cap B=\tilde{\mathfrak{m}}$. $I(\mathfrak{m}, \mathfrak{D})$ integrates $\mathfrak{D}$ and B is stable under $\mathfrak{D}$. Therefore $I(\mathfrak{m}, \mathfrak{D})\cap B$ integrates $\tilde{\mathfrak{D}}$. Hence $I(\mathfrak{m}, \mathfrak{D})\cap B\subset I(\tilde{\mathfrak{m}}, \tilde{\mathfrak{D}})$.

Conversely the ideal $\mathfrak{j}$ in A generated by $I(\tilde{\mathfrak{m}}, \tilde{\mathfrak{D}})$ is a subideal of $\mathfrak{m}$. $I(\tilde{\mathfrak{m}}, \tilde{\mathfrak{D}})$ integrates $\tilde{\mathfrak{D}}$; this means that $I(\tilde{\mathfrak{m}}, \tilde{\mathfrak{D}})$ as a subset of A is stable under $\mathfrak{D}$. Hence $\mathfrak{j}$ integrates $\mathfrak{D}$, and consequently $\mathfrak{j}\subset I(\tilde{\mathfrak{m}}, \tilde{\mathfrak{D}})$. This shows that $I(\tilde{\mathfrak{m}}, \tilde{\mathfrak{D}})\subset I(\mathfrak{m}, \mathfrak{D})$, and hence $I(\tilde{\mathfrak{m}}, \tilde{\mathfrak{D}})\subset I(\mathfrak{m}, \mathfrak{D})\cap B$.

§ 6. In the following discussion Ω will denote a universal domain to which the fields to be considered belong.

Let V be an n-dimensional vector space over Ω with a base fixed once for all. There is an associated base in $E(V)$. Generic coordinates in V will be denoted by z_i, $i=1, \dots, n$, and the generic coordinates in $E(V)$ by z_{ij}, $i=1, \dots, n$, $j=1, \dots, n$.

Let for $p=(\pi_{ij})\in E(V)$, $\mathfrak{m}_p$ denote the maximal ideal in $\Omega[z_{ij}]$ associated with p, i.e. the ideal generated by $z_{ij}-\pi_{ij}$. $\mathfrak{m}_p$ has a basis in every field K to which the π_{ij} belong. By $\mathfrak{m}_e$ we denote the maximal ideal of the unit element $e=(\delta_{ij})$, δ_{ij} Kronecker symbol. $\mathfrak{m}_e$ has a basis consisting of polynomials with rational coefficients.

For any set of derivations $\mathfrak{D}$ of $\Omega[z_{ij}]$ and $p\in E(V)$ we denote by $W(p, \mathfrak{D})$ the set of zeroes of $I(\mathfrak{m}_p, \mathfrak{D})$. From props. 6 and 7 it follows that $W(p, \mathfrak{D})$ is an irreducible variety through p, and defined over any field K which contains the coordinates of p and the coefficients of $\mathfrak{D}$. *In particular $W(e, \mathfrak{D})$ is an irreducible variety through e and it is defined over any field which contains the coefficients of $\mathfrak{D}$.*

An $n\times n$-matrix α_{ij} with $\alpha_{ij}\in\Omega$ induces in $\Omega[z_{ij}]$ a left invariant derivation $D=\sum f_{ij}\partial/\partial z_{ij}$, where $f_{ij}=\sum z_{ik}\alpha_{kj}$. D determines a field of tangent vectors on $E(V)$; it assigns to the point (ϱ_{ij}) the vector $\sum \varphi_{ij}\partial/\partial z_{ij}$, where $\varphi_{ij}=\sum \varrho_{ik}\alpha_{kj}$. This field of tangent vectors is invariant with respect to left multiplication. For the derivation D this is expressed by the equation $\lambda(p)D=D\lambda(p)$, where $\lambda(p)$ is the endomorphism of $\Omega[z_{ij}]$ induced by left multiplication by $p=(\pi_{ij})$ and described by $z_{ij}\to\sum \pi_{ik}z_{kj}$. Notice that if the α_{ij} belong to a field K then D has coefficients in K. We shall further not distinguish between "derivation" and "vector field".

From now on L will denote a set of $n\times n$ matrices with coefficients in K, and $\mathfrak{D}_L$ will denote the associated set of left invariant derivations. For a $p \in E(V)$ we set $W(p, L) = W(p, \mathfrak{D}_L)$. Geometrically speaking $W(p, L)$ is the minimal variety through p which admits in every simple point x the vectors of $\mathfrak{D}_L$ at x as tangent vectors. We shall say that a variety with this property is an *integrating variety* of the set of vector fields $\mathfrak{D}_L$.

Proposition 9. *For any $p, x \in E(V)$, $pW(x, L) \subset W(px, L)$. If p is invertible then $pW(x, L) = W(px, L)$.*

Proof: $\lambda(p)$ maps $\mathfrak{m}_{px}$ into $\mathfrak{m}_x$. Further $DI(\mathfrak{m}_{px}, \mathfrak{D}_L) \subset I(\mathfrak{m}_{px}, \mathfrak{D}_L)$ for any $D \in \mathfrak{D}_L$. From $D\lambda(p) = \lambda(p)\ D$ we find $D\lambda(p)I(\mathfrak{m}_{px}, \mathfrak{D}_L) = \lambda(p)DI(\mathfrak{m}_{px}, \mathfrak{D}_L) \subset \lambda(p)I(\mathfrak{m}_{px}, \mathfrak{D}_L)$. This shows that $\lambda(p)I(\mathfrak{m}_{px}, \mathfrak{D}_L)$ is a $\mathfrak{D}_L$-stable subset of $\mathfrak{m}$. Therefore $\lambda(p)I(\mathfrak{m}_{px}, \mathfrak{D}_L) \subset I(\mathfrak{m}_x, \mathfrak{D}_L)$ (1). Or set theoretically $pW(x, L) \subset W(px, L)$. If p is invertible we have likewise $\lambda(p^{-1})I(\mathfrak{m}_x, \mathfrak{D}_L) \subset I(\mathfrak{m}_{px}, \mathfrak{D}_L)$ or $\lambda(p)I(\mathfrak{m}_{px}, \mathfrak{D}_L) \supset I(\mathfrak{m}_x, \mathfrak{D}_L)$. This inclusion combined with (1) yields that $pW(x, L) = W(px, L)$ for invertible p.

Proposition 10. *$W(e, L) \cap GL(V)$ is an irreducible algebraic group defined over K.*

Proof: It remains to prove the group property. Let $p \in W(e, L) \cap GL(V)$. Both $W(p, L)$ and $W(e, L)$ pass through p and integrate the vector fields $\mathfrak{D}_L$. However $W(p, L)$ is a minimal integrating variety through p and therefore $W(p, L) \subset W(e, L)$. Further $W(p, L)$ and $W(e, L)$ are irreducible. By prop. 9 $W(p, L) = pW(e, L)$, and since p is invertible, $W(p, L)$ and $W(e, L)$ have same dimension. Therefore they must coincide. From this result it follows immediately that $W(e, L) \cap GL(V)$ is a group.

From the fact that $W(e, L) \cap GL(V)$ is a group it follows that e is a non singular point of $W(e, L)$. Further $W(e, L)$ is defined by equations with coefficients in K and e has coordinates in K. Therefore the space T of tangent vectors at e is defined by linear equations with coefficients in K. The solutions of these equations in K constitute therefore a linear Lie algebra over K, which will be denoted by $Z(L)$ the Zariski closure of L. In fact $Z(L)$ is defined for any set L of matrices with coefficients in K. From now on we assume that L is a Lie algebra (over K).

Notice that $Z(L)$ generates T over Ω. Further $W(e, L)$ being tangent to the vectors of $\mathfrak{D}_L$, $W(e, L)$ is tangent to the vectors of $\mathfrak{D}_L$ at e, i.e. $W(e, L)$ is tangent to L, or $L \subset Z(L)$.

From the entire construction it appears that if K' is an extension of K, and if $L', Z(L)'$ denote the corresponding extensions of L and $Z(L)$, then $Z'(L') = (Z(L))'$, $Z'(\ )$ denoting the closure operator in K'.

Let $L \subset L' \subset Z(L)$. Then $\mathfrak{D}_L \subset \mathfrak{D}_{L'}$ and hence $W(e, L) \subset W(e, L')$. However $W(e, L) \cap GL(V)$ integrates the vector fields $\mathfrak{D}_{Z(L)}$ and therefore also $\mathfrak{D}_{L'}$. $W(e, L) \cap GL(V)$ being Zariski dense in $W(e, L)$ if follows that $W(e, L)$ integrates $\mathfrak{D}_{L'}$, and so $W(e, L') \subset W(e, L)$. Consequently $W(e, L) = W(e, L')$, and this implies that $Z(L) = Z(L')$.

Suppose that L leaves invariant the subspace V': $z_{s+1}=\ldots=z_n=0$. The matrices belonging to L have then zeroes in the entries ij, $s+1\leqslant i\leqslant n$, $1\leqslant j\leqslant s$. One verfies easily that the set of derivations $\mathfrak{D}_L$ transforms the z_{ij} $(s+1\leqslant i\leqslant n,\ 1\leqslant j\leqslant s)$ linearly among themselves. Therefore the ideal generated by these z_{ij} integrates D_L, and since it is contained in $\mathfrak{m}_e$ it is $\subset I(m_e, D_L)$. Hence the matrices $\in W(e, L)$ have zeroes in the entry ij $(s+1\leqslant i\leqslant n,\ 1\leqslant j\leqslant s)$, and hence also the matrices of $Z(L)$, or V' is invariant under $Z(L)$. Similarly it follows that V' is annihilated by $Z(L)$ if it is annihilated by L. Maintaining the assumption on L for a moment, it may be checked that also the z_{ij} $(1\leqslant i\leqslant s,\ 1\leqslant j\leqslant s)$ are linearly transformed among themselves by $\mathfrak{D}_L$. The set of derivations induced by $\mathfrak{D}_L$ in $B=K[z_{ij}]$, $1\leqslant i\leqslant s$, $1\leqslant j\leqslant s$, is just the set $\mathfrak{D}_{L_{V'}}$. By proposition 8 one infers that $I(\mathfrak{m}_e, \mathfrak{D}_L)\cap B=I(\mathfrak{m}_{e_{V'}}, \mathfrak{D}_{L_{V'}})$. This leads to $(Z(L))_{V'}=Z(L_{V'})$.

Finally suppose that V has an algebra structure. The automorphisms of V are the zeroes in $GL(V)$ of the polynomials $Q_{f,u,v}$ in z_{ij} which are defined by $Q_{f,u,v}(Z)=f(Zu\cdot Zv)-f(Z(uv))$, Z being the matrix (z_{ij}), u, v being elements of V, f being a linear function on V. For any derivation Δ of V, the corresponding left invariant derivation D of the polynomial ring $\Omega[z_{ij}]$ acts on $Q_{f,u,v}$ by $DQ_{f,u,v}(Z)=f(Z\Delta u\cdot Zv)+f(Zu, Z\Delta v)-f(Z\Delta(uv))$. Since Δ is a derivation of the algebra structure we see that $DQ_{f,u,v}=$ $=Q_{f,\Delta u,v}+Q_{f,u,\Delta v}$. Hence the ideal $\mathfrak{q}$ generated by the $Q_{f,u,v}$ integrates $\mathfrak{D}_L$ if L is a Lie algebra of derivations. Since $\mathfrak{q}\subset\mathfrak{m}_e$, we see that $W(e, L)\cap$ $\cap\, GL(V)$ is an automorphism group of the algebra structure of V, and hence $Z(L)$ is again a Lie algebra of derivations of the algebra structure of V.

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