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What is beyond coherent pairs of orthogonal polynomials?

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Abstract

Usually, coherent pairs of orthogonal polynomials have been considered in the wider context of Sobolev orthogonality. In this paper, we focus our attention on the problem of coherence between two orthogonal polynomial sequences in terms of the corresponding linear functionals. We deduce some conditions about the linear functionals in order that the corresponding orthogonal polynomial sequences constitute a coherent pair.

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1. Introduction

Polynomials which are orthogonal with respect to a Sobolev inner product like

$$(f,g)_S = \int_I f(x)g(x)\,\mathrm{d}\mu_0(x) + \lambda \int_I f'(x)g'(x)\,\mathrm{d}\mu_1(x), \tag{1.1}$$

for $\lambda \geqslant 0$, have been introduced in connection with problems of smooth least square data fitting (see [7]).

In [1], a study of the case $\mathrm{d}\mu_0(x) = \mathrm{d}\mu_1(x) = \mathrm{d}x$, $x \in [-1,1]$ was introduced. More precisely, properties of their zeros as well as some differential operators related to them were found. In [2], a

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similar study was given for $d\mu_0(x) = d\mu_1(x) = e^{-x}dx$, $x \in [0, \infty)$. More recently, some extensions of the above problems were considered in [9, 11, 15]. Furthermore, a more general point of view was presented in [16], when $\mu_0(x)$ and $\mu_1(x)$ are semiclassical measures.

On the other hand, the first general approach to the case when μ_0 and μ_1 are general measures, was started by Iserles et al. in [6]. They introduced the concepts of *coherent pairs* and *symmetrically coherent pairs* of polynomials as a condition on the measures $\mu_0(x)$ and $\mu_1(x)$, in order that the orthogonal polynomials associated with the inner product (1.1) satisfy some special properties. In fact, Iserles et al. raised questions about necessary and sufficient conditions for the existence of two monic orthogonal polynomial sequences (MOPS) $\{P_n\}_n$ and $\{T_n\}_n$, related by

$$T_n(x) = \frac{P'_{n+1}(x)}{n+1} - \sigma_n \frac{P'_n(x)}{n}, \quad n \geqslant 1, \tag{1.2}$$

where $\sigma_1, \sigma_2, \ldots$ are nonzero real numbers. Also, they gave several examples of them, in particular, they showed that Laguerre polynomials are self-coherent and Gegenbauer polynomials constitute a simple case of a self-symmetrically coherent pair.

In [15], Meijer obtains properties of the zeros of the Sobolev orthogonal polynomials in the presence of coherence property. Furthermore, new interesting examples of coherent pairs and symmetrically coherent pairs are shown.

In the present paper, we will study the concept of coherence in a more general situation; we will say that two quasi-definite linear functionals u_0 and u_1 constitute a coherent pair if and only if the corresponding monic orthogonal polynomial sequences, say $\{P_n\}_n$ and $\{T_n\}_n$, satisfy a relation like (1.2). In the case when $\{u_0, u_1\}$ constitute a coherent pair, we will say that u_1 is a "companion" for u_0.

A first approach was given in [6] when $\{P_n\}_n$ is a positive-definite classical family (Hermite, Laguerre, Jacobi), with the restriction of the positivity of the companion measure. In [12], the result was extended to classical linear functionals (even for the Bessel case when the orthogonality is defined with respect to a quasi-definite linear functional).

A second result deals with a classical family of orthogonal polynomials $\{T_n\}_n$. In [12], necessary and sufficient conditions on the sequence $\{\sigma_n\}$ in order to let $\{P_n\}_n$ be a sequence of orthogonal polynomials are obtained.

More recently, in [11] the authors proved that all coherent pairs can be described in terms of semiclassical orthogonal polynomials. They prove that both polynomials are semiclassical and the linear functional corresponding to the first one is a rational modification of the second one.

In this paper, we solve the following problems:

1. Given a sequence $\{P_n\}_n$ of monic orthogonal polynomials, to find $\{\sigma_n\}$ such that the sequence $\{T_n\}_n$ given by (1.2) be orthogonal. Conversely, given a sequence $\{T_n\}_n$ of monic orthogonal polynomials, to find $\{\sigma_n\}$ such that the sequence $\{P_n\}_n$ in (1.2) be orthogonal.

2. To obtain the coherence in the symmetric case as a consequence of the usual coherence in terms of the symmetrization process.

2. Definitions and preliminary results

Let $\mathbb{P}$ denote the linear space of complex polynomials. Given a linear functional u on $\mathbb{P}$ and a family of monic polynomials $\{P_n\}_n$ satisfying

$$\deg(P_n) = n, \quad n = 0, 1, \ldots$$
$$\langle u, P_n P_m \rangle = k_n \delta_{n,m}, \quad k_n \in \mathbb{C} \setminus \{0\}, \ n, m = 0, 1, \ldots,$$

we will say that $\{P_n\}_n$ is a *monic orthogonal polynomial sequence* (MOPS) with respect to u.

If u is a linear functional on $\mathbb{P}$, and a MOPS $\{P_n\}_n$ for u exists, then u is called *regular* or *quasi-definite*. In this case, the polynomials $\{P_n\}_n$ of any MOPS satisfy a three term recurrence relation (see [3], p. 18]) like

$$P_0(x) = 1, \qquad P_1(x) = x - \beta_0, \qquad xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geqslant 1,$$

where $\{\beta_n\}_{n\geqslant 0}$ and $\{\gamma_n\}_{n\geqslant 1}$ are two sequences of complex numbers with $\gamma_n \neq 0$ for $n \geqslant 1$. It is important to remark that, according to Favard's Theorem (see [3, p. 21, Theorem 4.4]), the existence of such a relation characterizes completely a given MOPS.

Let us recall that a regular linear functional u is called *semiclassical* (see [5, 13]), if there exist two polynomials $\phi(x)$, $\psi(x)$, with $\deg(\phi) \geqslant 0$ and $\deg(\psi) \geqslant 1$, such that u satisfies the following distributional differential equation

$$D(\phi u) = \psi u. \tag{2.1}$$

We will say that $s \geqslant 0$ is the class of u when

$$s = \min\{\max\{\deg(\phi) - 2, \deg(\psi) - 1\}, \text{ for every pair of polynomials } \phi, \psi \text{ satisfying (2.1)}\}.$$

As it is well known, the *classical* functionals (i.e., those of Laguerre, Jacobi, Hermite and Bessel) are semiclassical with $\deg(\phi) \leqslant 2$ and $\deg(\psi) = 1$, and therefore of class 0. Many characterizations of the classical MOPS's are known (see e.g., [8]). For our purposes, we need only the following ones.

Theorem 2.1. *Let u be a regular linear functional, and denote by $\{P_n\}_n$ the MOPS associated with u. Then the following assertions are equivalent*:

(C.1) *$\{P_n\}_n$ is a classical family, that is, there exist two polynomials, ϕ and ψ, with* $\deg(\phi) \leqslant 2$, $\deg(\psi) = 1$, *such that the functional u satisfies the distributional differential equation*

$$D(\phi u) = \psi u.$$

(C.2) (Hahn [4]). *The monic polynomials*

$$\left\{ \tilde{P}_n(x) = \frac{P'_{n+1}(x)}{n+1} \right\}_n$$

constitute a MOPS with respect to the moment functional $\tilde{u} = \phi u$. Furthermore, $\{\tilde{P}_n\}_n$ is also a classical family of the same type as $\{P_n\}_n$, since $\tilde{u}$ satisfies the distributional equation

$$D(\phi \tilde{u}) = (\psi + \phi')\tilde{u}.$$

(C.3) (Marcellán et al. [8]). *There exist two sequences of complex parameters, d_n and e_n, such that*

$$P_n(x) = \frac{P'_{n+1}(x)}{n+1} + d_n \frac{P'_n(x)}{n} + e_n \frac{P'_{n-1}(x)}{n-1}, \quad n \geqslant 2. \tag{2.2}$$

3. Coherent pairs, the general case

Given a linear functional u_0, the search of the companions for u_0 arises in a natural way, but, also we can pose the converse problem: given a linear functional u_1, find the functionals u_0 such that $\{u_0, u_1\}$ is a coherent pair. These questions have been partially solved in [12], where the case is considered that one of the two functionals is a classical one. In this section, we will study the general case when $\{P_n\}_n$ and $\{T_n\}_n$ are arbitrary sequences of monic polynomials, satisfying

$$\deg(P_n) = \deg(T_n) = n, \quad n \geqslant 0,$$

as well as the coherence relation (1.2). If both of them are MOPS, we shall denote by

$$P_0(x) = 1, \qquad P_1(x) = x - \beta_0, \qquad xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geqslant 1,$$

$$T_0(x) = 1, \qquad T_1(x) = x - \tilde{\beta}_0, \qquad xT_n(x) = T_{n+1}(x) + \tilde{\beta}_n T_n(x) + \tilde{\gamma}_n T_{n-1}(x), \quad n \geqslant 1,$$

the corresponding three term recurrence relations.

On the other hand, since $\{P_n\}_n$ is a family of monic polynomials with $\deg(P_n) = n$, then the family of monic derivatives $\{(1/(n+1))P'_{n+1}\}_{n \geqslant 0}$ constitutes a basis for $\mathbb{P}$. We can expand the polynomial $P_n(x)$ in terms of the first derivatives

$$P_n(x) = \frac{P'_{n+1}(x)}{n+1} + \sum_{i=1}^{n} b_i^{(n)} \frac{P'_i(x)}{i}, \quad n \geqslant 0. \tag{3.1}$$

In this way, we can prove the following result.

Proposition 3.1. *Let $\{T_n\}_n$ be the MOPS associated with the moment functional u_1, and $\{P_n\}_n$ a sequence of monic polynomials with* $\deg(P_n) = n$, $n \geqslant 0$. *We assume that these sequences satisfy a coherence relation like* (1.2). *If the sequence $\{P_n\}_n$ is a MOPS, then*

(i) $T_n(c) \neq 0$, $n \geqslant 0$, *where* $c = \tilde{\beta}_0 + \tilde{\gamma}_1/\sigma_1$;

(ii) *the parameters σ_n, $n \geqslant 1$, satisfy the relation*

$$\sigma_{n+1} = \frac{\tilde{\gamma}_{n+1} - (1/\sigma_n)\, b_n^{(n+2)}/(n+2)}{T_{n+1}(c) + \sum_{k=1}^{n-1} \frac{1}{\sigma_{k+1}\sigma_k} \frac{b_k^{(k+2)}}{k+2} T_n(c) + \sum_{j=1}^{n-1} \prod_{k=j+1}^{n} \sigma_k \left(\sum_{i=1}^{j} \frac{1}{\sigma_{i+1}\sigma_i} \frac{b_i^{(i+2)}}{i+2} \right) T_j(c)} T_n(c). \tag{3.2}$$

Conversely, if

$$\tilde{\gamma}_{n+1} - \frac{1}{\sigma_n}\frac{b_n^{(n+2)}}{n+2} \neq 0, \quad n \geqslant 1, \tag{3.3}$$

taking $\sigma_1 \neq 0$, *in such a way that*

$$T_n(c) \neq 0, \quad n \geqslant 0, \quad c = \tilde{\beta}_0 + \frac{\tilde{\gamma}_1}{\sigma_1},$$

and given $\sigma_{n+1}, n \geqslant 1$ *from the recursive algorithm* (3.2), *then* $\{P_n\}_n$ *is a MOPS.*

In this situation, the parameters in the three term recurrence relation for the polynomials $\{P_n\}_n$ *satisfy*

$$\beta_1 = \tilde{\beta}_0 - \sigma_1 + b_1^{(1)},$$

$$\beta_{n+1} = \tilde{\beta}_n - \sigma_{n+1} + \sigma_n + \frac{b_{n+1}^{(n+1)}}{n+1}, \quad n \geqslant 1,$$

$$\gamma_{n+1} = \frac{n+1}{n}\left[\tilde{\gamma}_n + \sigma_n(\beta_n - \tilde{\beta}_n) + \frac{b_n^{(n+1)}}{n+1} - \sigma_n\frac{b_n^{(n)}}{n}\right], \quad n \geqslant 1,$$

and β_0, γ_1 *are arbitrary.*

Proof. Let $\{P_n\}_n$ be a MOPS. We can use both recurrence relations and the expansions of the polynomials $P_n(x)$ in terms of their first derivatives (3.1), to obtain

$$\begin{aligned} T_1(x) &= (x - \tilde{\beta}_0)T_0(x) = (x - \tilde{\beta}_0)P_1'(x) = (x - \beta_1)P_1'(x) + (\beta_1 - \tilde{\beta}_0)P_1'(x) \\ &= P_2'(x) - P_1(x) + (\beta_1 - \tilde{\beta}_0)P_1'(x) = \frac{P_2'(x)}{2} + \left(-b_1^{(1)} + \beta_1 - \tilde{\beta}_0\right)P_1'(x). \end{aligned}$$

By comparison with the coherence relation, for $n = 1$, we get

$$\beta_1 = \tilde{\beta}_0 - \sigma_1 + b_1^{(1)}. \tag{3.4}$$

If $n \geqslant 1$, we can use the same argument to deduce

$$\begin{aligned} T_{n+1}(x) = {} & \frac{P_{n+2}'(x)}{n+2} + \left[\beta_{n+1} - \tilde{\beta}_n - \sigma_n - \frac{b_{n+1}^{(n+1)}}{n+1}\right]\frac{P_{n+1}'(x)}{n+1} \\ & + \left[\frac{n}{n+1}\gamma_{n+1} - \sigma_n(\beta_n - \tilde{\beta}_n) - \tilde{\gamma}_n - \frac{b_n^{(n+1)}}{(n+1)} + \sigma_n\frac{b_n^{(n)}}{n}\right]\frac{P_n'(x)}{n} \\ & + \left[\tilde{\gamma}_n\sigma_{n-1} - \frac{n-1}{n}\gamma_n\sigma_n - \frac{b_{n-1}^{(n+1)}}{n+1} + \sigma_n\frac{b_{n-1}^{(n)}}{n}\right]\frac{P_{n-1}'(x)}{n-1} \\ & - \sum_{i=1}^{n-2}\left[\frac{b_i^{(n+1)}}{n+1} - \sigma_n\frac{b_i^{(n)}}{n}\right]\frac{P_i'(x)}{i}. \end{aligned}$$

By comparison with the coherence relation for $n+1$, $n \geqslant 1$, and including Eq. (3.4), the following system can be obtained

$$\beta_1 = \tilde{\beta}_0 - \sigma_1 + b_1^{(1)}, \tag{3.5}$$

$$\beta_{n+1} = \tilde{\beta}_n - \sigma_{n+1} + \sigma_n + \frac{b_{n+1}^{(n+1)}}{n+1}, \quad n \geqslant 1, \tag{3.6}$$

$$\gamma_{n+1} = \frac{n+1}{n}\left[\tilde{\gamma}_n + \sigma_n(\beta_n - \tilde{\beta}_n) + \frac{b_n^{(n+1)}}{n+1} - \sigma_n \frac{b_n^{(n)}}{n}\right], \quad n \geqslant 1, \tag{3.7}$$

$$\sigma_n \gamma_n = \frac{n}{n-1}\left[\sigma_{n-1}\tilde{\gamma}_n - \frac{b_{n-1}^{(n+1)}}{n+1} + \sigma_n \frac{b_{n-1}^{(n)}}{n}\right], \quad n \geqslant 2, \tag{3.8}$$

$$0 = \frac{b_i^{(n+1)}}{n+1} - \sigma_n \frac{b_i^{(n)}}{n}, \qquad 1 \leqslant i \leqslant n-2, \quad n \geqslant 2. \tag{3.9}$$

Now, we can substitute (3.6) in (3.7), and next in (3.8). Thus, dividing by $\sigma_n \sigma_{n-1}$

$$\frac{\tilde{\gamma}_{n+1}}{\sigma_{n+1}} + \tilde{\beta}_n + \sigma_n = \frac{\tilde{\gamma}_{n+2}}{\sigma_{n+2}} + \tilde{\beta}_{n+1} + \sigma_{n+1} - \frac{1}{\sigma_{n+2}\sigma_{n+1}} \frac{b_{n+1}^{(n+3)}}{n+3}, \quad n \geqslant 1, \tag{3.10}$$

holds. In a similar way, if we substitute (3.5) in (3.7) for $n=1$, and the result in (3.8), for $n=2$, if we divide by $\sigma_1\sigma_2$, we get

$$\frac{\tilde{\gamma}_1}{\sigma_1} + \tilde{\beta}_0 = \frac{\tilde{\gamma}_2}{\sigma_2} + \tilde{\beta}_1 + \sigma_1 - \frac{1}{\sigma_2\sigma_1}\frac{b_1^{(3)}}{3}. \tag{3.11}$$

Expressions (3.10) and (3.11) allow us to obtain

$$\frac{\tilde{\gamma}_1}{\sigma_1} + \tilde{\beta}_0 = \frac{\tilde{\gamma}_{n+1}}{\sigma_{n+1}} + \tilde{\beta}_n + \sigma_n - \sum_{k=1}^{n} \frac{1}{\sigma_{k+1}\sigma_k}\frac{b_k^{(k+2)}}{k+2}. \tag{3.12}$$

Hence, this expression does not depend on n and we will denote it by c. If we define the sequence

$$y_0 = 1, \qquad y_1 = \frac{\tilde{\gamma}_1}{\sigma_1} y_0,$$

$$y_{n+1} = \left(\frac{\tilde{\gamma}_{n+1}}{\sigma_{n+1}} - \sum_{k=1}^{n}\frac{1}{\sigma_{k+1}\sigma_k}\frac{b_k^{(k+2)}}{k+2}\right) y_n - \sum_{j=1}^{n-1}\prod_{k=j+1}^{n}\sigma_k \left(\sum_{i=1}^{j}\frac{1}{\sigma_{i+1}\sigma_i}\frac{b_i^{(i+2)}}{i+2}\right) y_j, \quad n \geqslant 1,$$

then Eq. (3.12) can be written in the following way

$$y_{n+1} = (c - \tilde{\beta}_n) y_n - \tilde{\gamma}_n y_{n-1}, \quad n \geqslant 0, \quad y_{-1} = 0.$$

Then, we obtain $y_n = T_n(c)$, $n \geqslant 0$.

Taking into account the expression for y_{n+1}, we can deduce that if

$$\tilde{\gamma}_{n+1} - \frac{1}{\sigma_n}\frac{b_n^{(n+2)}}{n+2} \neq 0, \quad n \geqslant 1, \quad \text{and} \quad y_n = T_n(c) \neq 0, \quad n \geqslant 0,$$

then σ_{n+1} can be computed in a recursive way by means of the expression (3.2). The values of the sequences $\{\beta_n\}_{n\geqslant 1}$ and $\{\gamma_n\}_{n\geqslant 2}$ are obtained from Eqs. (3.5)–(3.7). However, the values of β_0 and γ_1 are arbitrary.

Conversely, taking the parameter σ_1 in such a way that the number c, defined from (3.12), satisfies $T_n(c)\neq 0$, $n\geqslant 0$, for the values of the parameters $\{\sigma_n\}_{n\geqslant 1}$, $\{\beta_n\}_{n\geqslant 1}$ and $\{\gamma_n\}_{n\geqslant 2}$ given by relations (3.2), (3.5)–(3.7), we can easily show that the sequence of monic polynomials $\{P_n\}_n$, defined by (1.2), is a MOPS since it satisfies a three term recurrence relation. □

Remark. We must notice that a sufficient condition for hypothesis (3.3) to be fulfilled, is $b_n^{(n+2)}=0$, $n\geqslant 1$. Therefore, by using Eq. (3.9), we deduce $b_i^{(n+2)}=0$, $1\leqslant i\leqslant n$, and, thus

$$P_{n+2}(x)=\frac{P'_{n+3}(x)}{n+3}+b_{n+2}^{(n+2)}\frac{P'_{n+2}(x)}{n+2}+b_{n+1}^{(n+2)}\frac{P'_{n+1}(x)}{n+1},\quad n\geqslant 1.$$

In this way, we can conclude that the sequence $\{P_n\}_n$ is a classical MOPS, as a consequence of characterization (C.3) from Theorem 2.1.

By using the same type of arguments, we can solve the converse problem.

Theorem 3.2. *Let $\{P_n\}_n$ be a MOPS. Define a sequence $\{T_n\}_n$ of monic polynomials by mean of the coherence relation* (1.2) *where $\{\sigma_n\}_{n\geqslant 1}$ is a sequence of nonzero complex numbers. Then, $\{T_n\}_n$ is a MOPS if and only if*

$$\sigma_n^2\sigma_{n-1}+\sigma_n\sigma_{n-1}(\beta_n-\beta_{n+1})+\frac{n-1}{n}\sigma_n\gamma_n-\frac{n}{n+1}\sigma_{n-1}\gamma_{n+1}$$
$$+\sigma_n\sigma_{n-1}\frac{b_{n+1}^{(n+1)}}{n+1}+\sigma_{n-1}\left[\frac{b_n^{(n+1)}}{n+1}-\sigma_n\frac{b_n^{(n)}}{n}\right]+\frac{b_{n-1}^{(n+1)}}{n+1}-\sigma_n\frac{b_{n-1}^{(n)}}{n}\neq 0,\quad n\geqslant 2.$$

The parameters of the three term recurrence relation for the MOPS $\{T_n\}_n$, say $\{\tilde{\beta}_n\}_{n\geqslant 0}$ and $\{\tilde{\gamma}_n\}_{n\geqslant 1}$, are given by Eqs. (3.5)–(3.7).

4. Symmetrically coherent pairs

Obviously, when both functionals, u_0 and u_1, are symmetric, the coherence relation studied in the previous sections is meaningless. Thus, in the above mentioned work by Iserles et al. [6], they introduce the concept of *symmetric coherence*: two symmetric linear functionals $\{u_0,u_1\}$ constitute a *symmetrically coherent pair* if there exists a sequence of nonzero complex numbers $\sigma_1,\sigma_2,\ldots,$ satisfying

$$T_n(x)=\frac{P'_{n+1}(x)}{n+1}-\sigma_{n-1}\frac{P'_{n-1}(x)}{n-1},\quad n\geqslant 2. \tag{4.1}$$

In this situation, the linear functional u_1 is called a "symmetric companion" for u_0.

Let $\{P_n\}_n$ and $\{T_n\}_n$ be the MOPS associated to u_0 and u_1, respectively. Then, we have

$$P_n(-x)=(-1)^nP_n(x),\quad T_n(-x)=(-1)^nT_n(x),\quad \forall n\in\mathbb{N}.$$

and, in this way, (see [3]) we can write

$$P_{2n}(x) = U_n(x^2), \quad P_{2n+1}(x) = xU_n^*(x^2), \tag{4.2}$$

$$T_{2n}(x) = V_n(x^2), \quad T_{2n+1}(x) = xV_n^*(x^2). \tag{4.3}$$

Let v_0 and v_1 be the linear functionals defined by means of the relations

$$\langle v_0, x^n \rangle = \langle u_0, x^{2n} \rangle, \qquad \langle v_1, x^n \rangle = \langle u_1, x^{2n} \rangle, \quad n \geqslant 0, \tag{4.4}$$

then $\{U_n\}_n$ is the MOPS associated with v_0 and $\{U_n^*\}_n$ is the MOPS associated with xv_0. Analogously, $\{V_n\}_n$ is the MOPS associated with v_1 and $\{V_n^*\}_n$ is the MOPS associated with xv_1.

In the next theorem, we will show that symmetrically coherent pairs and coherent pairs are related by means of a symmetrization procedure. In fact, two symmetric MOPS constitute a symmetrically coherent pair if the even component of the first sequence and the odd component of the second sequence constitute a coherent pair.

Theorem 4.1. *Let $\{u_0, u_1\}$ be a symmetrically coherent pair. Then, with the above notations, $\{v_0, xv_1\}$ is a coherent pair, that is*

$$V_n^*(x) = \frac{U'_{n+1}(x)}{n+1} - \sigma_{2n}\frac{U'_n(x)}{n}, \quad \forall n \geqslant 1. \tag{4.5}$$

Moreover, the MOPS $\{U_n^*\}_n$ and $\{V_n\}_n$ are related in the following way:

$$V_n(x) = \frac{U_n^*(x)}{2n+1} - \sigma_{2n-1}\frac{U_{n-1}^*(x)}{2n-1} + 2x\left[\frac{(U_n^*)'(x)}{2n+1} - \sigma_{2n-1}\frac{(U_{n-1}^*)'(x)}{2n-1}\right], \quad \forall n \geqslant 1. \tag{4.6}$$

Conversely, if v_0, v_1, xv_0 and xv_1 are four regular linear functionals, let us suppose that the corresponding MOPS $\{U_n\}_n$, $\{V_n\}_n$, $\{U_n^*\}_n$ and $\{V_n^*\}_n$ are related by means of the conditions (4.5) *and* (4.6). *Then the sequences of polynomials $\{P_n\}_n$ and $\{T_n\}_n$ given by* (4.2) *and* (4.3), *constitute a symmetrically coherent pair.*

Proof. It suffices to substitute (4.2) and (4.3) into the symmetric coherence relation for the even and odd polynomials. □

The symmetric coherence relation for the MOPS gives an explicit relation between the corresponding linear functionals. The next result will be proved using a technique based in the properties of duality (see [8, 13]).

Since $\{P_n\}_n$ is a basis for $\mathbb{P}$, we can associate the corresponding dual basis $\{u_n^{(0)}, n \in \mathbb{N}\}$, where $u_n^{(0)}$ is the linear functional such that

$$\langle u_n^{(0)}, P_m \rangle = \delta_{n,m}.$$

In the same way, let $\{u_n^{(1)}\}_n$ be the dual basis associated to the MOPS $\{T_n\}_n$. On the other hand, the family of polynomials defined by $\{\tilde{P}_n = P'_{n+1}/n+1\}_n$, is also a basis for $\mathbb{P}$, with $\deg(\tilde{P}_n) = n$. Then, we can associate to it a dual basis $\{\tilde{u}_n^{(0)}, n \in \mathbb{N}\}$.

In this way, the definition of *symmetric coherent pairs* can be given in terms of the dual basis:

Lemma 4.2. *Let $\{u_0, u_1\}$ be a symmetric coherent pair. Then*

(i) $\tilde{u}_n^{(0)} = u_n^{(1)} - \sigma_{n+1} u_{n+2}^{(1)}, \quad n \geqslant 0$ (4.7)

(ii) $(n+1)u_{n+1}^{(0)} = \sigma_{n+1} D u_{n+2}^{(1)} - D u_n^{(1)}, \quad n \geqslant 0.$ (4.8)

Equivalently, (ii) *can be expressed as*:

$$(n+1)\frac{P_{n+1}(x)}{\langle u_0, P_{n+1}^2\rangle} u_0 = \sigma_{n+1} D\left[\frac{T_{n+2}(x)}{\langle u_1, T_{n+2}^2\rangle} u_1\right] - D\left[\frac{T_n(x)}{\langle u_1, T_n^2\rangle} u_1\right], \quad n \geqslant 0. \tag{4.9}$$

Proof. (i) It suffices to write

$$\tilde{u}_n^{(0)} = \sum_{m=0}^{\infty} \lambda_m^{(n)} u_m^{(1)}, \quad n \geqslant 0,$$

where

$$\lambda_m^{(n)} = \langle \tilde{u}_n^{(0)}, T_m\rangle = \langle \tilde{u}_n^{(0)}, \tilde{P}_m - \sigma_{m-1}\tilde{P}_{m-2}\rangle = \begin{cases} 1, & \text{if } m = n, \\ -\sigma_{n+1}, & \text{if } m = n+2, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Taking derivatives in the above expression, we get

$$D(\tilde{u}_n^{(0)}) = D(u_n^{(1)}) - \sigma_{n+1} D(u_{n+2}^{(1)}).$$

Finally, it is very easy to prove that $D(\tilde{u}_n^{(0)}) = -(n+1)u_{n+1}^{(0)}$. □

By using the same reasoning as in [11], we can deduce that if $\{u_0, u_1\}$ constitute a symmetrically coherent pair, both functionals are semiclassical and they are related by means of a rational expression. In fact,

Proposition 4.3. *Let $\{u_0, u_1\}$ be a symmetrically coherent pair. Then*

(i) *u_1 is a semiclassical functional of class less than or equal to* 2, *i.e., there exist two polynomials $\phi_1(x), \psi_1(x)$, with degrees less than or equal to* 4 *and* 3, *respectively, such that*

$$D(\phi_1 u_1) = \psi_1 u_1, \tag{4.10}$$

whose explicit expressions are

$$\phi_1(x) = 2\frac{P_2(x)}{\langle u_0, P_2^2\rangle} C_2(x) - \frac{P_1(x)}{\langle u_0, P_1^2\rangle} C_3(x), \qquad \psi_1(x) = 2\frac{P_2'(x)}{\langle u_0, P_2^2\rangle} C_2(x) - \frac{P_1'(x)}{\langle u_0, P_1^2\rangle} C_3(x),$$

where

$$C_n(x) = \sigma_{n-1}\frac{T_n(x)}{\langle u_1, T_n^2\rangle} - \frac{T_{n-2}(x)}{\langle u_1, T_{n-2}^2\rangle}, \quad n \geqslant 2.$$

(ii) *There exist two polynomials* $A_4(x)$ *and* $B_4(x)$ *with degrees less than or equal to* 4, *such that*:

$$A_4(x)u_0 = B_4(x)u_1,$$

defined by

$$A_4(x) = \phi_1(x), \qquad B_4(x) = C_2(x)C_3'(x) - C_2'(x)C_3(x).$$

(iii) u_0 *is also a semiclassical functional, of class at most* 10, *since it satisfies the distributional differential equation*

$$D(\phi_0 u_0) = \psi_0 u_0,$$

where $\phi_0(x)$ *and* $\psi_0(x)$ *are polynomials with degrees less than or equal to* 12 *and* 11, *respectively, whose expressions are given by*

$$\phi_0(x) = \phi_1^2(x)B_4(x), \quad \psi_0(x) = \left[2\phi_1(x)B_4'(x) + \psi_1(x)B_4(x)\right]\phi_1(x).$$

Proof. (i) From (4.9), for $n = 0$ and $n = 1$, we deduce the following system of linear equations:

$$\begin{aligned} &\frac{P_1(x)}{\langle u_0, P_1^2\rangle}u_0 = C_2'(x)u_1 + C_2(x)Du_1, \\ &2\frac{P_2(x)}{\langle u_0, P_2^2\rangle}u_0 = C_3'(x)u_1 + C_3(x)Du_1. \end{aligned} \qquad (4.11)$$

And it suffices to solve the previous system for u_1 and Du_1, to get the announced result.

(ii) It suffices to eliminate Du_1 in the above system.

(iii) From (i) and (ii), we get

$$\begin{aligned} D(\phi_1^2 B_4 u_0) &= D(\phi_1 B_4^2 u_1) = 2B_4B_4'\phi_1 u_1 + B_4^2 D(\phi_1 u_1) \\ &= 2B_4B_4'\phi_1 u_1 + B_4^2\psi_1 u_1 = \left[2\phi_1 B_4' + \psi_1 B_4\right]\phi_1 u_0. \qquad \square \end{aligned}$$

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