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Cohomology of Inverse Semigroups

HANS LAUSCH

Department of Mathematics, Monash University, Clayton, Australia 3168

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1. INTRODUCTION

D'Alarcao [3] and Coudron [2] investigated the following problem: Given a semilattice G of groups and an inverse semigroup S, what are the inverse semigroups U such that there is an idempotent separating surjective homomorphism from U to S with G as its kernel normal system? Their answer came out in terms of a certain action of S on G and a "factor system" condition, similar to the classical case of group extensions, but naturally more involved. Whereas Eilenberg and MacLane [5] could phrase the theory of group extensions in terms of cohomology theory, the corresponding extension problem for inverse semigroups was somehow left in the "wilderness," similar to Schreier's original paper [8] on group extensions. Only for a very special situation, cohomological notions have been introduced [9]. The purpose of this paper is to provide a cohomological framework for inverse semigroups which will not only fit the extension problem, but also discuss some apparently new notions such as complementation and inner automorphism for inverse semigroups.

In Section 2 we introduce the category of S-modules for inverse semigroups S: an inverse semigroup S is represented as a semigroup of certain endomorphisms of a semilattice A of abelian groups. Sections 3 and 4 are devoted to the free, projective, and injective objects in this category. In Section 5 we apply some general results of cohomology theory for abelian categories to the category of S-modules, and in Section 6 we set up various projective resolutions of a standard S-module $\mathbf{Z}_S$. Section 7 links d'Alarcao's and Coudron's results with cohomology theory for the case where G consists of abelian groups. It is interesting to note that one has to introduce a dummy identity element in S to tackle the extension problem. Theorem 7.5 is an improvement on d'Alarcao's and Coudron's results insofar as "factor systems" need not be defined on the whole of $S \times S$ but just a certain subset, in order to determine a unique extension. This fact can be neatly expressed in terms

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of a certain chain homotopy. Section 8 studies certain endomorphisms of semilattices of groups, generalizing group automorphisms and the group of outer automorphisms of a group. Section 9 provides an obstruction theory for extensions using the notions of Section 8, and interprets the third cohomology group for inverse semigroups. Section 10 develops the notions of an inner automorphism of an inverse semigroup and complementation of kernel normal systems which are applied in Section 11 to interpret first cohomology groups.

Two problems arise: (1) What are the semilattices A of abelian groups on which an inverse semigroup S can be represented, i.e., make A into an S-module? (2) Using left satellites of the first cohomology group of S, what are the conditions determining those S which admit a homology theory that can be interpreted as cohomology in negative dimensions (as in the case of finite groups)? We hope to attack these questions in another paper. For definitions and theorems concerning inverse semigroups, the reader is referred to [1].

Finally the author wants to thank Dr. T. E. Hall, Professor R. McFadden, and Professor G. B. Preston for their advice resulting from many helpful discussions.

2. The Category of S-Modules

Definition. Let S be an inverse semigroup and $E(S)$ its semilattice of idempotents. A semilattice A of (additively written) abelian groups together with a map $A \times S \to A$, denoted by $(a, s) \to as$, $a \in A$, $s \in S$ is called an *S-module* if

(i) there is an isomorphism θ from $E(S)$ to $E(A)$;

(ii) $(a_1 + a_2)\, s = a_1 s + a_2 s$, for all $a_1, a_2 \in A$, $s \in S$;

(iii) $a(s_1 s_2) = (a s_1)\, s_2$, for all $a \in A$, $s_1, s_2 \in S$;

(iv) $ae = a + e\theta$, for all $a \in A$, $e \in E(S)$;

(v) $(e\theta)\, s = (s^{-1} e s)\, \theta$, for all $e \in E(S)$, $s \in S$.

For $e\theta$ we will write 0_e.

Definition. If S an inverse semigroup, A and B are S-modules, then a map α: $A \to B$ is called an *S-morphism* if

(i) $(a_1 + a_2)\, \alpha = a_1\alpha + a_2\alpha$, for all $a_1, a_2 \in A$,

(ii) $(a\alpha)\, s = (as)\, \alpha$, for all $a \in A$, $s \in S$;

(iii) $0_e \alpha = 0_e$, for all $e \in E(S)$.

The set of all S-morphisms will be denoted by $\mathrm{Hom}_S(A, B)$. $\mathrm{Hom}_S(A, B)$ becomes an abelian group by $a(\alpha + \beta) = a\alpha + a\beta$, $a \in A$, $\alpha, \beta \in \mathrm{Hom}_S(A, B)$, $a(-\alpha) = -a\alpha$, $a \in A$, $\alpha \in \mathrm{Hom}_S(A, B)$, and the class of S-modules together with the sets $\mathrm{Hom}_S(A, B)$ form a category, denoted by $\mathrm{Mod}(S)$. If A is an S-module, $e \in E(S)$, then A_e will denote the set $\{a \in A \mid a - a = 0_e\}$. Clearly A_e is an abelian group. If $\alpha: A \to B$ is an S-morphism, then $\ker \alpha = \{a \in A \mid a\alpha = 0_e, \text{ if } a \in A_e\}$ is an S-submodule of A, called the *kernel* of α, and if B is a submodule of A, then $A/B = \{a + B_e \mid a \in A_e, e \in E(S)\}$ is called the *factor module* of A mod B if we define, for $e, e_1, e_2 \in E(S)$, $(a_1 + B_{e_1}) + (a_2 + B_{e_2}) = (a_1 + a_2) + B_{e_1 e_2}$, $a_1, a_2 \in A$, $(a + B_e)\, s = as + B_{s^{-1}es}$, $a \in A$, $s \in S$. A/B is an S-module and $a \to a + B_e$, $a \in A_e$, defines an S-morphism from A to A/B. We find easily, that $\mathrm{Mod}(S)$ is an abelian category. The zero object of $\mathrm{Mod}(S)$ is $E(S)$, additively written, with $(e, s) \to s^{-1}es$. Direct sums $A \oplus B$ in $\mathrm{Mod}(S)$ are given by $(A \oplus B)_e = A_e \oplus B_e$ with $(a, b)\, s = (as, bs)$, $a \in A_e$, $b \in B_e$, $s \in S$.

3. Free S-Modules

S will always denote an inverse semigroup.

Definition. Let Λ be a semilattice. A *Λ-set* is a disjoint union $T = \bigcup \{T_\lambda \mid \lambda \in \Lambda\}$ of sets T_λ, and if $T = \bigcup T_\lambda$ and $U = \bigcup U_\lambda$ are Λ-sets, a map $\alpha: T \to U$ with $T_\lambda \alpha \subseteq U_\lambda$ is called a *Λ-map*. The Λ-sets together with the Λ-maps form a category denoted by Set_Λ.

Remark. Every S-module A is an $E(S)$-set as $A = \bigcup \{A_e \mid e \in E(S)\}$ and every S-morphism is an $E(S)$-map.

Definition. An S-module F is said to be *free* over a subset $T \subseteq F$ if

(i) T is an $E(S)$-subset generating F, and

(ii) every $E(S)$-map from T to any S-module A extends uniquely to an S-morphism from F to A.

Proposition 3.1. *For every $E(S)$-set T there exists an S-module F which is free over a subset $\tilde{T}$ of F such that T and $\tilde{T}$ are isomorphic in the category $\mathrm{Set}_{E(S)}$.*

Proof. For any $e \in E(S)$, define F_e to be the abelian group freely generated by the pairs (t, s), where $s^{-1}s = e$, $t \in T_f$, for some $f \in E(S)$ such that $ss^{-1} \leqslant f$.

We define an addition and S-action on $\bigcup F_e$ by:

$$\sum_{s^{-1}s=e} n_{t,s}(t, s) + \sum_{s_1^{-1}s_1=e_1} n_{t_1,s_1}(t_1, s_1)$$

$$= \sum_{s^{-1}s=e} n_{t,s}(t, se_1) + \sum_{s_1^{-1}s_1=e_1} n_{t_1,s_1}(t_1, s_1e) \qquad \text{for} \quad e, e_1 \in E(S)$$

$$\sum_{s^{-1}s=e} n_{t,s}(t, s)s_1 = \sum_{s^{-1}s=e} n_{t,s}(t, ss_1), \qquad \text{for} \quad e \in E(S), \quad s_1 \in S,$$

where the sums are finite sums, $n_{t,s}$, $n_{t_1,s_1} \in \mathbf{Z}$; this definition makes F into an S-module. Let A be an S-module and α: $\{(t, e) \mid t \in T_e, e \in E(S)\} \to A$ an $E(S)$-map. Then, for $t \in T_f$, $f \in E(S)$, $\sum_{s^{-1}s=e} n_{t,s}(t, s)\,\psi = \sum_{s^{-1}s=e} n_{t,s}(t, f)\,\alpha s$ is an S-morphism from F to A extending α and is the only such extension. If we put $\tilde{T} = \{(t, e) \mid t \in T_e, e \in E(S)\}$, the map $t \to (t, e)$ is a bijection from T to $\tilde{T}$. Moreover $\tilde{T}$ generates F as an S-module.

DEFINITION. We say F is *freely generated* by T.

COROLLARY 3.2. *Free S-modules are projective in* Mod(S).

COROLLARY 3.3. *Every S-module is a homomorphic image of a free S-module.*

4. INJECTIVE S-MODULES

The purpose of this section is to show that Mod(S) has enough injectives.

DEFINITION. $\mathbf{Z}S$ denotes the S-module defined by $(\mathbf{Z}S)_e$ = abelian group freely generated by the symbols (s), where $s \in S$, $s^{-1}s = e$ with the operations defined by

$$\sum_{s^{-1}s=e} n_s(s) + \sum_{t^{-1}t=e_1} n_t(t) = \sum_{s^{-1}s=e} n_s(se_1) + \sum_{t^{-1}t=e_1} n_t(te), \qquad e, e_1 \in E(S)$$

$$\Big(\sum_{s^{-1}s=e} n_s(s)\Big) s_1 = \sum_{s^{-1}s=e} n_s(ss_1), \qquad e \in E(S), \quad s_1 \in S, \quad n_s \in \mathbf{Z}.$$

LEMMA 4.1. *Let J be an S-module. Then J is injective if and only if, for every S-submodule I of $\mathbf{Z}S$, every S-morphism from I to J extends to an S-morphism from $\mathbf{Z}S$ to J.*

Proof. The "only if" part is obvious. Suppose that A is an S-submodule of an S-module B, and ϕ: $A \to J$ an S-morphism. By Zorn's Lemma, we find an S-submodule A_0 of B containing A and an S-morphism $\phi_0 : A_0 \to J$ extending ϕ which does not extend to an S-morphism $\phi_1 : A_1 \to J$, for some S-submodule A_1 of B containing A_0 properly. Suppose $A_0 < B$. Then there exists $b \in B \backslash A_0$. Let C be the S-submodule of B generated by $\{A_0, b\}$ and suppose $b \in B_e$. Then $C_f = A_f + \langle bs \mid s \in S, s^{-1}es = f \rangle$, $f \in E(S)$. (By $\langle \varnothing \rangle$ we mean the trivial abelian group.) Define $I = E(\mathbf{Z}S) \cup \{\sum n_s(es) \mid s^{-1}es = f, f \in E(S), \sum n_s bs \in A_0\}$. Then I is an S-submodule of $\mathbf{Z}S$. Let α: $I \to A_0$ be the S-morphism defined by $(\sum_{s^{-1}es=f} n_s(es))\,\alpha = \sum_{s^{-1}es=f} n_s bs$. Then $\alpha\phi_0 : I \to J$ extends to an S-morphism ψ: $\mathbf{Z}S \to J$. This allows us to define an S-morphism χ: $C \to J$, by $(a + \sum_{s^{-1}es=f} n_s bs)\,\chi = a\phi_0 + \sum n_s(es)\,\psi$, which is well-defined as ψ extends $\alpha\phi_0$. Hence $A_0 = B$, and J is injective.

THEOREM 4.2. *Every S-module A can be embedded into an injective S-module.*

Proof. Let L be the direct sum of the S-modules $(\mathbf{Z}S)_\alpha$, $\alpha \in \mathrm{Hom}_S(I, A)$, I running through all S-submodules of $\mathbf{Z}S$, and $(\mathbf{Z}S)_\alpha \cong \mathbf{Z}S$. The element of $(\mathbf{Z}S)_\alpha$ corresponding to $\sum n_{s,\alpha}(s)$ in $\mathbf{Z}S$ will be denoted by $\sum n_{s,\alpha}(s, \alpha)$. Let K be the S-submodule of $A \oplus L$ generated by $\{(\sum n_{s,\alpha}(s)\,\alpha, -\sum n_{s,\alpha}(s, \alpha)) \mid \sum n_{s,\alpha}(s) \in I, \alpha \in \mathrm{Hom}_S(I, A), I$ an S-submodule of $\mathbf{Z}S\}$. Let $D(A) = (A \oplus L)/K$. Then $a \to (a, 0_e) + K$ is an embedding of A into $D(A)$: for suppose $(a, 0_e) \in K$, then $-\sum_{s,\alpha} n_{s,\alpha}(s, \alpha) = 0_e$, $\sum_{s,\alpha} n_{s,\alpha}(s)\,\alpha = a$, for suitable $n_{s,\alpha} \in \mathbf{Z}$, $a \in A_e$, $s^{-1}s = e$. But L_e is an abelian group freely generated by the elements (s, α) hence $n_{s,\alpha} = 0$, for all pairs s, α, whence $a = 0_e$. Moreover, every $\alpha \in \mathrm{Hom}_S(I, A)$ extends to an S-morphism $\tilde{\alpha}$: $\mathbf{Z}S \to D(A)$ by defining $(s)\,\tilde{\alpha} = (0_e, (s, \alpha)) + K$. Let ν be the least infinite ordinal whose cardinal is larger than that of $\mathbf{Z}S$. We define $D_1(A) = D(A)$, $D_{\beta+1}(A) = D(D_\beta(A))$, $D_\lambda(A) = \bigcup_{\mu<\lambda} D_\mu(A)$, if λ is a limiting ordinal. Then $D_\nu(A)$ is injective as, by the choice of ν, the image of every S-submodule I of $\mathbf{Z}S$ under any S-morphism α: $I \to D_\nu(A)$ is contained in some $D_\beta(A)$, $\beta < \nu$, and hence α extends to an S-morphism $\tilde{\alpha}$: $\mathbf{Z}S \to D_{\beta+1}(A) \subseteq D_\nu(A)$. Moreover, A can be embedded into $D_\nu(A)$.

Remark. The proof of this theorem copies exactly the construction of [4, p. 9].

5. CONSTRUCTION OF A COHOMOLOGY FUNCTOR ON MOD(S)

This section is devoted to the construction of a cohomology functor H_S from Mod(S) to the category of abelian groups which is characterized by the following properties: if $A \in$ Mod(S), then

(i) $H_S^i(A) = 0$, for $i < 0$,

(ii) $H_S^i(J) = 0$, if J is injective, $i \in \mathbf{Z}$,

(iii) $H_S^0(A) = \{\delta: E(S) \to A \mid \delta$ a map with $(e\delta)\, s = (s^{-1}es)\,\delta$, $e \in E(S)$, $s \in S\}$ and "pointwise" addition.

Note that in the case where S is a group, H_S is just the ordinary cohomology functor. As Mod(S) has enough injectives, by the uniqueness theorem ([6, p. 5]) for cohomology functors on abelian categories, there exists at most one such functor on Mod(S). We will establish the existence of H by standard methods [6, p. 25]:

LEMMA 5.1. *If $\mathfrak{A}$ and $\mathfrak{B}$ are two abelian categories and $Y: \mathfrak{A} \to$ chain complexes over $\mathfrak{B}$ an exact functor, then there exists a cohomological functor H from $\mathfrak{A}$ to $\mathfrak{B}$ such that $H^i(A) =$ ith homology of $Y(A)$.*

LEMMA 5.2. *If P is a projective S-module, J an injective S-module, and $0 \to A \to B \to C \to 0$ an exact sequence in* Mod(S), *then*

$$0 \to \mathrm{Hom}_S(P, A) \to \mathrm{Hom}_S(P, B) \to \mathrm{Hom}_S(P, C) \to 0$$

and

$$0 \to \mathrm{Hom}_S(C, J) \to \mathrm{Hom}_S(B, J) \to \mathrm{Hom}_S(A, J) \to 0$$

are exact sequences of abelian groups.

Proof. This is true for any abelian category.

COROLLARY 5.3. *If X is a chain complex of projective S-modules, then $A \to \mathrm{Hom}_S(X, A)$ is an exact functor from* Mod(S) *to the category of chain complexes over the category of abelian groups.*

COROLLARY 5.4. *If X is a projective resolution of $B \in$* Mod(S) *then the cohomology functor arising from Lemma* 5.1 *is trivial on the injective S-modules in positive dimensions.*

Hence all we have to do, is to find $B \in$ Mod(S) such that $\mathrm{Hom}_S(B, A)$ and $H_S^0(A)$ as defined at the beginning of this section are naturally isomorphic.

DEFINITION. By $\mathbf{Z}_S$ we will denote the S-module defined by $(\mathbf{Z}_S)_e \cong \mathbf{Z}$, the additive group of integers, whose elements are the integers labelled by $e \in E(S)$, such that

$$n_e + m_f = (n + m)_{ef}, \qquad n, m \in \mathbf{Z}, \quad e, f \in E(S)$$

$$n_e s = n_{s^{-1}es}, \qquad n \in \mathbf{Z}, \quad e \in E(S), \quad s \in S.$$

PROPOSITION 5.5. *$\mathrm{Hom}_S(\mathbf{Z}_S, A) \cong \{\delta: E(S) \to A \mid \delta$ a map with $e\delta \in A_e$ and $(e\delta)\, s = (s^{-1}es)\, \delta,\ e \in E(S), s \in S\}$ and there is a natural isomorphism between these groups.*

Proof. Let $\alpha \in \mathrm{Hom}_S(\mathbf{Z}_S, A)$ and denote the group on the right-hand side of $\cong$ by A^S. Define $\theta: \mathrm{Hom}_S(\mathbf{Z}_S, A) \to A^S$ by $e(\alpha\theta) = 1_e\alpha$. Then $\alpha\theta \in A^S$ as $[e_1(\alpha\theta)]\, e_2 = (1_{e_1}\alpha)\, e_2 = (1_{e_1}e_2)\, \alpha = 1_{e_1e_2}\alpha = (e_1e_2)(\alpha\theta)$, and $(ss^{-1})(\alpha\theta)\, s = 1_{ss^{-1}}\alpha s = (1_{ss^{-1}}s)\, \alpha = 1_{s^{-1}ss^{-1}s}\alpha = 1_{s^{-1}s}\alpha = (s^{-1}s)(\alpha\theta)$. Define $\theta^{-1}: A^S \to \mathrm{Hom}_S(\mathbf{Z}_S, A)$ by $n_e(\delta\theta^{-1}) = n(e\delta)$. Then $n_e(\delta\theta^{-1})\, s = n(e\delta)\, s = n(s^{-1}es)\, \delta = n_{s^{-1}es}(\delta\theta^{-1}) = (n_e s)(\delta\theta^{-1})$. Hence $\delta\theta^{-1} \in \mathrm{Hom}_S(\mathbf{Z}_S, A)$. Clearly, θ and θ^{-1} are homomorphisms and are the inverses of one another. The naturality of θ follows immediately.

COROLLARY 5.6. *There exists a cohomological functor, that we shall denote by H_S, from* Mod(S) *to the category of abelian groups satisfying conditions* (i)–(iii).

6. COMPUTATION OF $H_S{}^i(A)$

By the results of the previous section, we have to construct a projective resolution of $\mathbf{Z}_S$. Let $T_i(S) = \bigcup_{e\in E(S)} (T_i(S))_e$ be the $E(S)$-set defined by $T_i(S)_e = \{[f, s_1, s_2, ..., s_i] \mid s_1s_1^{-1} \leqslant f \in E(S),\quad s_rs_r^{-1} \leqslant s_{r-1}^{-1}s_{r-1},\quad r = 2,..., i,\ s_i^{-1}s_i = e,\quad s_r \in S,\quad r = 1,..., i\}$ for $i \geqslant 1$ and $T_0(S)_e = \{[e] \mid e \in E(S)\}$. $B_i(S)$ will denote the S-module freely generated by $T_i(S)$. Next we define S-morphisms $\partial_i: B_i(S) \to B_{i-1}(S)$, $i \geqslant 1$, by the values of the $E(S)$ maps

$$\tilde{\partial}_i: T_i(S) \to B_{i-1}(S): [f, s_1, s_2, ..., s_i]\tilde{\partial}_i$$

$$= ([f, s_1, s_2, ..., s_{i-1}], s_i) + \sum_{j=2}^{i} (-1)^{i-j+1}([f, s_1, ..., s_{j-1}s_j, ..., s_i], s_i^{-1}s_i)$$

$$+ (-1)^i([s_1^{-1}s_1, s_2, ..., s_i], s_i^{-1}s_i), \qquad \text{for} \quad i \geqslant 1.$$

Moreover, we define an S-morphism $\epsilon: B_0(S) \to \mathbf{Z}_S$ by the values of the $E(S)$-map $\tilde{\epsilon}: T_0(S) \to \mathbf{Z}_S : [e]\, \tilde{\epsilon} = 1_e,\ e \in E(S)$.

PROPOSITION 6.1. $\cdots \to B_i(S) \to^{\partial_i} B_{i-1}(S) \to \cdots \to B_1(S) \to^{\partial_1} B_0(S) \to^{\epsilon} \mathbf{Z}_S$ *is a projective resolution of the S module $\mathbf{Z}_S$.*

Proof. Let $(L_i)_e = \{([f, s_1, ..., s_i], t) \mid tt^{-1} \leqslant s_i^{-1}s_i,\ t^{-1}t = e,\ [f, s_1, ..., s_i] \in T_i(S)_{s_i^{-1}s_i}\}$, $e \in E(S)$; hence $(L_i)_e$ generates $B_i(S)_e$ freely as an abelian group. Then $L_i = \bigcup_{e\in E(S)} (L_i)_e$ is an $E(S)$-set. We define $E(S)$-maps $\tilde{\sigma}_i: L_i \to B_{i+1}(S)$, $i \geqslant 0$, by $([f, s_1, ..., s_i], t)\, \tilde{\sigma}_i = ([f, s_1, ..., s_i, t], t^{-1}t)$. Then $\tilde{\sigma}_i$ extends uniquely to an $E(S)$-map $\sigma_i : B_i(S) \to B_{i+1}(S)$ such that $\sigma_i \mid (B_i(S))_e$ are group homomorphisms. Furthermore we define an $E(S)$-map $\tau: \mathbf{Z}_S \to B_0(S)$ by $n_e\tau = n([e], e)$, $e \in E(S)$, then $\tau \mid (\mathbf{Z}_S)_e$ are group homomorphisms. We will show: $\tau\epsilon = id_{\mathbf{Z}S}$, $\epsilon\tau + \sigma_0\partial_1 = id_{B_0(S)}$, $\partial_i\sigma_{i-1} + \sigma_i\partial_{i+1} = id_{B_i(S)}$, $i \geqslant 1$. Once these equalities of $E(S)$-maps are established, a standard argument [7, p. 115] proves that $\to B_i(S) \to^{\partial_i} B_{i-1}(S) \to \cdots \to B_1(S) \to^{\partial_1} B_0 \to^{\epsilon} \mathbf{Z}_S \to 0$ is exact. We have $1_e\tau\epsilon = ([e], e)\,\epsilon = 1_e$; furthermore if $([f], s) \in B_0(S)$, then

$$\begin{aligned}([f], s)(\epsilon\tau + \sigma_0\partial_1) &= 1_{s^{-1}fs}\tau + ([f, s], s^{-1}s)\partial_1 \\ &= ([s^{-1}fs], s^{-1}fs) + ([f], s) - ([s^{-1}s], s^{-1}s) \\ &= ([f], s), \qquad \text{as} \quad ss^{-1} \leqslant f.\end{aligned}$$

Let $([f, s_1, ..., s_i], s) \in B_i(S)$. Then

$$\begin{aligned}&([f, s_1, ..., s_i], s)\, \partial_i\sigma_{i-1} \\ &\quad = ([f, s_1, ..., s_{i-1}], s_is)\sigma_{i-1} + \sum_{j=2}^{i} (-1)^{i-j+1} \cdot ([f, ..., s_{j-1}s_j, ..., s_i], s)\sigma_{i-1} \\ &\qquad + (-1)^i([s_1^{-1}s_1, s_2, ..., s_i], s)\sigma_{i-1} \\ &\quad = ([f, s_1, ..., s_{i-1}, s_is], s^{-1}s) + \sum_{j=2}^{i} (-1)^{i-j+1}([f, ..., s_{j-1}s_j, ..., s_i, s], s^{-1}s) \\ &\qquad + (-1)^i([s_1^{-1}s_1, s_2, ..., s_i, s], s^{-1}s).\end{aligned}$$

On the other hand,

$$\begin{aligned}([f, s_1, ..., s_i], s)\, \sigma_i\partial_{i+1} &= ([f, s_1, ..., s_i, s], s^{-1}s)\partial_{i+1} \\ &= ([f, s_1, ..., s_i], s) - ([f, s_1, ..., s_is], s^{-1}s) \\ &\quad + \sum_{j=2}^{i} (-1)^{i-j}([f, ..., s_{j-1}s_j, ..., s_i, s], s^{-1}s).\end{aligned}$$

Hence $\partial_i\sigma_{i-1} + \sigma_i\partial_{i+1} = id_{B_i}$, Q.E.D.

We may therefore use this resolution for computing $H^i(A)$, for $A \in \mathrm{Mod}(S)$. For practical purposes, however, we construct another resolution of $\mathbf{Z}_S$ for the case where S has an identity element 1:

Let $V_i(S) = \bigcup_{e \in E(S)} V_i(S)_e$ be the $E(S)$-set defined by

$$V_i(S)_e = \{[s_1, ..., s_i] \mid s_r \in S,\ r = 1, ..., i,\ s_i^{-1}s_{i-1}^{-1} \cdots s_1^{-1}s_1 \cdots s_i = e\}, \qquad i \geqslant 1$$

$C_i(S)$ will denote the S-module freely generated by $V_i(S)$, for $i \geqslant 1$, and $C_0(S) = \mathbf{Z}S$. We need the following.

LEMMA 6.2. *If S has an identity element* 1, *then* $\mathbf{Z}S$ *is a free S-module.*

Proof. Let $V_0(S)_e = \varnothing$ if $e \neq 1$, $V_0(S)_1 = \{[\ ]\}$, a one-point set. Then $V_0(S) = \bigcup_{e \in E(S)} V_0(S)_e$ is an $E(S)$-set. We define an $E(S)$-map $\tilde{\lambda}$: $V_0(S) \to \mathbf{Z}S$ by $[\ ]\tilde{\lambda} = (1)$. Then $\tilde{\lambda}$ extends uniquely to an S-morphism λ from the S-module F which is free on $V_0(S)$ to $\mathbf{Z}S$. Conversely define a map $\mu: \mathbf{Z}S \to F$ by $(s)\mu = ([\ ], s)$. μ is an S-morphism, and λ and μ are inverses of one another.

Next we define S-morphisms $\partial_i : C_i(S) \to C_{i-1}(S)$, $i \geqslant 2$ by the values of the $E(S)$-maps

$$\begin{aligned}\tilde{\partial}_i\colon V_i(S) \to C_{i-1}(S)\colon [s_1, \dots, s_i]\tilde{\partial}_i &= ([s_1, \dots, s_{i-1}], s_{i-1}^{-1} \cdots s_1^{-1}s_1 \cdots s_{i-1}s_i) \\ &\quad + \sum_{j=2}^{i} (-1)^{i-j+1}([s_1, \dots, s_{j-1}s_j, \dots, s_i], s_i^{-1} \cdots s_1^{-1}s_1 \cdots s_i) \\ &\quad + (-1)^{i-j+1}([s_2, \dots, s_i], s_i^{-1} \cdots s_1^{-1}s_1 \cdots s_i);\end{aligned}$$

$\partial_1 : C_1(S) \to C_0(S)$ is defined by the values of the $E(S)$-map $\tilde{\partial}_1 : V_1(S) \to C_0(S)$: $[s]\,\tilde{\partial}_1 = (s) - (s^{-1}s)$, and ϵ: $C_0(S) \to \mathbf{Z}_S$ by $(s)\,\epsilon = 1_{s^{-1}s}$. Again we prove

PROPOSITION 6.3. $\to C_i(S) \to^{\partial_i} C_{i-1}(S) \to \cdots \to C_1(S) \to^{\partial_i} C_0(S) \to^{\epsilon} \mathbf{Z}_S \to 0$ *is an exact sequence of S-modules.*

Proof. As before. Let $(M_i)_e = \{([s_1, \dots, s_i], t) \mid tt^{-1} \leqslant s_i^{-1} \cdots s_1^{-1}s_1 \cdots s_i,\ t^{-1}t = e\}$, where $e \in E(S)$, $i \geqslant 1$. Then $(M_i)_e$ generates $C_i(S)_e$ freely as an abelian group and $M_i = \bigcup_{e \in E(S)} (M_i)_e$ is an $E(S)$-set. We define $E(S)$-maps $\tilde{\sigma}_i : M_i \to C_{i+1}(S)$, $i \geqslant 1$, by

$$([s_1, \dots, s_i], t)\tilde{\sigma}_i = ([s_1, \dots, s_i, t], t^{-1}s_i^{-1} \cdots s_1^{-1}s_1 \cdots s_i t).$$

Then $\tilde{\sigma}_i$ extends uniquely to an $E(S)$-map $\sigma_i : C_i(S) \to C_{i+1}(S)$ such that $\sigma_i \mid C_i(S)_e$ are group homomorphisms. Additionally we define an $E(S)$-map $\sigma_0 : C_0(S) \to C_1(S)$ by $(s)\,\sigma_0 = ([s], s^{-1}s)$. Then $\sigma_0 \mid C_0(S)_e$ is a group homomorphism. Moreover $\tau: \mathbf{Z}_S \to C_0(S)$ defined by $n_e\tau = n(e)$ is an $E(S)$-map such that $\tau \mid (\mathbf{Z}_S)_e$ is a group homomorphism. Again it will be sufficient to show that

$$\tau\epsilon = id_{\mathbf{Z}S}, \qquad \epsilon\tau + \sigma_0\partial_1 = id_{C_0(S)}, \qquad \partial_i\sigma_{i-1} + \sigma_i\partial_{i+1} = id_{C_i}(S), \qquad i \geqslant 1.$$

The first identity is obvious. Let $(s) \in C_0(S)$, then

$$(s)(\epsilon\tau + \sigma_0\partial_1) = 1_{s^{-1}s}\tau + ([s], s^{-1}s)\partial_1$$
$$= (s^{-1}s) + (s) - (s^{-1}s) = (s).$$

Let $([s_1], s_2) \in C_1(S)$, then

$$([s_1], s_2)(\partial_1\sigma_0 + \sigma_1\partial_2)$$
$$= [(s_1s_2) - (s_1^{-1}s_1s_2)]\sigma_0 + ([s_1, s_2], s_2^{-1}s_1^{-1}s_1s_2)\partial_2$$
$$= ([s_1s_2], s_2^{-1}s_1^{-1}s_1s_2) - ([s_1^{-1}s_1s_2], s_2^{-1}s_1^{-1}s_1s_2) + ([s_1], s_1^{-1}s_1s_2)$$
$$- ([s_1, s_2], s_2^{-1}s_2^{-1}s_2s_2) + ([s_2], s_2^{-1}s_1^{-1}s_1s_2)$$
$$= ([s_1], s_2), \qquad \text{as} \quad s_2s_2^{-1} \leqslant s_1^{-1}s_1 .$$

Similarly, the general case holds. Q.E.D.

For computational purposes, we will require more projective resolutions of $\mathbf{Z}_S$: Let

$$W_i(S)_e = \{[s_1, ..., s_i] \mid 1 \neq s_r \in S,\ r = 1, ..., i,\ s_i^{-1} \cdots s_1^{-1}s_1 \cdots s_i = e\} \quad i \geqslant 1$$

and $W_i(S) = \bigcup_{e \in E(S)} W_i(S)_e$. Then the $E(S)$-set $W_i(S)$ freely generates an S-submodule $D_i(S)$ of $C_i(S)$. We put $D_0(S) = C_0(S)$. If we define ∂_i, σ_i as before and put $([s_1, ..., s_i], s) = 0_e$, $e = s^{-1} s_i^{-1}s_1^{-1} \cdots s_1^{-1}s_1 \cdots s_is$, if $s = 1$ or one $s_r = 1$, whenever this expression appears as an image of ∂_{i+1} or σ_{i-1}, then we obtain another projective resolution of $\mathbf{Z}_S$.

Let $X_i(S)_e = \{[s_1, ..., s_i] \mid 1 \neq s_r \in S,\ 1 \leqslant r \leqslant i,\ s_js_j^{-1} \leqslant s_{j-1}^{-1}s_{j-1},\ 2 \leqslant j \leqslant i,$ $s_i^{-1}s_i = e\}$, $i \geqslant 1$ and $X_i(S) = \bigcup_{e \in E(S)} X_i(S)_e$. Then the $E(S)$-set $X_i(S)$ freely generates an S-submodule $\bar{D}_i(S)$ of $D_i(S)$. We put $\bar{D}_0(S) = D_0(S)$. Then $\bar{D}_i(S)\,\partial_i \subseteq \bar{D}_{i-1}(S)$, $\bar{D}_i(S)\,\sigma_i \subseteq \bar{D}_{i+1}(S)$, $i \geqslant 1$. Also $\bar{D}_0(S)\,\sigma_0 \subseteq \bar{D}_1(S)$, as

$$\bar{D}_0(S)\,\sigma_0 = D_0(S)\,\sigma_0 \subseteq D_1(S) = \bar{D}_1(S).$$

Hence $\to \bar{D}_i(S) \to^{\partial_i} \bar{D}_{i-1}(S) \to \cdots \to \bar{D}_1(S) \to^{\partial_1} \bar{D}_0(S) \to^{\epsilon} \mathbf{Z}_S$ is a projective resolution of $\mathbf{Z}_S$.

If we define $\bar{C}_i(S)$ to be the S-submodule of $C_i(S)$ freely generated by

$$\{[s_1, ..., s_i] \mid s_r \in S,\ 1 \leqslant r \leqslant i,\ s_k \in E(S) \text{ implies}$$
$$s_{k-1}s_k \neq s_{k-1} \text{ or } s_ks_{k+1} \neq s_{k+1}\},$$

for $i \geqslant 1$, $\bar{C}_0(S) = C_0(S)$ and define ∂_i, σ_i as before, putting $([s_1, ..., s_i], s) = 0_e$,

$e = s^{-1}s_i^{-1} \cdots s_1^{-1}s_1 \cdots s_i s$, if $([s_1, ..., s_i], s) \notin \bar{C}_i(S)$ whenever such an element appears as an image under ∂_{i+1} or σ_{i-1}, we obtain another projective resolution $\to \bar{C}_i(S) \to^{\partial_i} \bar{C}_{i-1}(S) \to \cdots \to \bar{C}_1(S) \to^{\partial_1} \bar{C}_0(S) \to^{\epsilon} \mathbf{Z}_S$ over Mod(S).

7. Extensions and $H_S^2(A)$

The first application of the cohomology theory for inverse semigroups which we have developed deals with the following problem:

Let A be a semilattice of abelian groups and S an inverse semigroup. Find all inverse semigroups U such that there is an idempotent-separating homomorphism j from U onto S with $A \subseteq U$ and $A = \{u \in U \mid uj \in E(S)\}$.

Definition. (U, j) is called an *extension* of A by S.

The answer to this problem is well-known for groups [5]. For inverse semigroups, papers by d'Alarcao [3] and Coudron [2] have dealt with this question, but without the "structural" approach that was made so successfully for groups.

Definition. Two extensions (U, j) and $(\bar{U}, \bar{j})$ of A by S are called *equivalent* if there is a homomorphism μ: $U \to \bar{U}$ of inverse semigroups such that

(i) $\mu \mid A = id_A$ and

(ii) $\mu \bar{j} = j$.

Clearly, "being equivalent" is an equivalence relation on any set of extensions of A by S.

The following lemma is well-known (see [10]).

Lemma 7.1. *Let (U, j) be an extension of A by S and ρ: $S \to U$ a "transversal," i.e., a map ρ such that $\rho j = id_S$. Then every $u \in U$ can be written uniquely as $(s\rho)a$, $s \in S$, $a \in A$, such that $(s\rho)^{-1}(s\rho) = aa^{-1}$.*

Remark. That A is a semilattice of *abelian* groups, was not used in the proof.

Let (U, j) be an extension of A by S, $u \in U$, $a \in A$. Then $(u^{-1}au)j = (uj)^{-1}(aj)(uj) \in E(S)$, hence $u^{-1}au \in A$. As j is idempotent-separating and surjective, $j \mid E(U)$ is an isomorphism from $E(U)$ to $E(S)$. But by definition of A, $E(U) = E(A)$. Hence $\theta = (j \mid E(U))^{-1}$ is an isomorphism from $E(S)$ to $E(A)$.

If a_1, $a_2 \in A$, $u \in U$, then $u^{-1}(a_1a_2)u = u^{-1}a_1a_2uu^{-1}u = (u^{-1}a_1u)(u^{-1}a_2u)$ as $uu^{-1} \in A$ and A is abelian. If $a \in A$, u_1, $u_2 \in U$, then $(u_1u_2)^{-1} a(u_1u_2) =$

$u_2^{-1}(u_1^{-1}au_1)\,u_2$. If $a \in A$, $u \in U$, $uj \in E(S)$, then $u \in A$, hence $u^{-1}au = a(u^{-1}u) = a[(uj)(j \mid E(U))^{-1}]$. If $e \in E(S)$ and $u \in U$, then $[u^{-1}[e(j \mid E(U))^{-1}]\,u]\,j = (uj)^{-1}\,e(uj) \in E(S)$, *hence* $u^{-1}[e(j \mid E(U))^{-1}]\,u = [(uj)^{-1}\,e(uj)](j \mid E(U))^{-1}$. Suppose $u, u_1 \in U$, $uj = u_1 j$, then, by Lemma 7.1, $u = (uj\rho)\,a$, $u_1 = (u_1 j\rho)\,a_1 = (uj\rho)\,a_1$, for some $a, a_1 \in A$. Then, for $b \in A$, we have

$$\begin{aligned} u^{-1}bu &= a^{-1}[uj\rho]^{-1}\,b(uj\rho)\,a \\ &= (uj\rho)[(uj\rho)^{-1}\,a(uj\rho)]^{-1}\,b[(uj\rho)^{-1}\,a(uj\rho)][uj\rho)^{-1} = (uj\rho)\,b(uj\rho)^{-1}, \end{aligned}$$

as A is abelian. Hence $u^{-1}bu$ does not depend on a whence $u^{-1}bu = u_1^{-1}bu_1$ if $uj = u_1 j$. As j is surjective, we can make A an S-module by writing A additively and defining $as = (s\rho)^{-1}\,a(s\rho)$.

PROPOSITION 7.2. *If (U, j) and $(\bar{U}, \bar{j})$ are equivalent extensions of A by S, then the S-module structures of A arising from either extension are identical.*

Proof. Let $\rho\colon S \to U$, $\bar{\rho}\colon S \to \bar{U}$ be the maps such that $\rho j = \bar{\rho}\bar{j} = id_S$. If $s \in S$, then there exist $u \in U$, $\bar{u} \in \bar{U}$ such that $s\rho = u$, $s\bar{\rho} = \bar{u}$, and $\bar{u}\bar{j} = s = uj = u\mu\bar{j}$. Hence $[\bar{u}^{-1}(u\mu)]\,\bar{j} = s^{-1}s \in E(S)$ which implies $\bar{u}^{-1}(u\mu) \in A$. As $\bar{j}$ is idempotent-separating, we have $\bar{u}\bar{u}^{-1} = (u\mu)(u\mu)^{-1}$. Hence $uau^{-1} = (uau^{-1}) = (u\mu)\,a(u\mu)^{-1} = (u\mu)\,a(u\mu)^{-1}\,\bar{u}\bar{u}^{-1} = (u\mu)(u\mu)^{-1}\,\bar{u}a\bar{u}^{-1} = \bar{u}a\bar{u}^{-1}$, as A is abelian. Q.E.D.

The last proposition allows us to restate the extension problem: If A is an S-module, find all extensions (U, j) of A by S such that $u^{-1}au = a(uj)$, for all $a \in A$, $u \in U$.

Another problem is to find all S-modules with some underlying semilattice A of abelian groups. One may call this a representation problem—but we will restrict ourselves to the extension problem.

If (U, j) is an extension of A by S and $\rho\colon S \to U$ a map such that $\rho j = id_S$, and $s_1, s_2 \in S$, then $[(s_1\rho)(s_2\rho)]\,j = (s_1 s_2)\,\rho$. Hence $(s_1\rho)(s_2\rho) = (s_1 s_2)\,\rho[(s_1, s_2)\,\alpha]$, by Lemma 7.1, where $\alpha\colon S \times S \to A$ is a map such that $(s_1, s_2)\,\alpha \in A_{s_2^{-1}s_1^{-1}s_1 s_2}$. If $s_1, s_2, s_3 \in S$, we compute $(s_1\rho)(s_2\rho)(s_3\rho)$ in two different ways:

$$\begin{aligned} [(s_1\rho)(s_2\rho)](s_3\rho) &= [(s_1 s_2)\,\rho][(s_1 s_2)\,\alpha](s_3\rho) \\ &= [(s_1, s_2)\,\rho](s_3\rho)(s_3\rho)^{-1}[(s_1, s_2)\,\alpha](s_3\rho) \\ &= [(s_1 s_2 s_3)\,\rho][(s_1 s_2, s_3)\,\alpha](s_3\rho)^{-1}[(s_1, s_2)\,\alpha](s_3\rho); \end{aligned}$$

on the other hand,

$$\begin{aligned} (s_1\rho)[(s_2\rho)(s_3\rho)] &= (s_1\rho)[(s_2 s_3)\,\rho][(s_2, s_3)\,\alpha] \\ &= [(s_1 s_2 s_3)\,\rho][(s_1, s_2 s_3)\,\alpha][(s_2, s_3)\,\alpha]. \end{aligned}$$

As both factorizations of $(s_1\rho)(s_2\rho)(s_3\rho)$ satisfy the conditions of Lemma 7.1, we obtain, in additive S-module notation:

$$(s_1, s_2)\,\alpha s_3 - (s_1, s_2 s_3)\,\alpha + (s_1 s_2, s_3)\,\alpha - (s_2, s_3)\,\alpha = 0_e, \tag{7.1}$$

where $e = s_3^{-1}s_2^{-1}s_1^{-1}s_1s_2s_3$.

Suppose (U, j) and $(\bar{U}, \bar{j})$ are two equivalent extensions of A by S, and $\mu\colon U \to \bar{U}$ is a homomorphism such that $\mu \mid A = id_A$ and $j = \bar{j}$. (U, j) defines a map $\alpha\colon S \times S \to A$ and $(\bar{U}, \bar{j})$ a map $\bar{\alpha}\colon S \times S \to A$ satisfying (7.1). Let $\rho\colon S \to U$, $\bar{\rho}\colon S \to \bar{U}$ be maps with $\rho j = \bar{\rho}\bar{j} = id_S$, and let $s \in S$. Then, by Lemma 7.1, $s\rho\mu = s\bar{\rho})(s\beta)$ where $\beta\colon S \to A$ is a map such that $s\beta \in A_{s^{-1}s}$. If $s_1, s_2 \in S$, then $[(s_1s_2)\,\rho][(s_1, s_2)\,\alpha] = (s_1\rho)(s_2\rho)[(s_1, s_2)\,\alpha]$ whence

$$\begin{aligned}[(s_1s_2)\,\rho\mu][(s_1, s_2)\,\alpha] &= (s_1\rho\mu)(s_2\rho\mu) = (s_1\bar{\rho})(s_1\beta)(s_2\bar{\rho})(s_2\beta)\\ &= (s_1\bar{\rho})(s_2\bar{\rho})[(s_2\bar{\rho})^{-1}(s_1\beta)(s_2\bar{\rho})](s_2\beta)\\ &= (s_1s_2)\,\bar{\rho}[(s_1, s_2)\,\bar{\alpha}][(s_2\bar{\rho})^{-1}(s_1\beta)(s_2\bar{\rho})](s_2\beta).\end{aligned}$$

On the other hand,

$$[(s_1s_2)\,\rho\mu][(s_1, s_2)\,\alpha] = [(s_1s_2)\,\bar{\rho}][(s_1s_2)\,\beta][s_1, s_2)\,\alpha].$$

As both factorizations of $[(s_1s_2)\,\rho\mu][(s_1, s_2)\,\alpha]$ satisfy the conditions of Lemma 7.1, we obtain, in additive S-module notation:

$$(s_1, s_2)\,\alpha - (s_1, s_2)\,\bar{\alpha} = (s_1\beta)\,s_2 - (s_1s_2)\,\beta + s_2\beta. \tag{7.2}$$

Before interpreting equations (7.1) and (7.2) in terms of cohomology, we need the following construction: Let S be an inverse semigroup. Define $S^I = S \cup \{I\}$, where I is a symbol and $I \notin S$. On S^I we define a multiplication $*$ by:

$$\begin{aligned} s_1 * s_2 &= s_1s_2, && \text{if } s_1, s_2 \in S,\\ s * I &= I * s = s, && \text{if } s \in S^I.\end{aligned}$$

This definition makes S^I an inverse semigroup with identity I containing S such that the maximal subgroups of S^I are those of S and the trivial group $\{I\}$.

For $A \in \mathrm{Mod}(S)$, we construct an S^I-module A^0 as follows: $A^0 = A \cup \{0_I\}$, where 0_I is a symbol. A^0 becomes an S^I-module containing A by defining an addition $+$:

$$\begin{aligned} a_1 + a_2 &= a_1 + a_2, && \text{if } a_1, a_2 \in A\\ a + 0_I &= 0_I + a = a, && \text{if } a \in A^0\end{aligned}$$

and S^I-action $\circ$:

$$\begin{aligned} a \circ s &= as, && \text{if } a \in A,\ s \in S,\\ a \circ I &= a, && \text{if } a \in A^0,\\ 0_I \circ s &= 0_{s^{-1}s}, && \text{if } s \in S.\end{aligned}$$

In order to avoid proliferation of operation symbols, we will write $\cdot$ and $^{-1}$ for the operations on S^I, and $+$, $\cdot$ for the S^I-module operations on A^0.

Let $\cdots \to D_3(S^I) \to^{\partial_3} D_2(S^I) \to^{\partial_2} D_1(S^I) \to^{\partial_1} D_0(S^I) \to^{\epsilon} \mathbf{Z}_{S^I}$ by the projective resolution of $\mathbf{Z}_{S^I}$ in Mod(S^I). We compute ∂_3 and ∂_2: if $[s_1, s_2, s_3] \in W_3(S^I)$, then

$$[s_1, s_2, s_3]\partial_3 = ([s_1, s_2], s_2^{-1}s_1^{-1}s_1s_2s_3) - ([s_1, s_2s_3], s_3^{-1}s_2^{-1}s_1^{-1}s_1s_2s_3) + ([s_1s_2, s_3], s_3^{-1}s_2^{-1}s_1^{-1}s_1s_2s_3) - ([s_2, s_3], s_3^{-1}s_2^{-1}s_1^{-1}s_1s_2s_3).$$

If $[s_1, s_2] \in W_2(S^I)$, then

$$[s_1, s_2]\partial_2 = ([s_1], s_1^{-1}s_1s_2) - ([s_1s_2], s_2^{-1}s_1^{-1}s_1s_2) + ([s_2], s_2^{-1}s_1^{-1}s_1s_2).$$

We note that $W_i(S^I)_I = \varnothing$ as S^I has I as its only unit. Hence, for any $A \in$ Mod(S), $\mathrm{Hom}_{S^I}(D_i(S^I), A^0)$ can be identified with the group of all $E(S)$-maps from $W_i(S^I)$ to A^0. ∂_i induces homomorphisms $\partial_i^*: \mathrm{Hom}_{S^I}(D_i(S^I), A^0) \to \mathrm{Hom}_{S^I}(D_{i+1}(S^I), A^0)$ and $H^i_{S^I}(A^0) = \ker \partial^*_{i+1}/\mathrm{im}\, \partial_i^*$. Hence, for $i = 2$, ker ∂_3^* is just the group of all mappings α satisfying (7.1) whereas im ∂_1^* is the group of all mappings α: $S \times S \to A$ with β: $S \to A$, $s\beta \in A_{s^{-1}s}$ satisfying $(s_1\beta)\, s_2 - (s_1s_2)\, \beta + s_2\beta = (s_1, s_2)\, \alpha$. Hence Eq. (7.2) means that $\alpha - \bar{\alpha} \in im\ \partial_1^*$. We summarize our results in the following.

PROPOSITION 7.3. *Each equivalence class of extensions of A by S determines an element of $H^2_{S^I}(A^0)$.*

Now let $\alpha \in \ker \partial_2^*$ and $U = \{(s, a) \mid s \in S,\ \ a \in A,\ \ (s^{-1}s)\, \theta = aa^{-1}\}$. Then α: $S \times S \to A$ is a map satisfying (7.1) and if we define a multiplication on U by $(s_1, a_1)(s_2, a_1) = (s_1s_2, (s_1, s_2)\, \alpha + a_1s_2 + a_2)$, then Eq. (7.1) shows that U is a semigroup. Furthermore $E(U) = \{(e, -(e, e)\, \alpha \mid e \in E(S)\}$. If $e_1, e_2 \in E(S)$, then $(e_1, -(e_1, e_1)\, \alpha)(e_2, -(e_2, e_2)\, \alpha) = (e_1e_2, (e_1, e_2)\, \alpha - (e_1, e_1)\, \alpha e_2 - (e_2, e_2)\, \alpha) = (e_2, -(e_2, e_2)\, \alpha)(e_1, -(e_1, e_1)\, \alpha)$, as $0_{e_1e_2} = \{([e_1, e_2, e_1], e_1e_2) - ([e_2, e_1, e_1], e_1e_2) - ([e_1, e_1, e_2], e_1e_2)\}\, \partial_3 = ([e_1, e_2], e_1e_2) - ([e_2, e_1], e_1e_2)$. Hence the idempotents of $E(U)$ commute. Finally we show that (s, a) has $(s^{-1}, -(ss^{-1}, ss^{-1})\, \alpha - (s, s^{-1})\, \alpha - as^{-1})$ as its inverse. We have

$$\begin{aligned}(s, a)(s^{-1}, &-(ss^{-1}, ss^{-1})\, \alpha - (s, s^{-1})\, \alpha - as^{-1} \\ &= (ss^{-1}, ss^{-1})\, \alpha)(ss^{-1}, -(ss^{-1}, ss^{-1})\, \alpha)(s, a) \\ &= (s, (ss^{-1}, s)\, \alpha - (ss^{-1}, ss^{-1})\, \alpha s + a).\end{aligned}$$

But $0_{s^{-1}s} = ([ss^{-1}, ss^{-1}, s], s^{-1}s)\, \partial_3 = ([ss^{-1}, ss^{-1}], s) - ([ss^{-1}, s], s^{-1}s)$. Hence $(s, a) = (s, (ss^{-1}, s)\, \alpha - (ss^{-1}, ss^{-1})\, \alpha s + a)$. Further $(s^{-1}, -(ss^{-1}, ss^{-1})\, \alpha - (s, s^{-1})\, \alpha - as^{-1})(s, a) = (s^{-1}s, (s^{-1}, s)\, \alpha - (ss^{-1}, ss^{-1})\, \alpha s - (s, s^{-1})\, \alpha s)$ and $(s^{-1}s, (s^{-1}, s)\, \alpha - (ss^{-1}, ss^{-1})\, \alpha s - (s, s^{-1})\, \alpha s)(s^{-1}, -(ss^{-1}, ss^{-1})\, \alpha - (s, s^{-1})\, \alpha -$

$as^{-1}) = (s^{-1}, (s^{-1}s, s^{-1})\,\alpha + (s^{-1}, s)\,\alpha s^{-1} - (ss^{-1}, ss^{-1})\,\alpha - (s, s^{-1})\,\alpha - (ss^{-1}, ss^{-1})\,\alpha - (s, s^{-1})\,\alpha - as^{-1})$. But

$$\begin{aligned}0_{ss^{-1}} &= \{([s^{-1}, s, s^{-1}], ss^{-1}) + ([s^{-1}, ss^{-1}, ss^{-1}], ss^{-1})\}\partial_3 \\ &= ([s^{-1}, s], s^{-1}) + ([s^{-1}s, s^{-1}], ss^{-1}) - ([s, s^{-1}], ss^{-1}) - ([ss^{-1}, ss^{-1}], ss^{-1}).\end{aligned}$$

This proves that (s, a) has an inverse. Hence U is an inverse semigroup. Let $j\colon U \to S$ be defined by $(s, a)\,j = s$, then j is a homomorphism and is idempotnent-separating; $(s, a)\,j \in E(S)$ if and only if $s \in E(S)$. Let $\bar{A} = \{(e, a) \mid e \in E(S), (e, a) \in U\}$ and define a map $\nu\colon A \to \bar{A}$ by $a\nu = (e, a - (e, e)\,\alpha)$, for $a \in A_e$. If $b \in A_f$, $f \in E(S)$, then

$$\begin{aligned}(a\nu)(b\nu) &= (e, a - (e, e)\,\alpha)(f, b - (f, f)\,\alpha) \\ &= (ef, (e, f)\,\alpha - (e, e)\,\alpha - (f, f)\,\alpha + a + b)\end{aligned}$$

whereas $(a + b)\,\nu = (ef, a + b - (ef, ef)\,\alpha)$. But

$$\begin{aligned}0_{ef} &= \{([e, f, ef], ef) - ([e, e, f], ef) - ([f, f, e], ef)\}\,\partial_3 \\ &= ([e, f], ef) + ([ef, ef], ef) - ([e, e], ef) - ([f, f], ef).\end{aligned}$$

Hence $(a + b)\,\nu = (a\nu)(b\nu)$ and ν is a homomorphism from A to $\bar{A}$. Clearly ν is bijective, thus $A \cong \bar{A}$ as semilattices of abelian groups. Finally, $(as)\,\nu = (s^{-1}es, as - (s^{-1}es, s^{-1}es)\,\alpha)$, for $a \in A_e$ and

$$\begin{aligned}&(s, 0_{s^{-1}s})^{-1}(a\nu)(s, 0_{s^{-1}s}) \\ &\quad= (s^{-1}, -(ss^{-1}, ss^{-1})\alpha - (s, s^{-1})\alpha)(e, a - (e, e)\alpha)(s, 0_{s^{-1}s}) \\ &\quad= (s^{-1}, (s^{-1}, e)\alpha - (ss^{-1}, ss^{-1})\,\alpha e - (s, s^{-1})\,\alpha e - (e, e)\alpha + a)(s, 0_{s^{-1}s}) \\ &\quad= (s^{-1}es, (s^{-1}e, s)\alpha + (s^{-1}, e)\,\alpha s \\ &\qquad - (ss^{-1}, ss^{-1})\,\alpha es - (s, s^{-1})\,\alpha es - (e, e)\,\alpha s + as) \\ &\quad= (s^{-1}es, as - (s^{-1}es, s^{-1}es)\alpha),\end{aligned}$$

as

$$\begin{aligned}y = \{&([s^{-1}e, s], s^{-1}es) + ([s^{-1}, e], es) - ([ss^{-1}, ss^{-1}], es) \\ &- ([s, s^{-1}], es) - ([e, e], s) + ([s^{-1}es, s^{-1}es], s^{-1}es)\} \in \ker \partial_2,\end{aligned}$$

whence $y = z\partial_3$ for some $z \in D_3(S^I)$ and therefore $y\alpha = (z\partial_3)\,\alpha = z(\alpha\partial_3{}^*) = 0_{s^{-1}es}$.

Thus from each $\alpha\colon S \times S \to A$ which satisfies (7.1), arises an equivalence class of extensions of A by S.

Now we show that if $\alpha \in \ker \partial_3{}^*$ and β: $S \dashrightarrow A$ is a map satisfying $s\beta \in A_{s^{-1}s}$, then α and $\alpha + \beta\partial_2{}^*$ yield equivalent extensions.

Define a map μ: $U \to U_1$ by $(s, a)\,\mu = (s, a - s\beta)$, where U is the extension determined by α and U_1 the extension arising from $\alpha + \beta\partial_2{}^*$. The properties of ∂_2 and the definition of μ ensure

(i) $(e, a - (e, e)\,\alpha)\,\mu = (e, a - (e, e)(\alpha + \beta\partial_2{}^*))$, for $a \in A_e$,

(ii) μ is a homomorphism,

(iii) (s, a) and $(s, a)\,\mu$ have the same projections in S.

Hence U and U_1 are equivalent.

Summing up, we have constructed a map η:{equivalence classes of extensions of A by S} $\to H^2_{SI}(A^0)$ and a map ζ in the opposite direction.

One finds easily that η and ζ are inverses of one another. Hence

THEOREM 7.4. *If S is an inverse semigroup and A an S-module, then the set of equivalence classes of extensions of A by S is in one-to-one correspondence with the abelian group $H^2_{SI}(A^0)$.*

THEOREM 7.5. *Each extension of A by S is uniquely determined by a map $\bar\alpha$: $X_2(S) \to A$ satisfying* (7.1), *where* $X_2(S) = \{(s_1, s_2) \mid s_1, s_2 \in S,\ s_2 s_2^{-1} \leqslant s_1^{-1} s_1\}$, *i.e., any two maps α_1, α_2 from $S \times S$ to A satisfying* (7.1) *and extending $\bar\alpha$, determine equivalent extensions, and $\bar\alpha$ always extends to α: $S \times S \to A$ satisfying* (7.1).

Proof. Let

$$D(S)\colon \mathbf{Z}_S \xleftarrow{\ \epsilon\ } D_0(S) \xleftarrow{\ \partial_1\ } D_1(S) \xleftarrow{\ \partial_2\ } D_2(S) \longleftarrow \cdots$$

$$\bar D(S)\colon \mathbf{Z}_S \xleftarrow{\ \epsilon\ } \bar D_0(S) \xleftarrow{\ \partial_1\ } \bar D_1(S) \xleftarrow{\ \partial_2\ } \bar D_2(S) \longleftarrow \cdots$$

be the two projective resolutions of $\mathbf{Z}_S$ in Section 6. $\bar D(S)$ is a subcomplex of $D(S)$, hence there is a chain transformation ι from $\bar D(S)$ to $D(S)$ which is the inclusion map in each dimension. By the comparison theorem there exists a chain transformation χ: $D(S) \to \bar D(S)$ lifting $id_{\mathbf{Z}_S}$. Hence $\iota\chi$: $\bar D(S)_S \to \bar D(S)$ is a chain transformation lifting $id_{\mathbf{Z}_S}$, and $\iota\chi$ and $id_{\bar D(S)}$ are chain homotopic. Moreover, we may choose χ such that $[s]\,\chi = [s]$, $s \in S$, i.e., χ is the identity in dimension 1.

As $\iota\chi$ and $id_{\bar D(S)}$ are chain homotopic, there exist S-morphism ζ: $\bar D(S) \dashrightarrow \bar D(S)$, $i \geqslant 0$, such that $\iota\chi - id_{\bar D(S)} = \zeta\partial + \partial\zeta$. Let $\bar\alpha$ be a map from $X_2(S)$ to A satisfying (7.1). Define α: $S \times S \to A$ by $\alpha = \chi(id_{\bar D_2(S)} - \partial_2\zeta)\,\bar\alpha$. Then α satisfies (7.1), as $\partial_3\alpha = \partial_3\chi\bar\alpha - \partial_3\partial_2\chi\zeta\bar\alpha = \chi\partial_3\bar\alpha$, and $\partial_3\bar\alpha$ is the zero-morphism.

Furthermore

$$\begin{aligned}\iota\alpha &= \iota\chi\bar{\alpha} - \iota\chi\partial_2\zeta\bar{\alpha} = \bar{\alpha} + \zeta\partial_3\bar{\alpha} + \partial_2\zeta\bar{\alpha} - \iota\chi\partial_2\zeta\bar{\alpha} \\ &= \bar{\alpha} + \partial_2\zeta\bar{\alpha} - \partial_2\iota\chi\zeta\bar{\alpha} = \alpha\end{aligned}$$

as $\iota\chi$ is the identity in dimension 1. Hence α extends $\bar{\alpha}$. Let α_1, α_2 be extensions of $\bar{\alpha}$, satisfying (7.1). Then $\bar{\alpha} = \iota\alpha_1 = \iota\alpha_2$. As $\chi\iota\colon D(S) \to D(S)$ is a chain transformation lifting $id_{\mathbf{Z}_S}$, there exist S-morphisms $\eta\colon D_i(S) \to D_{i+1}(S)$ such that $\chi\iota - id_{D(S)} = \eta\partial + \partial\eta$. But $\chi\bar{\alpha} = \chi\iota\alpha_r = \alpha_r + \eta\partial_3\alpha_r + \partial_2\eta\alpha_r = \alpha_r + \partial_2(\eta\alpha_r)$, $r = 1, 2$. Hence $\alpha_2 - \alpha_1 = \partial_2[\eta(\alpha_1 - \alpha_2)]$. Thus α_1 and α_2 determine equivalent extensions, by Theorem 7.4. Q.E.D.

Remark. The condition that χ is the identity in dimension 1, shows that one can take $\zeta\colon \bar{D}_1(S) \to \bar{D}_2(S)$ to be the zero morphism. The same condition implies that $\chi\colon D_2(S) \to \bar{D}_2(S)$ may be defined by

$$\begin{aligned}(s_1, s_2)\chi &= ([s_1, s_2], s_2^{-1}s_1^{-1}s_1s_2)\,\partial_2\chi\sigma_1 \\ &= ([s_1, s_1^{-1}s_1s_2], s_2^{-1}s_1^{-1}s_1s_2) - ([s_1s_2, s_2^{-1}s_1^{-1}s_1s_2], s_2^{-1}s_1^{-1}s_1s_2) \\ &\quad + ([s_2, s_1^{-1}s_1^{-1}s_1s_2], s_2^{-1}s_1^{-1}s_1s_2).\end{aligned}$$

Hence we have

$$(s_1, s_2)\alpha = (s_1, s_1^{-1}s_1s_2)\bar{\alpha} - (s_1s_2, s_2^{-1}s_1^{-1}s_1s_2)\bar{\alpha} + (s_2, s_2^{-1}s_1^{-1}s_1s_2)\bar{\alpha}.$$

If we remember that $(s_1, s_2)\,\alpha = [(s_1s_2)\,\rho]^{-1}(s_1\rho)(s_2\rho)$ and substitute this into formula (7.3), we obtain a direct proof of Theorem 7.5. Such a proof, however, would have looked accidental and would not have revealed the chain homotopies responsible for this result.

8. Endomorphisms of Semilattices of Groups

Let G be a semilattice of groups. Then, in general, the semigroup End G of endomorphisms of G is not an inverse semigroup.

Definition. $\alpha \in \text{End}\, G$ is *relatively invertible* if there exists $\bar{\alpha} \in \text{End}\, G$, $e_\alpha \in E(G)$ such that

(i) $g\alpha\bar{\alpha} = ge_\alpha$, for all $g \in G$,

(ii) $g\bar{\alpha}\alpha = g(e_\alpha\alpha)$, for all $g \in G$,

(iii) $e_\alpha\alpha$ is a right identity on $G\alpha$, e_α is a right identity on $G\bar{\alpha}$.

The set of all relatively invertible endomorphisms of G will be denoted by end G.

PROPOSITION 8.1. end G *is an inverse semigroup and there is an isomorphism* $\tau: E(G) \to E(\text{end } G)$ *with* $g(e\tau) = ge$.

Proof. Let α, $\beta \in \text{end}(G)$, and $g \in G$. Then $g\alpha\beta\bar{\beta}\bar{\alpha} = [(g\alpha)\, e_\beta]\, \bar{\alpha} = (g\alpha\bar{\alpha})(e_\beta\bar{\alpha}) = ge_\alpha(e_\beta\bar{\alpha})$. Further

$$\begin{aligned} g([e_\alpha(e_\beta\bar{\alpha})]\, \alpha\beta) &= g(e_\alpha\alpha\beta)(e_\beta\bar{\alpha}\alpha\beta) = g(e_\alpha\alpha\beta)[e_\beta(e_\alpha\alpha)]\, \beta \\ &= g(e_\alpha\alpha\beta)(e_\beta\beta) = g(e_\beta\beta)(e_\alpha\alpha\beta) = (g\bar{\beta}\beta)(e_\alpha\alpha\beta) \\ &= [(g\bar{\beta})(e_\alpha\alpha)]\, \beta = g\bar{\beta}\bar{\alpha}\alpha\beta. \end{aligned}$$

Also

$$\begin{aligned} (g\alpha\beta)([e_\alpha(e_\beta\bar{\alpha})]\, \alpha\beta) &= (g\alpha\beta)(e_\alpha\alpha\beta)(e_\beta\bar{\alpha}\alpha\beta) = [(g\alpha)(e_\alpha\alpha)]\, \beta(e_\beta(e_\alpha\alpha))\, \beta \\ &= (g\alpha\beta)(e_\beta\beta) = g\alpha\beta \end{aligned}$$

whereas $(g\bar{\beta}\bar{\alpha})\, e_\alpha(e_\beta\bar{\alpha}) = (g\bar{\beta}\bar{\alpha})(e_\beta\bar{\alpha}) = ((g\bar{\beta})\, e\beta)\, \bar{\alpha} = g\bar{\beta}\bar{\alpha}$. Hence $\alpha\beta \in \text{end } G$. By definition of end G, $\alpha\bar{\alpha}\alpha = \alpha$, $\bar{\alpha}\alpha\bar{\alpha} = \bar{\alpha}$, for $\alpha \in \text{end } G$, and $\bar{\alpha} \in \text{end } G$. Let $\alpha \in E(\text{end } G)$, then $\alpha\alpha = \alpha$ implies $(\bar{\alpha}\alpha)(\alpha\bar{\alpha}) = \bar{\alpha}$, hence $\bar{\alpha} \in E(\text{end } G)$. Moreover, $g\alpha = g\alpha\bar{\alpha}\alpha = g(\alpha\bar{\alpha})(\bar{\alpha}\alpha) = ge_\alpha(e_\alpha\alpha)$. Hence $\alpha \in E(\text{end } G)$ if and only if $g\alpha = ge$, for some $e \in E(G)$. Define $\tau: E(G) \to E(\text{end } G)$ by $g(e\tau) = ge$. Obviously τ is an isomorphism of semilattices.

DEFINITION. $\alpha \in \text{end } G$ is called a *relatively invertible inner endomorphism* of G if $g\alpha = h^{-1}gh$, for some $h \in G$ and all $g \in G$. The set of all relatively invertible inner endomorphisms will be denoted by *in* G.

Remark. By a kernel normal system K of an inverse semigroup S we shall mean an inverse subsemigroup K of S with $E(K) = E(S)$ and $s^{-1}ks \in K$, for all $s \in S$, $k \in K$. This is a slight modification of the definition in [1].

PROPOSITION 8.2. *Let G be a semilattice of groups. Then*

(i) *in G is a semilattice of groups under composition,*

(ii) *in G is a kernel normal system of* end G, *determining an idempotent separating homomorphism of end G,*

(iii) $Z(G) = \{z \in G \mid zg = gz$, *for* $g \in G\}$ *is the kernel normal system of an identity separating homomorphism from G onto in G.*

Proof. Define $\lambda: G \to in\ G$, by $g(h\lambda) = h^{-1}gh$. Obviously, $h\lambda \in in\ G$, for every $h \in G$, hence λ is surjective. As $(h_1h_2^{-1})\, g(h_1h_2) = h_2^{-1}(h_1^{-1}gh_1)\, h_2$, λ is a homomorphism. Let $e_1, e_2 \in E(G)$, then $e_1^{-1}ge_1 = e_2^{-1}ge_2$, for all $g \in G$, implies $e_1 = e_2e_1 = e_2$ hence λ is idempotent separating. Let $h^{-1}h^{-1}ghh = h^{-1}gh$, for all $g \in G$, then $h^{-1}ghh^{-1}h = h^{-1}hg$, that is $gh = hg$, for all $g \in G$, whence $h \in Z(G)$. Conversely if $h \in Z(G)$, then $h^{-1}h^{-1}ghh = h^{-1}(h^{-1}h)\, gh = (h^{-1}hh^{-1})\, gh = h^{-1}gh$ whence $h\lambda \in E(in\, G)$. Thus (i) and (iii) hold. Let $\alpha \in \text{end } G$, $h\lambda \in in\, G$, $g \in G$, then $g\bar{\alpha}(h\lambda)\, \alpha = (h^{-1}(g\alpha)\, h)\, \alpha = (h^{-1}\alpha)(g\bar{\alpha}\alpha)(h\alpha) = (h^{-1}\alpha)\, g(e_\alpha\alpha)(h\alpha) = g[(e_\alpha h)\, \alpha\lambda]$. Hence $\bar{\alpha}(h\lambda)\, \alpha \in in\, G$, and $in\, G$ is a kernel normal system for an idempotent separating congruence.

Remark. As usual, the image of the idempotent separating homomorphism end G determined by $in\, G$ will be denoted by end $G/in\, G$.

9. Existence of Extensions

Let G be a semilattice of (not necessarily abelian) groups and S an inverse semigroup. A pair (U, j) consisting of an inverse semigroup U and a homomorphism $j\colon U \to S$ such that j is idempotent separating, surjective and has G as its kernel normal system, is called an *extension* of G by S. If $(\bar{U}, \bar{j})$ is another extension of G by S, we say (U, j) and $(\bar{U}, \bar{j})$ are *equivalent* if there is a homomorphism $\mu\colon U \to \bar{U}$ such that $\mu \mid G = id_G$ and $\mu\bar{j} = j$. As for abelian G, "being equivalent" is an equivalence relation on any set of extensions of G by S. We note that $\nu\colon U \to \text{end } G, g(u\nu) = u^{-1}gu$ is an idempotent separating homomorphism, and if $u_1 j = u_2 j$, $u_1, u_2 \in U$, then, by Lemma 7.1, $u_2 = u_1 h$, for some $h \in G$ with $u_1^{-1}u_1 = hh^{-1}$. Hence $u_2\nu = (u_1 h)\, \nu = (u_1\nu)(h\nu)$, thus $u_1\nu$ and $u_2\nu$ are mapped to the same element of end $G/in\, G$ under the idempotent separating homomorphism determined by $in\, G$. Hence every extension (U, j) determines a map $\psi\colon S \to \text{end } G/in\, G$, and ψ is an idempotent separating homomorphism because $u_1 j = s_1$, $u_2 j = s_2$, $uj = s_1 s_2$, u, $u_1 u_2 \in U$ implies $u_1 u_2 = ug$, for some $g \in G$ with $u^{-1}u = gg^{-1}$, by Lemma 7.1. We will call ψ the *abstract kernel* of (U, j). Let $\rho\colon S \to U$ be a map with $\rho j = id_S$ abd $E(S)\, \rho \subseteq E(U)$. Then $s\rho\nu \in \text{end } G$ represents $s\psi \in \text{end } G/in\, G$, $e\rho\nu \in E(\text{end } G)$, for $e \in E(S)$, and $(e\rho)(s\rho\nu) = (s\rho)^{-1}(e\rho)(s\rho) = (s^{-1}es)\, \rho$ as $(s\rho)^{-1}(e\rho)(s\rho) \in E(U)$.

If $s_1, s_2 \in S$, then $(s_1\rho)(s_2\rho) = [(s_1 s_2)\, \rho][(s_1, s_2)\, \alpha]$, for some map $\alpha\colon S \times S \to G$ with $[(s_1 s_2)\, \rho]^{-1}[(s_1 s_2)\, \rho] = [(s_1, s_2)\, \alpha][(s_1, s_2)\, \alpha]^{-1}$. Associativity of U implies

$$(s_1 s_2 s_3)\, \rho[(s_1 s_2, s_3)\, \alpha][(s_1, s_2)\, \alpha(s_3\rho\nu)] = (s_1 s_2 s_3)\, \rho[(s_1, s_2 s_3)\, \alpha][(s_2, s_3)\, \alpha].$$

Both sides of this equation satisfy the conditions of Lemma 7.1, hence

$$[(s_1 s_2, s_3)\, \alpha][(s_1, s_2)\, \alpha(s_3\rho\nu)] = [(s_1, s_2 s_3)\, \alpha][(s_2, s_3)\, \alpha], \quad \text{for all } s_1, s_2, s_3 \in S. \tag{9.1}$$

Also, as ν: $U \to$ end G is a homomorphism, we have

$$(s_1\rho\nu)(s_2\rho\nu) = [(s_1s_2)\,\rho\nu][(s_1, s_2)\,\alpha\nu]. \tag{9.2}$$

We note that $g\nu = g\lambda$, the relatively invertible inner automorphism of G induced by g. As $e \in E(U)$ if $e \in E(S)$, $e\rho\nu \in E(\text{end } G)$. Even more, as $e\rho \in E(G)$, $\theta = \rho \mid E(S)$ is an isomorphism from $E(S)$ to $E(G)$ and $e\rho\nu = e\theta\lambda$. The equations $(e\rho)(e\rho)(s\rho) = (e\rho)(s\rho)$ and $(s\rho)(e\rho)(e\rho) = (s\rho)(e\rho)$, $s \in S$, $e \in E(S)$, yield $(e, es)\,\alpha = (s^{-1}es)\,\theta$, $(se, e)\,\alpha = (es^{-1}s)\,\theta$. (9.3)

The following theorem is essentially Coudron's result [2].

THEOREM 9.1. *Given a semilattice G of groups, an inverse semigroup S, an isomorphism θ: $E(S) \to E(G)$, a map ϕ: $S \to$ end G, and a map α: $S \times S \to G$ satisfying*

(i) $e\phi = (e\theta)\,\lambda$, *the element of in G induced by e,*

(ii) $(e, es)\,\alpha = (s^{-1}es)\,\theta$, *for* $e \in E(S)$, $(se, e)\,\alpha = (es^{-1}s)\,\theta$, *for* $e \in E(S)$,

(iii) $[(s_1s_2, s_3)\,\alpha][(s_1, s_2)\,\alpha(s_3\phi)] = [(s_1, s_2s_3)\,\alpha][(s_2, s_3)\,\alpha]$,

(iv) $[(s_1, s_2)\,\alpha] \in G_{(s_2^{-1}s_1^{-1}s_1s_2)\theta}$,

(v) $(s_1\phi)(s_2\phi) = [(s_1s_2)\,\phi][(s_1, s_2)\,\alpha\lambda]$,

(vi) $(e\theta)(s\phi) = (s^{-1}es)\,\theta$.

Then the set $U = \{(s, g) \mid s \in S,\ g \in G,\ (s^{-1}s)\,\theta = gg^{-1}\}$ becomes an inverse semigroup under the multiplication defined by

$$(s_1, g_1)(s_2\, g_2) = (s_1s_2, [(s_1, s_2)\,\alpha][g_1(s_2\phi)]\, g_2).$$

$(s, g)\,j = s$ defines an idempotent separating, surjective homomorphism j: $U \to S$, and $g\kappa = ((gg^{-1})\,\theta^{-1}, g)$ an injective homomorphism κ: $G \to U$. Identifying G with $G\kappa$, (U, j) is an extension of G by S.

Proof. Associativity follows from (iii), (iv), and (v). $E(U) = \{(e, e\theta) \mid e \in E(S)\}$ by (i) and (ii). An inverse of (s, g) is $(s^{-1}, g^{-1}(s^{-1}\phi)[(s, s^{-1})\,\alpha]^{-1})$ by (i)–(vi), putting $s_1 = s$, $s_2 = s^{-1}$, $s_3 = s$ in (iii). The idempotents of U commute, by (i), (iii), and (vi), putting, in (iii), $s_1 = s_2 = e_1 \in E(S)$, $s_3 = e_2 \in E(S)$, then $s_1 = e_1$, $s_2 = e_2$, $s_3 = e_1e_2$, and then $s_1 = e_2$, $s_1 = e_1$, $s_3 = e_1e_2$. j is obviously idempotent separating and surjective, by (i) and (iii), κ is a homomorphism, and κ is injective. The kernel normal system for j is $G\kappa$, by definition of j. Q.E.D.

THEOREM 9.2. *Let $(\bar{U}, \bar{j})$ be an extension of G by S with abstract kernel ψ: $S \to$ end $G/$in G, γ: end $G/$in $G \to$ end G a map with $E(\text{end } G/\text{in } G)\,\gamma \subseteq E(\text{end } G)$, k: end $G \to$ end $G/$in G the idempotent separating, surjective homo-*

morphism associated with the kernel normal system in G, *and* $\gamma k =$ *identity on* end $G/in\, G$. *Then if* $\phi = \psi\gamma$, $(\bar{U}, \bar{j})$ *is equivalent to an extension* (U, j) *described in Theorem* 9.1.

Proof. If $\nu: \bar{U} \to \text{end } G$ is the homomorphism defined by $g(\bar{u}\nu) = \bar{u}^{-1}g\bar{u}$, then we can define a map $\rho: S \to \bar{U}$ with $\rho\bar{j} = id_S$ such that $s\rho\nu = s\phi$. $(\bar{U}, \bar{j})$ determines a map $\alpha: S \times S \to G$ satisfying (9.1), (9.2), (9.3). Hence α satisfies (ii), (iii), (iv), (v) of Theorem 9.1 with $\rho\nu = \phi$, and ϕ satisfies (i) and (vi). Hence α and ϕ determine an extension (U, j) of Theorem 9.1. Now define $\mu: U \to \bar{U}$ by $(s, g)\mu = (s\rho)g$. Then μ is a homomorphism, by the definition of the product in U. If $g \in G$, then $((gg^{-1})\theta^{-1}, g)\mu = gg^{-1}g = g$, and $(s, g)\bar{j} = [(s\rho)g]\bar{j} = s = sj$. Hence (U, j) and $(\bar{U}, \bar{j})$ are equivalent.

Q.E.D.

Now suppose that $\psi: S \to \text{end } G/in\, G$ is an idempotent separating homomorphism. As in Theorem 9.2, define $\gamma: \text{end } G/in\, G \to \text{end } G$ to be a map with $E(\text{end } G/in\, G) \subseteq E(\text{end } G)$, $k: \text{end } G \to \text{end } G/in\, G$, the homomorphism determined by the kernel normal system *in* G, such that γk is the identity on end $G/in\, G$, and let $\phi: S \to \text{end } G$ be the map $\phi = \psi\gamma$. Suppose $(e\theta)(s\phi) = (s^{-1}es)\theta$. We say ϕ is a *transversal* for ψ. Then

$$(s_1\phi)(s_2\phi) = [(s_1s_2)\phi][(s_1, s_2)\alpha\lambda], \tag{9.4}$$

where $\alpha: S \times S \to G$ is a map and $\lambda: G \to in\, G$ is defined, as before, by $g(h\lambda) = h^{-1}gh$, and $(s_1s_2)\alpha \in G_{s_2^{-1}s_1^{-1}s_1s_2}$. Putting $s_1 = e \in E(S)$, $s_2 = es$, we find

$$(es)\phi = [(es)\phi][(e, es)\alpha\lambda],$$

hence $(e, es)\alpha\lambda \in E(in\, G)$. Therefore we may put $(e, es)\alpha = (s^{-1}es)\theta$. Similarly, put $(se, e)\alpha = (es^{-1}s)\theta$. We compute $(s_1\phi)(s_2\phi)(s_3\phi)$ in two different ways:

$$\begin{aligned}
[(s_1\phi)(s_2\phi)](s_3\phi) &= [(s_1s_2)\phi][(s_1, s_2)\alpha\lambda](s_3\phi) \\
&= [(s_1s_2)\phi](s_3\phi)(s_3\phi)^{-1}[(s_1, s_2)\alpha\lambda](s_3\phi) \\
&= [(s_1s_2s_3)\phi][(s_1s_2, s_3)\alpha\lambda](s_3\phi)^{-1}[(s_1, s_2)\alpha\lambda](s_3\phi);
\end{aligned}$$

$$\begin{aligned}
(s_1\phi)[(s_2\phi)(s_3\phi)] &= (s_1\phi)[(s_2s_3)\phi][(s_2, s_3)\alpha\lambda] \\
&= [(s_1s_2s_3)\phi][(s_1, s_2s_3)\alpha\lambda][(s_2, s_3)\alpha\lambda].
\end{aligned}$$

By the proof of Proposition 8.2,

$$(s_3\phi)^{-1}[(s_1, s_2)\alpha\lambda](s_3\phi) = [e_{s_3\phi}(s_1, s_2)\alpha](s_3\phi)\lambda,$$

but $e_{s_3\phi} = (s_3s_3^{-1})\theta$, as k is idempotent separating. Further $[(s_3s_3^{-1})\theta](s_3\phi) = s_3\phi$, as $(s_3s_3^{-1}, s_3)\alpha = (s_3^{-1}s_3)\theta$. Hence $[(s_1s_2, s_3)\alpha][(s_1, s_2)\alpha(s_3\phi)]$ and

$[(s_1, s_2s_3)\,\alpha][(s_2, s_3)\,\alpha]$ have the same image under λ: $G \to in\ G$ by Lemma 7.1. As $Z(G)$ is the kernel normal system for λ, there exists a map β: $S \times S \times S \to Z(G)$ such that

$$[(s_1s_2, s_3)\,\alpha][(s_1, s_2)\,\alpha(s_3\phi)] = [(s_1, s_2s_3)\,\alpha] \cdot [(s_2, s_3)\,\alpha][(s_1, s_2, s_3)\,\beta] \quad (9.5)$$

and $(s_1, s_2, s_3)\,\beta \in Z(G)_e$, $e = s_3^{-1}s_2^{-1}s_1^{-1}s_1s_2s_3$. Let $e \in E(S)$ and put $s_1 = e$, $s_2 = es_2$, then $(e, es_2, s_3)\,\beta = (s_3^{-1}s_2^{-1}es_2s_3)\,\theta$; if $s_1 = s_1e$, $s_2 = e$, $s_3 = es_3$, then $(s_1e, e, es_3)\,\beta = (s_3^{-1}es_1^{-1}s_1s_3)\,\theta$; if $s_2 = s_2e$, $s_3 = e$, then $(s_1, s_2e, e)\,\beta = (es_2^{-1}s_1^{-1}s_1s_2)\,\theta$, using $(se, e)\,\alpha = (es^{-1}s)\,\theta$, $(e, es)\,\alpha = (s^{-1}es)\,\theta$.

LEMMA 9.3. *Let* ψ: $S \to$ end $G/in\ G$ *be an idempotent separating homomorphism. Then* $Z(G)$ *becomes an* S*-module if we write* $Z(G)$ *additively,* θ: $E(S) \simeq E(G)$, *and define*

$$zs = z(s\phi),$$

where ϕ *is a transversal for* ψ *and* $(e\theta)(s\phi) = (s^{-1}es)\,\theta$. *Any two such transversals for* ψ *determine the same* S*-module structure of* $Z(G)$.

Proof. Let $\chi \in$ end G, $z \in Z$, $g \in G$. Then, using the definition of end G,

$$\begin{aligned} g(z\chi) &= g(z\chi)(e_x\chi) \\ &= g(e_x\chi)(z\chi) \\ &= (g\bar{\chi})\,\chi(z\chi) \\ &= [(g\bar{\chi})\,z]\,\chi \\ &= [z(g\bar{\chi})]\,\chi \\ &= (z\chi)(g\bar{\chi}\chi) \\ &= (z\chi)\,g(e_x\chi) \\ &= (z\chi)(e_x\chi)\,g \\ &= (z\chi)\,g. \end{aligned}$$

Hence $Z(G)$ is invariant under end G, hence $z(s\phi) \in Z(G)$. That $Z(G)$ is then an S-module, is straightforward. Let ϕ_1, ϕ_2 be two transversals for ψ. Then $s\phi_1 = (s\phi_2)(g\lambda)$ for some $g \in G$ with $s^{-1}s = gg^{-1}$. Hence $z(s\phi_1) = z(s\phi_2)(g\lambda) = g^{-1}z(s\phi_2)\,g = z(s\phi_2)(g^{-1}g) = z(s\phi_2)(gg^{-1})$. If $z \in Z(G)_e$, then $z(s\phi_2) \in Z(G)_{s^{-1}es}$ and $gg^{-1} = s^{-1}s$ acts as a right identity for the elements of $Z(G)_{s^{-1}es}$. Hence $z(s\phi_1) = z(s\phi_2)$, for all $z \in Z(G)$, $s \in S$. Q.E.D.

Every $\alpha \in \mathrm{Hom}_{S^I}(\bar{C}_3(S^I), Z(G)^0)$ can be regarded as a map from $S \times S \times S$ to $Z(G)^0$ as I is the identity of S^I and hence cannot appear in any $[s_1, s_2, s_3]$ of the free generating set for $\bar{C}_3(S^I)$. As neither $[e, es_2, s_3]$,

$[s_1 e, e, es_3]$, nor $[s_1, s_2 e, e]$, $e \in E(S)$ are in $\bar{C}_3(S^I)$, the elements of $\mathrm{Hom}_{S^I}(\bar{C}_3(S^I), Z(G)^0)$ are in one-to-one correspondence with all those maps $\alpha: S \times S \times S \to Z(G)^0$ which satisfy: $(e, es_2, s_3)\,\alpha$, $(s_1 e, e, es_3)\,\alpha$, and $(s_1, s_2 e, e)\,\alpha$ are idempotents. Referring to such maps, we will identify them with the corresponding 3-cochains of S^I in $Z(G)^0$. We have thus shown

LEMMA 9.4. *Let $\psi: S \to \text{end}\; G/\text{in}\; G$ be an idempotent separating homomorphism such that, for one (and hence for every) transversal ϕ for ψ, we have $(e\theta)(s\phi) = (s^{-1}es)\,\theta$, for $e \in E(S)$. Then every transversal ϕ for ψ determines an element $\beta_\phi \in \mathrm{Hom}_{S^I}(\bar{C}_3(S^I), Z(G)^0)$ and ψ is the abstract kernel for an extension of G by S if and only if β_ϕ is the zero-homomorphism for some transversal ϕ for ψ.*

LEMMA 9.5. *Under the hypothesis and with the notation of Lemma 9.4, $\beta_\phi \in \ker \partial_4^*$, for all transversals ϕ for ψ. Here $\partial_4^*: \mathrm{Hom}_{S^I}(\bar{C}_3(S^I), Z(G)^0) \to \mathrm{Hom}_{S^I}(\bar{C}_4(S^I), Z(G)^0)$ is, as usual, the homomorphism induced by*

$$\partial_4 : \bar{C}_4(S^I) \to \bar{C}_3(S^I).$$

Proof. We do not go into detail as the proof is a repetition of [7], IV, Lemma 8.4. The idea is to express

$$L = (s_1 s_2 s_3, s_4)\,\alpha[(s_1 s_2, s_3)\,\alpha[(s_1, s_2)\,\alpha(s_3\phi)]](s_4\phi)$$

in two ways, first by using formula (9.5) repeatedly, beginning with $[(s_1, s_2)\,\alpha](s_3\phi)$. Then

$$L = (s_1, s_2 s_3 s_4)\,\alpha \cdot (s_2, s_3 s_4)\,\alpha \cdot (s_3 s_4)\,\alpha \cdot (s_2, s_3, s_4)\,\beta \cdot (s_1, s_2 s_3, s_4)\,\beta \cdot (s_1, s_2, s_3)\,\beta(s_4\phi).$$

Using formula (9.4) first, to evaluate $(s_3\phi)(s_4\phi)$, we get

$$L = (s_1 s_2 s_3, s_4)\,\alpha \cdot (s_1 s_2, s_3)\,\alpha(s_4\phi) \cdot [(s_3, s_4)\,\alpha]^{-1} \cdot (s_1, s_2)\,\alpha(s_3 s_4)\,\phi \cdot (s_3, s_4)\alpha$$

and then applying formula (9.5) to get rid of all terms involving ϕ, we obtain

$$L = (s_1, s_2 s_3 s_4)\,\alpha \cdot (s_2, s_3 s_4)\,\alpha \cdot (s_3, s_4)\,\alpha \cdot (s_1 s_2, s_3, s_4)\,\beta \cdot (s_1, s_2, s_3 s_4)\,\beta.$$

As in both expressions for L, both the product of the terms involving α, and the product of the terms involving β are in the same maximal subgroup G_e, $e = s_4^{-1} s_3^{-1} s_2^{-1} s_1^{-1} s_1 s_2 s_3 s_4$, we can cancel the α-terms. Writing the β-terms additively, and observing that $z(s\phi) = zs$, $z \in Z(G)$, $s \in S$, if $Z(G)$ is regarded as an S-module, we find that $(s_1, s_2, s_3)\beta s_4 - (s_1, s_2, s_3s_4)\beta + (s_1, s_2s_3, s_4)\beta - (s_1s_2, s_3, s_4)\beta - (s_2, s_3, s_4)\beta \in E(Z(G))$, hence $\beta \in \ker \partial_4{}^*$. Q.E.D.

The subsequent two lemmas are again adaptations of [7], IV, Lemmas 8.5 and 8.6.

LEMMA 9.6. *Under the hypothesis and with the notation of Lemma* 9.4, *a change of* α *in* (9.5) *produces, for fixed* ϕ, *an element* $\beta' \in \ker \partial_4{}^*$ *such that* $\beta' - \beta \in im\, \partial_3{}^*$.

Proof. Let $\alpha'\colon S \times S \to G$ be another map satisfying (9.5) with $(s_1, s_2)\alpha' \in G_{s_2^{-1}s_1^{-1}s_1s_2}$ and $(s, se)\alpha'$, $(e, es)\alpha' \in E(G)$, for $e \in E(S)$. As $(s_1, s_2)\alpha\lambda$ and $(s_1, s_2)\alpha'\lambda$ are in the same maximal subgroup of *in G*, there is a map $\tau\colon S \times S \to Z(G)$ with

$$(s_1, s_2)\alpha' = [(s_1, s_2)\alpha][(s_1, s_2)\tau] \tag{9.6}$$

such that $(s_1, s_2)\tau \in Z(G)_{s_2^{-1}s_1^{-1}s_1s_2}$, and $(se, e)\tau$, $(e, es)\tau$ are elements of $E(Z(G))$, for $e \in E(S)$. Substitution of (9.6) into (9.5) yields:

$$[(s_1s_2, s_3)\tau][(s_1, s_2)\tau(s_3\phi)] = [(s_1, s_2s_3)\tau][(s_2, s_3)\tau]\cdot[(s_1, s_2, s_3)(\beta' - \beta).]$$

In additive S-module notation, this means

$$(s_1, s_2, s_3)(\beta' - \beta) = (s_1, s_2, s_3)\,\partial_3\tau.$$

Hence $\beta' - \beta \in im\, \partial_3{}^*$. Q.E.D.

LEMMA 9.7. *Under the hypothesis and with the notation of Lemma* 9.4, *a change of the transversal* ϕ *for* ψ *admits a choice for a new* α' *replacing* α, *such that* $\beta \in \ker \partial_4{}^*$ *remains unchanged.*

Proof. Let ϕ' be another transversal for ψ with $e\phi' \in E(\text{end } G)$. Hence $s\phi' = (s\phi)(s\eta\lambda)$, where $\eta\colon S \to G$, such that $e\eta \in E(G)$ and $(s\eta)(s\eta)^{-1} = (s^{-1}s)\,\theta$. Then

$$\begin{aligned}(s_1\phi')(s_2\phi') &= (s_1\phi)(s_1\eta\lambda)(s_2\phi)(s_2\eta\lambda)\\ &= (s_1\phi)\cdot(s_2\phi)\cdot[e_{s_2\phi}(s_1\eta)](s_2\phi)\lambda\cdot(s_2\eta\lambda)\\ &= (s_1s_2)\,\phi[(s_1, s_2)\alpha\cdot(s_1\eta)(s_2\phi)\cdot(s_2\eta)]\lambda.\end{aligned}$$

On the other hand,

$$(s_1s_2)\,\phi' = (s_1s_2)\,\phi[(s_1s_2)\,\eta\lambda], \qquad \text{hence} \qquad (s_1s_2)\,\phi = (s_1s_2)\,\phi'[(s_1s_2)\,\eta]^{-1}\,\lambda.$$

Hence

$$(s_1\phi')(s_2\phi') = (s_1s_2)\,\phi'([(s_1s_2)\,\eta]^{-1} \cdot (s_1 , s_2)\,\alpha \cdot (s_1\eta)(s_2\phi) \cdot (s_2\eta))\,\lambda.$$

Now choose α': $S \times S \to G$ by

$$(s_1s_2)\,\eta(s_1 , s_2)\,\alpha' = (s_1 , s_2)\,\alpha \cdot (s_1\eta)(s_2\phi) \cdot (s_2\eta). \tag{9.7}$$

Then, for $e \in E(S)$, $s \in S$, we have

$$(se)\,\eta(se, e)\,\alpha' = (se)\,\eta.$$

As all three elements are in $G_{(es^{-1}s)\theta}$, we have $(se, e)\,\alpha' = (es^{-1}s)\,\theta$. Similarly $(e, es)\,\alpha' = (s^{-1}es)\,\theta$. We substitute α' into formula (9.5). Using formulae (9.4) and (9.7), we find that α' determines the same $\beta \in \ker \partial_4{}^*$ as α. Q.E.D.

Summarizing we obtain the following.

THEOREM 9.8. *Let G be a semilattice of groups, S an inverse semigroup, θ: $E(S) \cong E(G)$, ψ: $S \to$ end $G/$in G an idempotent separating homomorphism such that ψ has a transversal ϕ with $(e\theta)(s\phi) = (s^{-1}es)\,\theta$, for all $e \in E(S)$. Then ψ determines an element β' of $H^3_{SI}(Z(G)^0)$ if we regard $Z(G)$ as an S-module via $zs = z(s\phi)$. Then ψ is an abstract kernel of an extension of G by S if and only if $\beta' = 0$.*

Proof. If ψ is an abstract kernel, then (9.1) shows that $\beta' = 0$. Conversely, let ψ satisfy the hypothesis of the theorem and $\beta' = 0$. Then ψ has a transversal ϕ which determines α and $\beta \in \operatorname{im} \partial_3{}^*$ satisfying (9.5). Then there exists γ: $S \times S \to Z(G)$ with $(s_1 , s_2)\,\gamma \in Z(G)_{s_2^{-1}s_1^{-1}s_1s_2}$

$$(s_1 , s_2 , s_3)\,\beta = = (s_1 , s_2)\,\gamma(s_3\phi) \cdot [(s_1 , s_2s_3)\,\gamma]^{-1} \cdot (s_1s_2 , s_3)\,\gamma \cdot [(s_2 , s_3)\,\gamma]^{-1}$$

and $(se, e)\,\gamma$, $(e, es)\,\gamma$ are idempotents, if $e \in E(S)$. Let α': $S \times S \to G$ be defined by $(s_1 , s_2)\,\alpha' = (s_1 , s_2)\,\alpha[(s_1 , s_2)\,\gamma]^{-1}$. Then $(s_1 , s_2)\,\alpha'\lambda = (s_1 , s_2)\,\alpha\lambda$, as $(s_1 , s_2)\,\gamma \in Z(G)$, hence $[(s_1 , s_2)\,\gamma]^{-1}\,\lambda = (s_2^{-1}s_1^{-1}s_1s_2)\,\theta\lambda$. Also $(se, e)\,\alpha'$ and $(e, es)\,\alpha'$ are idempotents, if $e \in E(S)$. Hence ϕ and α' satisfy(i)–(vi) of Theorem 9.1, therefore by Theorem 9.2, ψ is an abstract kernel.

COROLLARY 9.9. *Let G be a semilattice of groups with $Z(G) = E(G)$, S an inverse semigroup, θ: $E(S) \cong E(G)$, and ψ: $S \to$ end $G/$in G an idempotent separating homomorphism with a transversal $\bar{\phi}$ satisfying $(e\theta)(s\bar{\phi}) = (s^{-1}es)\,\theta$, for all $e \in E(S)$. Then ψ has a transversal ϕ which together with a map α: $S \times S \to G$ determines an extension of G by S, where $(s_1\phi)(s_2\phi) = [(s_1s_2)\,\phi][(s_1 , s_2)\,\alpha\lambda]$, i.e., ϕ, α satisfy* (i)–(vi) *of Theorem* 9.1.

Proof. If $Z(G) = E(G)$, then $H^3_{S^I}(Z(G)^0) = 0$, hence any ψ is an abstract kernel.

Remark. Theorem 9.8 does not tell us under what conditions ψ exists, with a transversal ϕ satisfying $(e\theta)(s\phi) = (s^{-1}es)\,\theta$, for all $e \in E(S)$ (see [2]). Not even if G is abelian can we obtain any information from cohomology theory. Certainly a necessary condition for the existence of such a ψ is that $G_{(ss^{-1})\theta} \cong G_{(s^{-1}s)\theta}$, for all $s \in S$. The main problem, however, is to find all semilattices G of groups for which the set of idempotent separating homomorphisms from S to end $G/in\ G$ is nonempty, i.e., we are confronted with a representation-theoretical problem. Here we note a marked difference between inverse semigroups and groups: if G and S are groups, $s\psi$ = class of id_G, for all $s \in S$, always gives rise to at least one extension of G by S, namely $S \times G$.

THEOREM 9.10. *Let ψ be the abstract kernel of the extension (U, j) of G by S. Then $H^2_{S^I}(Z(G)^0)$ acts as a regular permutation group on the set of equivalence classes of extensions of G by S with abstract kernel ψ.*

Proof. Let ϕ be a transversal of G by S, and $\alpha: S \times S \to G$ a function satisfying conditions (i)–(vi) of Theorem 9.1. If $\partial_3 : \bar{C}_3(S^I) \to \bar{C}_2(S^I)$ is the S^I-morphism of Section 6 and

$$\partial_3{}^*: \mathrm{Hom}_{S^I}(\bar{C}_2(S^I), Z(G)^0) \to \mathrm{Hom}_{S^I}(\bar{C}_3(S^I), Z(G)^0)$$

the homomorphism induced by ∂_3, then $\beta \in \ker \partial_3{}^*$ can be identified with a function $\beta: S \times S \to Z(G)$ with $(s_1, s_2)\,\beta \in Z(G)_{(s_2^{-1}s_1^{-1}s_1s_2)\theta}$, $(s_1e, e)\,\beta$ and $(e, es_2)\,\beta$ are idempotents of $Z(G)$, for $e \in E(S)$, and $[(s_1, s_2)\,\beta(s_3\phi)] \cdot [(s_1, s_2s_3)\,\beta]^{-1}[(s_1s_2, s_3)\,\beta][(s_2, s_3)\,\beta]^{-1}$ is an idempotent of $Z(G)$. Replacement of α by $\alpha': S \times S \to G$, $(s_1, s_2)\,\alpha' = [(s_1, s_2)\,\alpha][(s_1, s_2)\,\beta]$ with ϕ fixed, yields another extension as $(s_1, s_2)\,\beta \in Z(G)$. If $\beta \in im\ \partial_2{}^*$, where $\partial_2 : \bar{C}_2(S^I) \to \bar{C}_1(S^I)$, as usual, then α and α' yield equivalent extensions, for a similar reason as in the abelian case. Hence $H^2_{S^I}(Z(G)^0)$ acts on the equivalence classes of extension of G by S with abstract kernel ψ. If α and α' yield equivalent extensions, again an argument similar to the abelian case shows that $\beta \in im\ \partial_2{}^*$, hence $H^2_{S^I}(Z(G)^0)$ acts faithfully and fixed-point free. The transitivity of $H^2_{S^I}(Z(G)^0)$ can be shown as follows: Let α', ϕ satisfy (i)–(vi) of Theorem 9.1. By (9.4) and Lemma 7.1, $\alpha\lambda = \alpha'\lambda$. Hence $(s_1, s_2)\,\beta = [(s_1, s_2)\,\alpha]^{-1}[(s_1, s_2)\,\alpha] \in Z(G)$. Then $\beta \in \mathrm{Hom}_{S^I}(\bar{C}_2(S^I), Z(G)^0)$. We show $\beta\partial_3{}^*$ is the zero-morphism. Substitute this equation into equation (iii) of Theorem 9.1 and use the fact that $(s_1, s_2)\,\beta \in Z(G)$. Then

$$[(s_1, s_2)\,\beta(s_3\phi)][(s_1, s_2s_3)\,\beta]^{-1} \cdot [(s_1s_2, s_3)\,\beta][(s_2, s_3)\,\beta]^{-1} \in E(Z(G))$$

which, in additive notation just means $\beta \in \ker \partial_3{}^*$.

10. Conjugation and Complementation

Let S be an inverse semigroup and $K(S)$ the kernel normal system of the maximal idempotent separating congruence. We know ([1, p. 70]) that $K(S) = \{s \in S \mid se = es, \text{ for all } e \in E(S)\}$.

Proposition 10.1. *Let π be a map from $E(S)$ to $K(S)$ satisfying*

(i) $e\pi \in K(S)_e$

(ii) $(e_1\pi)\, e_2 = (e_1e_2)\, \pi$, *for all* $e_1, e_2 \in E(S)$.

Then $\tau_\pi : s \to [(ss^{-1})\, \pi]^{-1}\, s[(s^{-1}s)\, \pi]$ *is an automorphism of S such that* $e\tau_\pi = e$, *for all* $e \in E(S)$.

Proof. Let $s_1, s_2 \in S$. Then

$$\begin{aligned}(s_1s_2)\tau_\pi &= [(s_1s_2s_2^{-1}s_1^{-1})\pi]^{-1}\, s_1s_2[(s_2^{-1}s_1^{-1}s_1s_2)\pi]\\ &= [(s_1s_1^{-1})\pi]^{-1}\, s_1s_2s_2^{-1}s_1^{-1}s_1s_2s_2^{-1}s_1^{-1}s_1s_2[(s_2^{-1}s_2)\pi]\\ &= [(s_1s_1^{-1})\pi]^{-1}\, s_1s_2[(s_2^{-1}s_2)\pi].\end{aligned}$$

On the other hand,

$$\begin{aligned}(s_1\tau_\pi)(s_2\tau_\pi) &= [(s_1s_1^{-1})\pi]^{-1}\, s_1[(s_1^{-1}s_1)\pi][(s_2s_2^{-1})\pi]^{-1}\, s_2[(s_2^{-1}s_2)\pi]\\ &= [(s_1s_1^{-1})\pi]^{-1}\, s_1[(s_1^{-1}s_1)\pi]\, s_2s_2^{-1}s_1^{-1}s_1[(s_2s_2^{-1})\pi]^{-1}\, s_2[(s_2^{-1}s_2)\pi]\\ &= [(s_1s_1^{-1})\pi]^{-1}\, s_1[(s_1^{-1}s_1s_2s_2^{-1})\pi][(s_1^{-1}s_1s_2s_2^{-1})\pi]^{-1}\, s_2[(s_2^{-1}s_2)\pi]\\ &= [(s_1s_1^{-1})\pi]^{-1}\, s_1s_1^{-1}s_1s_2s_2^{-1}s_2[(s_2^{-1}s_2)\pi]\\ &= [(s_1s_1^{-1})\pi]^{-1}\, s_1s_2[(s_2^{-1}s_2)\pi].\end{aligned}$$

Hence τ_π is an endomorphism. Furthermore, if $e \in E(S)$, then $e\tau_\pi = (e\pi)^{-1}\, e(e\pi) = e(e\pi)^{-1}(e\pi) = e$. Let $t \in S$ and $s = [(tt^{-1})\, \pi]\, t[(t^{-1}t)\, \pi]^{-1}$. Then $s\tau_\pi = [(tt^{-1})\, \pi]^{-1}[(tt^{-1})\, \pi]\, t[(t^{-1}t)\, \pi]^{-1}[(t^{-1}t)\, \pi] = tt^{-1}tt^{-1}t = t$. Hence τ_π is surjective. Suppose for $s, t \in S$, $s\tau_\pi = t\tau_\pi$. Then

$$\begin{aligned}t &= [(tt^{-1})\, \pi][(ss^{-1})\, \pi]^{-1}\, s[(s^{-1}s)\, \pi][(t^{-1}t)\, \pi]\\ &= tt^{-1}ss^{-1}ss^{-1}st^{-1}t\\ &= tt^{-1}st^{-1}t;\end{aligned}$$

hence $tt^{-1} \leqslant st^{-1}ts^{-1}$ and $t^{-1}t \leqslant s^{-1}tt^{-1}s \leqslant s^{-1}st^{-1}t$. Therefore $t^{-1}t \leqslant s^{-1}s$, similarly $tt^{-1} \leqslant ss^{-1}$ and, by symmetry, $t^{-1}t = s^{-1}s$, $tt^{-1} = ss^{-1}$ whence $t = tt^{-1}st^{-1}t = ss^{-1}ss^{-1}s = s$. Hence τ_π is injective. Q.E.D.

PROPOSITION 10.2.

(a) $L(S) = \{\pi\colon E(S) \to K(S) \mid e\pi \in K(S)_e\,, (e_1\pi)\, e_2 = (e_1e_2)\, \pi,$ *for all* $e_1\,, e_2 \in E(S)\}$

is a group under the multiplication

$$e(\pi_1\pi_2) = (e\pi_1)(e\pi_2)$$

(b) *In* $S = \{\tau_\pi \mid \pi \in L(S)\}$ *is a group of idempotent preserving automorphisms of* S.

(c) *The map* $\pi \to \tau_\pi$ *is a group homomorphism and In* $S \simeq L(S)/Y(S)$, *where* $Y(S) = \{\pi \in L(S) \mid s[(s^{-1}s)\,\pi] = [(ss^{-1})\,\pi]\, s\}$

(d) *In* S *is a normal subgroup of* Aut S, *the group of all automorphisms of* S.

Proof. (a) $(e\pi_1)(e\pi_2) \in K(S)e$ and $(e_1\pi_1)(e_1\pi_2)\, e_2 = (e_1\pi_1)\, e_2(e_1\pi_2)\, e_2 = [(e_1e_2)\,\pi_1][(e_1e_2)\,\pi_2]$. Hence $\pi_1\pi_2 \in L(S)$. Clearly this multiplication is associative, $e \to e$ is the identity of $L(S)$, and $e \to (e\pi)^{-1}$ is the inverse of π.

(b) and (c) $\pi \to \tau_\pi$ is surjective and

$$s\tau_{\pi_1}\tau_{\pi_2} = [(ss^{-1})\pi_2]^{-1}[(ss^{-1})\pi_1]^{-1}\, s[(s^{-1}s)\pi_1][(s^{-1}s)\pi_2] = s\tau_{\pi_1\pi_2}\,.$$

Furthermore $[(ss^{-1})\,\pi]^{-1}\, s[(s^{-1}s)\,\pi] = s$ is equivalent to $\pi \in Y(S)$.

(d) Let $\alpha \in \text{Aut } S$, $\pi \in L(S)$, then

$$\begin{aligned} s\alpha^{-1}\tau_\pi\alpha &= \{[(ss^{-1})\,\alpha^{-1}\pi]^{-1}(s\alpha^{-1})[(s^{-1}s)\,\alpha^{-1}\pi]\\ &= [(ss^{-1})\,\alpha^{-1}\pi\alpha]^{-1}\, s[(s^{-1}s)\,\alpha^{-1}\pi\alpha]. \end{aligned}$$

We have to show that $\alpha^{-1}\pi\alpha \in L(S)$. But $e\alpha^{-1}\pi\alpha \in K(S)_{e\alpha^{-1}\alpha} = K(S)_e\,$, and if $e_1\,, e_2 \in E(S)$, then

$$\begin{aligned} (e_1\alpha^{-1}\pi\alpha)\, e_2 &= (e_1\alpha^{-1}\pi\alpha)(e_2\alpha^{-1}\alpha) = [(e_1\alpha^{-1}\pi)(e_2\alpha^{-1})]\\ &= [(e_1\alpha^{-1})(e_2\alpha^{-1})]\, \pi\alpha = (e_1e_2)\, \alpha^{-1}\pi\alpha. \end{aligned}$$

Hence $\alpha^{-1}\tau_\pi\alpha = \tau_{\alpha^{-1}\pi\alpha} \in In\ S$. Q.E.D.

Remark. We note that there are two distinct notions which are both generalizations of the group of inner automorphisms, namely *in* G for semilattices G of groups and *In* S, for inverse semigroups in general.

DEFINITION. Let S be an inverse semigroup, G a kernel normal system in S and U an inverse subsemigroup of S such that

$$S = UG, \qquad U \cap G = E(S).$$

We say U is a *complement* of G in S.

DEFINITION. Let U be an inverse subsemigroup of G and $\pi \in L(S)$. Then $U\tau_\pi$ is called a *conjugate* of U in S.

PROPOSITION 10.3. *If G is a kernel normal system of an inverse semigroup S and U is a complement of G in S, then every conjugate of U is a complement of G in S.*

Proof. Let $\pi \in L(S)$. Since $S = UG$, we have $S = S\tau_\pi = (U\tau_\pi)(G\tau_\pi) = (U\tau_\pi)\, G$ as $g \in G$ implies

$$g\tau_\pi = [(gg^{-1})\,\pi]^{-1}\, g[(g^{-1}g)\,\pi] = [(g^{-1}g)\,\pi]^{-1}\, g[(g^{-1}g)\,\pi] \in G$$

because G is a kernel normal system. $U \cap G = E(S)$ implies $E(S) = U\tau_\pi \cap G\tau_\pi = U\tau_\pi \cap G$. Q.E.D.

PROPOSITION 10.4. *If U is a complement of a kernel normal system G in an inverse semigroup S, then S is an extension of G by U.*

Proof. Let $s \in S$, then $s = ug$, $u \in U$, $g \in G$, hence $s = (ugg^{-1})(u^{-1}ug)$. Define $j\colon S \to U$ by $sj = ugg^{-1}$. Then j is well-defined as $(ugg^{-1})^{-1}(ugg^{-1}) = gg^{-1}u^{-1}u = (u^{-1}ug)(u^{-1}ug)^{-1}$ and Lemma 7.1 applies. Let $s_1 = u_1g_1$, $s_2 = u_2g_2$, $u_1, u_2 \in U$, $g_1, g_2 \in G$, $u_1^{-1}u_1 = g_1g_1^{-1}$, $u_2^{-1}u_2 = g_2g_2^{-1}$. Then $(s_1s_2)j = [(u_1u_2)(u_2^{-1}g_1u_2g_2)]j = u_1u_2 = (s_1j)(s_2j)$ as $u_2^{-1}u_1^{-1}u_1u_2 = u_2^{-1}g_1g_1^{-1}u_2 = (u_2^{-1}g_1g_2)(u_2^{-1}g_1u_2g_2)^{-1}$. Hence j is a homomorphism, and is clearly surjective. Moreover, $e \in E(S)$ implies $ej = e$, and $sj \in E(U)$ if and only if $s = eg$, $g \in G$, $gg^{-1} = e$, i.e., $s \in G$. Hence j is an idempotent separating homomorphism with G as kernel normal system.

11. AN INTERPRETATION OF H_S^1

Suppose S is an inverse semigroup and A an S-module. We define an S^I-module A^1 by $A_e^1 = A_e$, if $e \in E(S)$

$$A_I^1 = H^0_{E(S)}(A) = \{\delta\colon E(S) \to A \mid (e_1\delta)e_2 = (e_1e_2)\delta,$$
$$\text{for all } e_1, e_2 \in E(S), \text{ and } e\delta \in A_e\}$$

and $a \dotplus \delta = a + e\delta$, for $a \in A_e$, $aI = a$, for all $a \in A^1$, $\delta s = (ss^{-1})\,\delta s$, for $s \in S$, extending the S-module structure of A. We verify quickly that A^1 becomes an S^I-module by this definition, and using the projective resolution $\bar{C}(S^I) \to^{\partial_i} \bar{C}_{i-1}(S^I) \to \cdots \to \bar{C}_0(S^I) \to Z_{S^I} \to^{\epsilon} 0$, we see that $H^i_{S^I}(A^0) = H^i_{S^I}(A^1)$, for $i \geqslant 2$. Moreover, ker $\partial_2{}^*$ can be identified with the group of all maps $\alpha\colon S \to A$ such that $s\alpha \in A_{s^{-1}s}$ and $(s_1\alpha)\,s_2 + (s_2\alpha) = (s_1s_2)\,\alpha$ while im $\partial_1{}^*$ is the group of all maps $\alpha\colon S \to A$ such that there exists some $\delta \in H^0_{E(S)}(A)$ with $s\alpha = (ss^{-1})\,\delta s - (s^{-1}s)\,\delta$.

THEOREM 11.1. *Let S be an inverse semigroup and A a kernel normal system in S consisting of abelian groups. If A is complemented in S, then In S acts on the set of complements of A in S by conjugation and the orbits of In S are in one-to-one correspondence with the elements of $H^1_{U^I}(A^1)$ where U is any complement of A in S and A is regarded as a U-module as usual.*

Proof. Let U be a complement of A in S and V another such. Then $u \in U$ can be written uniquely as $u = v(u\alpha)^{-1}$, $v \in V$, $v^{-1}v = (u\alpha)^{-1}(u\alpha)$, for some $u\alpha \in A_{u^{-1}u}$. As V is a complement of A, every $v \in V$ is of the form $u(u\alpha)$, and α is a map from U to A with $u\alpha \in A_{u^{-1}u}$. Then $u_1(u_1\alpha)\,u_2(u_2\alpha) = u_1u_2[(u_1u_2)\,\alpha]$, for all $u_1, u_2 \in U$. On the other hand $u_1(u_1\alpha)\,u_2(u_2\alpha) = u_1u_2u_2^{-1}(u_1\alpha)\,u_2(u_2\alpha)$. Hence $(u_1u_2)\,\alpha = [u_2^{-1}(u_1\alpha)\,u_2](u_2\alpha)$. In U-module notation this means that $\alpha \in \ker \partial_2{}^*$. The same computation in the opposite direction shows that any $\alpha \in \ker \partial_2{}^*$ gives rise to another complement $V = \{u(u\alpha) \mid u \in U\}$ of A in S. Suppose that $V = U\tau_\pi$, for some $\pi \in L(S)$. Define a map $\delta_\pi : E(U) \to A$ by: if $e\pi = u_e a_e$, $u_e^{-1}u_e = a_e a_e^{-1}$, $u_e \in U$, $a_e \in A$, then $e\delta_\pi = ae \in A_e$. If $t \in U$, then $t\tau_\pi = [(tt^{-1})\,\pi]^{-1}\,t[(t^{-1}t)\,\pi] = a^{-1}_{tt^{-1}}tu_{t^{-1}t}a_{t^{-1}t}$. Note that $(e_1\delta_\pi)\,e_2 = (e_1e_2)\,\delta_\pi$, and that $A \subseteq K(S)$. Hence $\tau\delta_\pi \in In\ S$, and $t\tau_\pi = t\tau_{\delta_\pi}$, for all $t \in U$. Moreover $\delta_\pi \in H^0_{E(U)}(A)$. Let $\alpha \in \ker \partial_2{}^*$ be the map associated with V. Then

$$u\tau_\pi = u\delta_\pi = uu^{-1}[(uu^{-1})\,\delta_\pi]^{-1}\,u[(u^{-1}u)\,\delta_\pi].$$

Hence

$$u^{-1}[(uu^{-1})\,\delta_\pi]^{-1}\,u[(u^{-1}u)\,\delta_\pi] = u. \tag{11.1}$$

In U-module notation, we have $u\alpha = -[(uu^{-1})\,\delta_\pi u - (u^{-1}u)\,\delta_\pi]$, for all $u \in U$. Therefore $\alpha \in \operatorname{im} \partial_1{}^*$. Conversely if α satisfies (11.1), for some $\delta \in H^0_{E(S)}(A)$ in place of δ_π, then $u(u\alpha) = u\tau_\delta$, for all $u \in U$, hence we have established a one-to-one correspondence between the orbits of *In* S and the elements of $H^1_{U^I}(A^1)$.

THEOREM 11.2. *Let A be an S-module, and $H^2_{S^I}(A^0) = 0$. Then every extension (U, j) of A by S is such that A is complemented in U.*

Proof. By the proof of Proposition 7.2, there is a transversal $\rho: S \to U$ which is an injective homomorphism. It follows that $S\rho$ is a complement of A in U.

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