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Differential Games and *BV* Functions

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1. INTRODUCTION

In [5], the main result (Theorem 10.1, p. 51) is that for a two-person zero-sum differential game which ends at a predetermined time T and satisfies the Isaacs condition (and other reasonable conditions), its value function $V = V(t, x)$ (in the sense of Friedman) exists. These games are the ones studied in the present paper. Roughly, $V(t_0, x_0)$ is the payoff if the game starts at time $t_0 \leqslant T$ in state $x_0 \in \mathbb{R}^m$ and both players play optimally thereafter. A broad class of optimal control problems which end at time T constitute a special case of this situation, since adding a powerless fictitious second player (to the one player in the optimal control problem) guarantees that the Isaacs condition will be satisfied. The results of this paper seem to be new even in this case.

The existence and nature of optimal controls has never been adequately established. What one is really looking for (see [9, p. 27]) is a pair of $\mathbb{R}^p$-valued functions $\Phi(t, x)$ and $\Psi(t, x)$, such that if at any time t and any state x the first player always plays the control $\Phi(t, x)$ and the second plays the control $\Psi(t, x)$, then both will be playing optimally.

We will call the set of all points (t, x) at which $\nabla_x V = (\partial V/\partial x_1, \dots, \partial V/\partial x_m)$ fails to be continuous the *singular set* for the differential game (this corresponds to the *singular surfaces* of [11, p. 66]). Study of the singular set is absolutely crucial for the problem of determining optimal controls, since different regimes of optimal control are separated by the singular set (see [3, pp. 353–356] and [9, p. 43]) and since optimal control in regions where $\nabla_x V$ is continuous is comparatively easy (see [9, p. 41]).

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If the result of the conjecture stated in this paper is correct, then $\partial V/\partial x_1, \ldots, \partial V/\partial x_m$ are each of bounded variation on the (t, x)-space, so that for the first time strong theorems about the discontinuity set of BV functions (e.g., [6, Theorem 4.5.9; compare Sect. 4.5.7 with Lemma 2 of the present paper to understand the change of notation] and [16]) can be applied to study the singular set. This paper proves that the conjecture is true in case $m = 1$. The proof given generalizes to the $m > 1$ case at every point but one (in the proof of Lemma 7, since there is no maximum principle available for strongly coupled parabolic systems of equations), giving some support to the conjecture.

The only hypothesis in the conjecture which is not reasonably standard is that the Hamiltonian function $H = H(t, x, p)$ has mild x-variation. This concept is defined and argued for near the beginning of Section 4.

Lemma 10 is of some independent interest. It shows for the first time that (at least in the $m = 1$ case with mild x-variation) $\partial W^k/\partial x$ converges in L^1_{loc} to $\partial W/\partial x$ as $k \to \infty$, where (apart from a time-reversal) W is the value function of the game ($W(t, x) \equiv V(T - t, x)$) and W^k is an approximation to W obtained by applying the method of vanishing viscosity to the Hamilton–Jacobi equation W satisfies.

2. Some Basic Assumptions, Terminology and Estimates

Let T be a positive number, fixed throughout the paper. This paper deals with a zero-sum differential game between two players, $\mathscr{P}_y$ and $\mathscr{P}_z$, which ends at time $t = T$. Compact spaces Y and Z are given, the control sets for $\mathscr{P}_y$ and $\mathscr{P}_z$, respectively. (For simplicity, assume that they are subsets of some Euclidean space $\mathbb{R}^p$.) At any time t, $x(t) \in \mathbb{R}^m$ denotes the state of the game at time t. Starting at any initial time $t_0 \in [0, T]$ and at any initial state $x_0 \in \mathbb{R}^m$, $x(t)$ evolves according to the following initial-value problem for a system of ordinary differential equations:

$$\frac{dx}{dt}(t) = f(t, x(t), y(t), z(t)) \quad \text{a.e. for } t_0 \leqslant t \leqslant T \tag{2.1}$$

$$x(0) = x_0. \tag{2.2}$$

Here $f\colon [0, T] \times \mathbb{R}^m \times Y \times Z \to \mathbb{R}^m$ is a given continuous function, $y(\cdot)$ is a measurable function with values in Y (the control function for $\mathscr{P}_y$), and $z(\cdot)$ is a measurable function with values in Z (the control function for $\mathscr{P}_z$). It is required that $x(\cdot)$ be absolutely continuous so that dx/dt is well defined a.e. for $t_0 \leqslant t \leqslant T$. We assume that

(A) There is an integrable function $k(\cdot)$ on $[0, T]$ such that

$$x \cdot f(t, x, y, z) \leqslant k(t)(1+|x|^2) \qquad \text{for } t \in [0, T],\ x \in \mathbb{R}^m,\ y \in Y, \text{ and } z \in Z,$$

where $\cdot$ and $|\,|$ denote the dot product and norm in $\mathbb{R}^m$, respectively.

(B) For every $R > 0$ there exists a constant $A_R > 0$ such that

$$|f(t, x, y, z) - f(t, x^*, y, z)| \leqslant A_R |x - x^*|, \qquad \text{whenever} \quad t \in [0, T],$$
$$x \in \mathbb{R}^m \text{ with } |x| \leqslant R,\ x^* \in \mathbb{R}^m \text{ with } |x^*| \leqslant R,\ y \in Y, \text{ and } z \in Z.$$

By [9, p. 2], the assumptions above insure that (2.1) and (2.2) have a unique solution.

LEMMA 1. *Let $\delta > 0$ be chosen so that over any subinterval of $[0, T]$ of length less than δ the integral of $k(\cdot)$ is less then $\frac{1}{4}$. Let N be an integer larger than T/δ. Then for any $R \geqslant 1$, any $t_0 \in [0, T]$, and any solution $x(\cdot)$ of equations* (2.1) *and* (2.2),

$$|x(t_0)| \leqslant R \textit{ implies that } |x(t)| \leqslant 2^N R \textit{ for } t_0 \leqslant t \leqslant T.$$

Idea of Proof

Assumption (A) is used together with

$$|x(t)|^2 = |x(t_0)|^2 + \int_{t_0}^{t} 2x(s) \cdot f(s, x(s), y(s), z(s))\, ds.$$

It is shown that $|x(t_0)| \leqslant R$ implies that $|x(t)| \leqslant 2R$ for $t_0 \leqslant t \leqslant t_0 + (T - t_0)/N$, and so on by induction, considering an interval of the form $t_0 + n(T - t_0)/N \leqslant t \leqslant t_0 + (n+1)(T - t_0)/N$ in each induction step, for $0 \leqslant n \leqslant N - 1$.

Let $g\colon \mathbb{R}^m \to \mathbb{R}$ and $h\colon [0, T] \times \mathbb{R}^m \times Y \times Z \to \mathbb{R}$ be continuous functions. For given control functions $y(\cdot)$ and $z(\cdot)$ as above, define the payoff functional $P[y, z]$ as follows:

$$P[y, z] = g(x(T)) + \int_{t_0}^{T} h(t, x(t), y(t), z(t))\, dt. \tag{2.3}$$

The aim of $\mathscr{P}_y$ is to maximize this payoff. The aim of $\mathscr{P}_z$ is to minimize this.

We will always assume that the *Isaacs condition* holds, i.e., that for each fixed $(t, x, p) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}^m$ we have (with $\cdot$ indicating the dot product in $\mathbb{R}^m$)

$$\min_{z \in Z} \max_{y \in Y} [f(t, x, y, z) \cdot p + h(t, x, y, z)]$$
$$= \max_{y \in Y} \min_{z \in Z} [f(t, x, y, z) \cdot p + h(t, x, y, z)]. \tag{2.4}$$

The common value of both sides of (2.4) defines $H(t, x, p)$, the Hamiltonian function of the differential game. Under the assumptions stated above the *value function* V: $[0, T] \times \mathbb{R}^m \to \mathbb{R}$ of the game (in the sense of Friedman) is defined [9, p. 13]. Roughly, $V(t_0, x_0)$ is the maximum payoff that $\mathscr{P}_y$ can force, given that the game starts at time t_0 in state x_0.

We will also assume for every $R > 0$ that the constant A_R of assumption (B) above also satisfies

(C) $|g(x) - g(x^*)| \leqslant A_R |x - x^*|$,

(D) $|h(t, x, y, z) - h(t, x^*, y, z)| \leqslant A_R |x - x^*|$,

whenever $x \in \mathbb{R}^m$ with $|x| \leqslant R$, $x^* \in \mathbb{R}^m$ with $|x^*| \leqslant R$, $t \in [0, T]$, $y \in Y$, and $z \in Z$. Then, defining $V(T, x) = g(x)$ for each $x \in \mathbb{R}^m$, V is continuous on $[0, T] \times \mathbb{R}^m$ and uniformly Lipschitz continuous on compact subsets of $[0, T) \times \mathbb{R}^m$ [9, p. 10]. Also, V satisfies the Hamilton–Jacobi equation (or Isaacs equation)

$$\frac{\partial V}{\partial t} + H\left(t, x, \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_m}\right) = 0 \tag{2.5}$$

at almost all points $(t, x) \in [0, T] \times \mathbb{R}^m$ [9, p. 12].

It turns out in what follows that we are interested in the values of $V(t, x)$ only for $|x| \leqslant Q$, where $Q \geqslant 1$ is some (arbitrarily) large number. Clearly we can modify the definitions of $f(t, x, y, z)$, $g(x)$, and $h(t, x, y, z)$ for $|x| > 2^N Q$ so that for some $P > 2^N Q$ we have $f(t, x, y, z) = g(x) = h(t, x, y, z) = 0$ whenever $|x| \geqslant P$, $t \in [0, T]$, $y \in Y$, and $z \in Z$. Clearly the values of $V(t, x)$ and $H(t, x, p)$ are unchanged by this for $|x| \leqslant Q$, but that $V(t, x) = H(t, x, p) = 0$ whenever $|x| \geqslant P$, $t \in [0, T]$, and $p \in \mathbb{R}^m$. Clearly we can do this so that the modified functions f, g, and h are continuous on their respective domains, with assumptions (A), (B), (C), and (D) giving (for some constants A and B) that

$$|f(t, x, y, z)| \leqslant B, \qquad |g(x)| \leqslant B, \qquad |h(t, x, y, z)| \leqslant B, \tag{2.6}$$

$$|f(t, x, y, z) - f(t, x^*, y, z)| \leqslant A\,|x - x^*|, \tag{2.7}$$

$$|g(x) - g(x^*)| \leqslant A\,|x - x^*|, \tag{2.8}$$

$$|h(t, x, y, z) - h(t, x^*, y, z)| \leqslant A\,|x - x^*|, \tag{2.9}$$

whenever $t \in [0, T]$, $x \in \mathbb{R}^m$, $x^* \in \mathbb{R}^m$, $y \in Y$, and $z \in Z$. With some effort, one can obtain the following estimates:

$$|H(t, x, 0)| \leqslant B, \tag{2.10}$$

$$|H(t, x, p) - H(t, x, p^*) \leqslant B\,|p - p^*|, \tag{2.11}$$

$$|H(t, x, p) - H(t, x^*, p)| \leqslant (A\,|p| + A)\,|x - x^*|, \tag{2.12}$$

whenever $t \in [0, T]$, $x \in \mathbb{R}^m$, $x^* \in \mathbb{R}^m$, $p \in \mathbb{R}^m$, and $p^* \in \mathbb{R}^m$. For $t > T$, define $f(t, x, y, z) = f(T, x, y, z)$ and $h(t, x, y, z) = h(T, x, y, z)$ for all $x \in \mathbb{R}^m$, $y \in Y$, and $z \in Z$. For $t < 0$, define $f(t, x, y, z) = f(0, x, y, z)$ and $h(t, x, y, z) = h(0, x, y, z)$ for all $x \in \mathbb{R}^m$, $y \in Y$, and $z \in Z$. The resulting functions f, h, and H are continuous on their new domains, and properties (2.6)–(2.12) continue to hold (with $t \in \mathbb{R}$ replacing $t \in [0, T]$).

Let $l \in C^\infty(\mathbb{R})$ with $l(x) \geqslant 0$ for all $x \in \mathbb{R}$, $l(x) \equiv 0$ for $|x| \geqslant 1$, $\int_{-\infty}^{\infty} l(x)\, dx = 1$, and $l(x) = l(-x)$ for all $x \in \mathbb{R}$. For any $\varepsilon > 0$, define $l_\varepsilon(x) = l(x/\varepsilon)/\varepsilon$ for all $x \in \mathbb{R}$. For any locally integrable function $F = F(t, x, p)$ defined on $\mathbb{R}^3$, we define its *ε-mollification* $[F]_\varepsilon = [F]_\varepsilon(t, x, p)$ as follows:

$$[F]_\varepsilon(t, x, p) = \int_{\mathbb{R}^3} l_\varepsilon(t-\tau)\, l_\varepsilon(x-\xi)\, l_\varepsilon(p-\pi)\, F(\tau, \xi, \pi)\, d\tau\, d\xi\, d\pi.$$

We define ε-mollifications of functions of a different number of variables similarly.

Define

$$\mathscr{H}(t, x, p) = H(T-t, x, p) \qquad \text{and} \qquad \mathscr{H}^j = [\mathscr{H}]_{1/j}$$

for each $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, $p \in \mathbb{R}^m$, $j = 1, 2, \ldots$. Clearly each $\mathscr{H}^j \in C^\infty(\mathbb{R}^{2m+1})$ and $\mathscr{H}^j \to \mathscr{H}$ uniformly on any compact subset of $\mathbb{R}^{2m+1}$ as $j \to \infty$. Clearly each of the $\mathscr{H}^j$ vanishes for $|x| \geqslant P + m^{1/2}$. Fairly easy estimates show that

$$|\mathscr{H}^j(t, x, p) - \mathscr{H}^j(t, x^*, p)| \leqslant A(|p| + m^{1/2} + 1)\, |x - x^*|, \tag{2.13}$$

$$|\mathscr{H}^j(t, x, p) - \mathscr{H}^j(t, x, p^*)| \leqslant B\, |p - p^*| \tag{2.14}$$

for all $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, $x^* \in \mathbb{R}^m$, $p \in \mathbb{R}^m$, $p^* \in \mathbb{R}^m$, and $j = 1, 2, \ldots$. We can use the above estimates to apply [10, Theorem 2.1] to get for any $0 < \varepsilon < 1$ and for $j = 1, 2, \ldots$, a unique bounded solution $W^{\varepsilon, j}$ of

$$\frac{\partial W^{\varepsilon,j}}{\partial t} - \mathscr{H}^j\left(t, x, \frac{\partial W^{\varepsilon,j}}{\partial x_1}, \ldots, \frac{\partial W^{\varepsilon,j}}{\partial x_m}\right) = \varepsilon \Delta W^{\varepsilon,j} \quad \text{for } 0 < t \leqslant T,\ x \in \mathbb{R}^m, \tag{2.15}$$

$$W^{\varepsilon,j}(0, x) = g(x) \qquad \text{for } x \in \mathbb{R}^m. \tag{2.16}$$

Following the proof of [10, Theorem 2.1] we obtain constants K_0 and K, independent of $0 < \varepsilon < 1$ and $j = 1, 2, \ldots$, such that

$$|W^{\varepsilon,j}(t, x)| \leqslant K_0, \tag{2.17}$$

$$\left|\frac{\partial W^{\varepsilon,j}}{\partial x_i}(t, x)\right| \leqslant K, \tag{2.18}$$

for $0<\varepsilon<1$, $j=1, 2,..., 0<t\leqslant T$ (Caution: There is a time-reversal when going from our situation to that of [10].), $x\in\mathbb{R}^m$, and $i=1, 2,..., m$. In fact, defining $\gamma=B+Bm^{1/2}$, $\gamma_R=\max(A+Am^{1/2}, B)$, $\gamma^\wedge=\max(A, B)$, any δ such that $\delta>\gamma+\gamma/(\gamma^\wedge+1)$, any γ' such that $2\gamma_R(1+|v|)|v|\leqslant\gamma'(1+|v|^2)$ for all $v\in\mathbb{R}^m$, and any η with $\eta>3\gamma'+\gamma'/m(\gamma^\wedge)^2$, we may take $K_0=(\gamma^\wedge+1)\,e^{\delta T}$ and $K=m^{1/2}\gamma^\wedge e^{\eta T/2}$.

3. Functions of Class BV

The following definition is that of [16, p. 226]:

DEFINITION. Let Ω be an open subset of $\mathbb{R}^n$. Then a locally integrable function $f\colon\Omega\to\mathbb{R}$ is of *bounded variation* if and only if there exist set functions $\mu_1,..., \mu_n$—each of which is a signed measure of finite total variation when restricted to the Borel subsets of any compact subset of Ω—such that

$$\int_\Omega f\,\frac{\partial\phi}{\partial x_i}\,dx_1\cdots dx_n=-\int_\Omega \phi\,d\mu_i(x) \tag{3.1}$$

for every $\phi\in C_0^\infty(\Omega)$ and $i=1,..., n$. The set of all such functions is denoted $BV(\Omega)$.

Thus, roughly, $f\in BV(\Omega)$ if and only if its first order derivatives (in the sense of distribution theory) are signed measures. If A and B are Borel subsets of Ω which are *not* contained in a compact subset of Ω, it could happen that (e.g.) $\mu_1(A)=\infty$ and $\mu_1(B)=-\infty$, so that $\mu_1(A\cup B)$ cannot be defined; this explains the awkwardness in the description of $\mu_1,..., \mu_n$ above.

Compare the following lemma with [6, Sect. 4.5.7]:

LEMMA 2. *Let f be a real-valued function which is locally integrable on an open subset Ω of $\mathbb{R}^n$. Then $f\in BV(\Omega)$ if and only if there exists a sequence $f_j\in C^\infty(\Omega)$, $j=1, 2,...$, with*

$$\lim_{j\to\infty}\int_K |f_j-f|\,dx=0 \qquad \textit{and} \quad \liminf_{j\to\infty}\int_K |\nabla f_j|\,dx<\infty \tag{3.2}$$

for every compact $K\subset\Omega$, where $|\nabla f_j|=[(\partial f_j/\partial x_1)^2+\cdots+(\partial f_j/\partial x_n)^2]^{1/2}$.

Idea of Proof

($\Rightarrow$) Assume that $f\in BV(\Omega)$. Let $K_1, K_2,...$ be a sequence of compact subsets of Ω, with Ω as their union and with each K_i containing all previous ones in its interior. For $j=1, 2,...$, define f_j to be the j^{-1}-

mollification of the function which is identically equal to f on K_j and identically equal to 0 elsewhere. The first part of (3.2) can be proved by showing that each of the three integrals on the right-hand side of

$$\int_K |f_j - f|\, dx \leqslant \int_K |f_j - h_j|\, dx + \int_K |h_j - h|\, dx + \int_K |h - f|\, dx$$

can be made arbitrarily small, where h is a continuous function with h_j as its j^{-1}-mollification. To prove the second part of (3.2) it suffices to obtain a bound for each $\int_K |\partial f_j/\partial x_i|\, dx$. Replacing f_j by its integral formula, taking $\partial/\partial x_i$ inside that integral, using (3.1), finding a simple upper bound involving $|\mu_i|$ (the total variation measure which comes from μ_i), applying the Fubini theorem, and using the fact that $|\mu_i|$ is finite on compact subsets of Ω gives the second part of (3.2).

($\Leftarrow$) Assume the existence of a sequence $f_j \in C^\infty(\Omega)$ with the properties stated. Let W be any bounded open set with closure $\bar{W} \subset \Omega$. We may assume (by choosing a subsequence if necessary) that there is a constant C such that $\int_{\bar{W}} |\nabla f_j|\, dx \leqslant C$ for $j = 1, 2, \ldots$. Fix i with $1 \leqslant i \leqslant n$. For any $\varphi \in C_0^\infty(W)$ define

$$L(\varphi) = -\int_W \frac{\partial \varphi}{\partial x_i} f\, dx.$$

An easy estimate shows that for any φ and $\varphi^* \in C_0^\infty(W)$ we have $|L(\varphi) - L(\varphi^*)| \leqslant C\|\varphi - \varphi^*\|$, where $\|\cdot\|$ denotes the supremum norm. Let $C_0(W)$ denote the completion of $C_0^\infty(W)$ under the norm $\|\cdot\|$. Since L is uniformly continuous on $C_0^\infty(W)$, it has a unique extension to $C_0(W)$. The Riesz representation theorem ([15, Theorem 6.19], e.g.) then gives the existence of a unique regular Borel signed measure $\mu_i|_W$ for which (3.1) holds for every $\varphi \in C_0^\infty(W)$. If $W \subset U \subset \bar{U} \subset \Omega$ for some bounded open set U, then the uniqueness portion of the above result shows that $\mu_i|_W$ and $\mu_i|_U$ agree on any Borel subset of W. Thus a set function μ_i as required is defined.

Compare the following lemma with [12, p. 218]:

LEMMA 3. *Let f be a real-valued function which is locally integrable on an open subset Ω of $\mathbb{R}^n$. Then $f \in BV(\Omega)$ if and only if for every compact $K \subset \Omega$ there exist positive constants C_K and δ_K such that*

$$\int_K |f(x+y) - f(x)|\, dx \leqslant C_K |y| \tag{3.3}$$

for every $y \in \mathbb{R}^n$ with $|y| \leqslant \delta_K$.

Idea of Proof

($\Rightarrow$) Assume that $f \in BV(\Omega)$. Let $K \subset \Omega$ be compact. Select any compact $K^* \subset \Omega$ having K in its interior, and take δ_K to be the distance from K to the boundary of K^*. Using Lemma 2, select a sequence $f_j \in C^\infty(\Omega)$, $j = 1, 2, \ldots$, with $\lim_{j \to \infty} \int_{K^*} |f_j - f| \, dx = 0$ and $\int_{K^*} |\nabla f_j| \, dx \leqslant C$ for some constant C and $j = 1, 2, \ldots$. Starting with

$$\int_K |f_j(x+y) - f_j(x)| \, dx = \int_K \left| \int_0^1 \sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(x+ty) \, y_i \, dt \right| dx$$

for $|y| \leqslant \delta_K$, we easily see after simple estimations that we can take $C_K = nC$.

($\Leftarrow$) Assume that for every compact $K \subset \Omega$ there exist C_K and δ_K so that (3.3) holds. For each $j = 1, 2, \ldots$, define f_j as in the first part of the proof of Lemma 2. Tedious but straightforward calculations show that (3.2) holds for every compact $K \subset \Omega$. Thus by Lemma 2, $f \in BV(\Omega)$.

4. A Conjecture and Some Essential Lemmas

Definition. We say that $H = H(t, x, p)$ has mild x-variation if and only if $H_{x_i} = \partial H / \partial x_i$ is continuous for $i = 1, 2, \ldots, m$ and there is a constant C such that

$$|H_{x_i}(t, x, p) - H_{x_i}(t, x^*, p^*)| \leqslant C|x - x^*| + C|p - p^*|$$

for $i = 1, 2, \ldots, m$, $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, $x^* \in \mathbb{R}^m$, $p \in \mathbb{R}^m$, *and* $p^* \in \mathbb{R}^m$.

Conjecture. Make the assumptions of Section 2. Assume that H has mild x-variation. Assume that g has Hölder continuous second derivatives. Then for $i = 1, 2, \ldots, m$,

$$\frac{\partial V}{\partial x_i} \in BV((0, T) \times \mathbb{R}^m).$$

Our main result (in Sect. 5) is that the conjecture is true for $m = 1$. Since we will use the characterization of BV functions given in Lemma 3 to prove this, clearly our consideration (a few lines after (2.5)) of the values of $V(t, x)$ only for $|x| \leqslant Q$ is justified.

It is almost certain the requirement that H have mild x-variation is not a best-possible assumption in the conjecture. However, the proof given here requires essentially this assumption, the assumption is true in the important case in which the Hamiltonian function $H(t, x, p)$ is independent of x, and *some* assumption about the functions $H_{x_i}(t, x, p)$ must be made for the conclusion of the conjecture to be true, as the following counterexample shows:

COUNTEREXAMPLE. Let F be a Lipschitz function of compact support in $\mathbb{R}^m$, with (for some i with $1 \leqslant i \leqslant m$) $\partial F/\partial x_i$ *not* in $BV(\mathbb{R}^m)$. Consider the differential game with $f \equiv 0$, $g \equiv 0$, and $h(t, x, y, z) \equiv F(x)$. Then $H(t, x, p) \equiv F(x)$, so $H_{x_i}(t, x, p) \equiv (\partial F/\partial x_i)(x)$. We will show that $\partial V/\partial x_i$ is not in $BV((0, T) \times \mathbb{R}^m)$. Suppose, for contradiction, that $\partial V/\partial x_i \in BV((0, T) \times \mathbb{R}^m)$. Let K be any compact subset of $\mathbb{R}^m$. Let $K^* = \{(t, x); T/3 \leqslant t \leqslant 2T/3, x \in K\}$. By Lemma 3 there exist constants C^* and δ^* such that

$$\int_{T/3}^{2T/3} \int_K \left| \frac{\partial V}{\partial x_i}(t, x+y) - \frac{\partial V}{\partial x_i}(t, x) \right| dx\, dt \leqslant C^* |y|$$

for every $y \in \mathbb{R}^m$ with $|y| \leqslant \delta^*$. Since $f \equiv 0$ and $g \equiv 0$, we clearly have $V(t, x) \equiv (T-t) F(x)$, so $\partial V/\partial x_i \equiv (T-t)(\partial F/\partial x_i)$. Substituting this into the above integral and performing the t-integration, we obtain

$$\int_K \left| \frac{\partial F}{\partial x_i}(x+y) - \frac{\partial F}{\partial x_i}(x) \right| dx \leqslant \frac{6C^*}{T^2} |y|$$

for every $y \in \mathbb{R}^m$ with $|y| \leqslant \delta^*$. Since this can be done for every compact set $K \subset \mathbb{R}^m$, by Lemma 3 we have $(\partial F/\partial x_i) \in BV(\mathbb{R}^m)$, a contradiction.

If H has mild x-variation, it is quite easy to show that

$$\frac{\partial^2}{\partial x_k\, \partial x_i} \mathscr{H}^j \quad \text{and} \quad \frac{\partial^2}{\partial p_k\, \partial x_i} \mathscr{H}^j \quad \text{are bounded uniformly for } j = 1, 2, \ldots, \tag{4.1}$$

for $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, $p \in \mathbb{R}^m$, $i = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, m$. This is the form of the assumption we will use in what follows.

Since our final result will be proved only for the $m = 1$ case, we will assume that $m = 1$ from now on, using x for x_1 and p for p_1. There appears to be only one point in what follows for which a generalization of the proof to the $m > 1$ case can not be made. That point will be identified when it occurs (Lemma 7).

LEMMA 4. *Make the assumptions of the conjecture for* $m = 1$. *Then*

(a) *each* $W^{\varepsilon,j} = W^{\varepsilon,j}(t, x)$ *is a* C^∞ *function on* $(0, T] \times \mathbb{R}$.

(b) $W^{\varepsilon,j}$, $\partial W^{\varepsilon,j}/\partial x$, $\partial^2 W^{\varepsilon,j}/\partial x^2$, *and* $\partial W^{\varepsilon,j}/\partial t$ *are bounded and Hölder continuous on* $[0, T] \times \mathbb{R}$. (*The bounds depend on* $0 < \varepsilon < 1$ *and* $j = 1, 2, \ldots$.)

(c) *For any* ε *and* j *with* $0<\varepsilon<1$ *and* $j=1,2,\ldots$, *there is a constant* $C_{\varepsilon,j}$ *such that for* $0\leqslant t\leqslant T$ *and* $x\in\mathbb{R}$ *with* $|x|>P+2$ *we have*

$$\left|\frac{\partial^2 W^{\varepsilon,j}}{\partial x^2}\right|+\left|\frac{\partial^3 W^{\varepsilon,j}}{\partial x^3}\right|\leqslant C_{\varepsilon,j}e^{-x^2/5T}.$$

Proof. For (a), apply [8, Theorem 11, p. 74]. For (b), apply [13, Theorem 8.1, p. 495]. For (c), let $v^{\varepsilon,j}=\partial^2 W^{\varepsilon,j}/\partial x^2$. From (2.15) and (2.16) for $m=1$ and the fact that $\mathscr{H}^j$ and g vanish for $|x|\geqslant P+1$, we have after two differentiations that

$$\frac{\partial v^{\varepsilon,j}}{\partial t}=\varepsilon\frac{\partial^2 v^{\varepsilon,j}}{\partial x^2}\qquad\text{for}\quad 0<t\leqslant T,\ |x|\geqslant P+1,$$

$$v^{\varepsilon,j}(0,x)=0\qquad\text{for}\quad |x|\geqslant P+1.$$

But we have the explicit formulas

$$U(t,x)=\frac{1}{2}\frac{x}{(\varepsilon\pi)^{1/2}}\int_0^t\frac{1}{(t-\tau)^{3/2}}e^{-x^2/4\varepsilon(t-\tau)}\Phi(\tau)\,d\tau$$

$$\frac{\partial U}{\partial x}(t,x)=\frac{1}{2(\varepsilon\pi)^{1/2}}\int_0^t\frac{1}{(t-\tau)^{3/2}}e^{-x^2/4\varepsilon(t-\tau)}\Phi(\tau)\,d\tau$$
$$+\frac{1}{2}\frac{x}{(\varepsilon\pi)^{1/2}}\int_0^t\frac{1}{(t-\tau)^{3/2}}\frac{-2x}{4\varepsilon(t-\tau)}e^{-x^2/4\varepsilon(t-\tau)}\Phi(\tau)\,d\tau$$

for the unique bounded solution of $\partial U/\partial t=\varepsilon\partial^2U/\partial x^2$ for $0<t\leqslant T$, $x>0$, $U(0,x)=0$ for $x\geqslant 0$, and $U(t,0)=\Phi(t)$ for $0\leqslant t\leqslant T$, provided that $\Phi\colon[0,T]\to\mathbb{R}$ is continuous with $\Phi(0)=0$. Making the appropriate changes of coordinates ($\Phi(t)\equiv v^{\varepsilon,j}(t,P+1)$ for one of them, $\Phi(t)\equiv v^{\varepsilon,j}(t,-(P+1))$ for the other) and using the boundedness of Φ gives (c) after some tedious but straightforward estimates.

DEFINITION. For $0<\varepsilon<1$ and $j=1,2,\ldots$, let $u^{\varepsilon,j}=\partial W^{\varepsilon,j}/\partial x$ and let $v^{\varepsilon,j}=\partial^2 W^{\varepsilon,j}/\partial x^2$.

LEMMA 5. *Make the assumptions of the conjecture for* $m=1$. *Let* $0<\tau<T$. *Let* $G=G(t,x)$ *be continuous on* $[0,\tau]\times\mathbb{R}$ *and let* G, $\partial G/\partial x$, $\partial^2G/\partial x^2$, *and* $\partial G/\partial t$ *be continuous and bounded on* $(0,\tau)\times\mathbb{R}$. *Then*

$$0 = -\int_{\mathbb{R}} (v^{\varepsilon,j}G)|_{t=\tau}\,dx + \int_{\mathbb{R}} (v^{\varepsilon,j}G)|_{t=0}\,dx$$
$$+ \int_0^\tau \int_{\mathbb{R}} \frac{\partial^2 \mathscr{H}^j}{\partial x^2}(t, x, u^{\varepsilon,j})G\,dx\,dt$$
$$+ \int_0^\tau \int_{\mathbb{R}} v^{\varepsilon,j}\left[\frac{\partial G}{\partial t} + \varepsilon\frac{\partial^2 G}{\partial x^2} - \frac{\partial \mathscr{H}^j}{\partial p}(t, x, u^{\varepsilon,j})\frac{\partial G}{\partial x}\right.$$
$$\left. + \frac{\partial^2 \mathscr{H}^j}{\partial p\,\partial x}(t, x, u^{\varepsilon,j})G\right]dx\,dt. \tag{4.2}$$

Proof. Differentiate (2.15) twice with respect to x (with $m=1$) to obtain

$$\frac{\partial v^{\varepsilon,j}}{\partial t} - \frac{\partial}{\partial x}\left[\frac{\partial \mathscr{H}^j}{\partial x}(t, x, u^{\varepsilon,j}) + \frac{\partial \mathscr{H}^j}{\partial p}(t, x, u^{\varepsilon,j})\,v^{\varepsilon,j}\right] - \varepsilon\frac{\partial^2 v^{\varepsilon,j}}{\partial x^2} = 0$$

for $0 < t < T$ and $x \in \mathbb{R}$. Multiplying both sides of the above by G, integrating over $(\delta, \tau-\delta)\times(-M, M)$, doing several integrations by parts, and taking the limit first as $M \to \infty$ and then as $\delta \to 0^+$ (using (a), (b), and (c) of Lemma 4 and the fact that $\mathscr{H}^j \equiv 0$ for $|x| \geqslant P+1$) gives (4.2).

For $0<\varepsilon<1$, $j=1, 2,\ldots$, and $0<\tau<T$, define $\beta(x) = \beta^{\varepsilon,j,\tau}(x) = \operatorname{sign}[v^{\varepsilon,j}(\tau, x)]$ for all $x \in \mathbb{R}$. (For any $y>0$, sign $y = +1$. For any $y<0$, sign $y = -1$. For $y=0$, sign $y=0$.) For each $h>0$, let $[\beta]_h$ denote the h-mollification of β. For the G of Lemma 5 we take $G = G^{\varepsilon,j,\tau,h}$ which is the solution of

$$\frac{\partial G}{\partial t} + \varepsilon\frac{\partial^2 G}{\partial x^2} - \frac{\partial \mathscr{H}^j}{\partial p}(t, x, u^{\varepsilon,j})\frac{\partial G}{\partial x} + \frac{\partial^2 \mathscr{H}^j}{\partial p\,\partial x}(t, x, u^{\varepsilon,j})G = 0, \tag{4.3}$$

$$G(\tau, x) = [\beta]_h(x), \tag{4.4}$$

where (4.3) holds for $0 \leqslant t < \tau$, $x \in \mathbb{R}$ and where (4.4) holds for $x \in \mathbb{R}$.

LEMMA 6. *For each ε, j, τ, and h with $0<\varepsilon<1$, $j=1, 2,\ldots$, $0<\tau<T$, and $h>0$, the solution $G = G^{\varepsilon,j,\tau,h}$ of (4.3) and (4.4) above exists and satisfies the conditions for the G of Lemma 5.*

Proof. Apply [13, Theorem 8.1, p. 495] after a time-reversal.

LEMMA 7. *Let M be a constant such that $|\partial^2\mathscr{H}^j/\partial p\,\partial x| \leqslant M$ for all $j=1, 2,\ldots$, $t \in \mathbb{R}$, $x \in \mathbb{R}$, and $p \in \mathbb{R}$. Let $G = G^{\varepsilon,j,\tau,h}$ be the function described above. Then*

$$|G^{\varepsilon,j,\tau,h}(t, x)| \leqslant e^{MT} \tag{4.5}$$

for all $h>0$, $0<\varepsilon<1$, $j=1, 2,\ldots$, $0<\tau<T$, $0 \leqslant t \leqslant \tau$, and $x \in \mathbb{R}$.

Proof. Let $\mathscr{L}^{\varepsilon,j}[G]$ be defined by the left-hand side of (4.3). For any $F = F(t, x)$, let

$$Q^{\varepsilon,j}[F] \equiv F_t + \varepsilon \frac{\partial^2 F}{\partial x^2} - \frac{\partial \mathscr{H}^j}{\partial p}(t, x, u^{\varepsilon,j}) \frac{\partial F}{\partial x} + 2MF.$$

Then for any $h > 0$, $0 < \varepsilon < 1$, $j = 1, 2, \ldots$, $0 < \tau < T$, we have for $G = G^{\varepsilon,j,\tau,h}$ that

$$Q^{\varepsilon,j}[G^2] \geqslant 2G\mathscr{L}^{\varepsilon,j}[G] = 0. \tag{4.6}$$

Using a maximum principle (e.g., [14, Theorem 10, p. 183] with time direction reversed, with $u = e^{-2M(\tau - t)}G^2 - 1$ and taking $L[F] \equiv F_t + \varepsilon F_{xx} - \mathscr{H}^j_p(t, x, u^{\varepsilon,j}) F_x$ and $h \equiv 0$, so that $L[u] = e^{-2M(\tau - t)}Q^{\varepsilon,j}[G^2] \geqslant 0$) together with the fact that $|G^{\varepsilon,j,\tau,h}(\tau, x)| \leqslant 1$ for all $x \in \mathbb{R}$, we obtain the estimate (4.5).

Remark. It is here that a generalization to the $m > 1$ case seems not to be possible. In the $m > 1$ case, a parabolic system of equations, coupled in the first derivative terms, replaces (4.3). No maximum principle seems to be available in this situation [14, p. 192, Remark (ii)].

LEMMA 8. *There is a constant E such that*

$$\int_{\mathbb{R}} |v^{\varepsilon,j}(\tau, x)| \, dx \leqslant E \tag{4.7}$$

for all $0 < \varepsilon < 1$, $j = 1, 2, \ldots$, *and* $0 < \tau < T$.

Proof. Substituting (4.3) and (4.4) into (4.2) and using (4.5), (4.1), and (2.16) together with the fact that $\mathscr{H}^j = 0$ and $g = 0$ for $|x| \geqslant P + 1$ shows the existence of a constant E such that

$$\int_{\mathbb{R}} v^{\varepsilon,j}(\tau, x)[\operatorname{sign} v^{\varepsilon,j}]_h(\tau, x) \, dx \leqslant E \tag{4.8}$$

for all $0 < \varepsilon < 1$, $j = 1, 2, \ldots$, $0 < \tau < T$, and $h > 0$. Taking the limit as $h \to 0^+$ and using the dominated convergence theorem applied to $v^{\varepsilon,j}(\tau, \cdot) \in L^1(\mathbb{R})$ on $\{x \in \mathbb{R}; v^{\varepsilon,j}(\tau, x) \neq 0\}$, we obtain (4.7).

LEMMA 9. *Select any sequence $\varepsilon_1, \varepsilon_2, \ldots$, with $\varepsilon_j \to 0^+$. Define $W^j = W^{\varepsilon_j,j}$ for $j = 1, 2, \ldots$. Then*

(a) *the functions $W^j(t, x)$ converge as $j \to \infty$ to a limit function $W(t, x)$, uniformly on compact subsets of $[0, T] \times \mathbb{R}$, and there is a constant M_2 such that*

$$|W(t^*, x^*) - W(t, x)| \leqslant M_2[|t^* - t|^{1/2} + |x^* - x|] \tag{4.9}$$

for all $t \in [0, T]$, $t^* \in [0, T]$, $x \in \mathbb{R}$, *and* $x^* \in \mathbb{R}$, *and*

$$W(0, x) = g(x) \qquad \text{for all} \quad x \in \mathbb{R}. \tag{4.10}$$

(b) *W is a viscosity solution of*

$$\frac{\partial W}{\partial t} - \mathscr{H}\left(t, x, \frac{\partial W}{\partial x}\right) = 0 \tag{4.11}$$

for $0 < t \leqslant T$ *and* $x \in \mathbb{R}$.

(c) *W is the unique bounded uniformly continuous viscosity solution of* (4.11) (*for* $0 < t \leqslant T$ *and* $x \in \mathbb{R}$) *and* (4.10).

(d) $V(T-t, x)$ *is a bounded uniformly continuous viscosity solution of* (4.11) (*for* $0 < t \leqslant T$ *and* $x \in \mathbb{R}$) *and* (4.10), *where V is the value function of the differential game in the sense of Friedman* (*after the modifications in Section* 2 *are made which ensure that* $f = g = h = 0$ *for* $|x| \geqslant P$). *Thus*

$$W(t, x) = V(T-t, x) \tag{4.12}$$

for $0 \leqslant t \leqslant T$ *and* $x \in \mathbb{R}$.

Proof. From [7, the first two paragraphs of p. 1004 and the sentence just after Theorem 4 on p. 1001], we obtain (a). From [4, Proposition IV.1 on p. 41 and Proposition V.1 on p. 48], we obtain (b). From [4, Theorem V.2(ii) on pp. 49 and 50], we obtain (c). In [2, Theorem 3.1], (d) is proved under slightly stronger hypotheses than ours. To obtain (d) under our assumptions, because of [4, Proposition IV.1, p. 41] it suffices to show that (after time-reversal) $V(t, x)$ is the limit of vanishing viscosity approximate solutions of problems of the type [4, $(4.1)_\varepsilon$, p. 41]. Using the construction on p. 23 of [9] we may get for a given (t_0, x_0) an approximate game with upper value $V_{kn}^+(t_0, x_0)$ close to $V(t_0, x_0)$. Lemma 3.2 of [9] applies to this approximate game so that vanishing viscosity approximate solutions $W_\varepsilon^+(t_0, x_0)$ are close to $V_{kn}^+(t_0, x_0)$ and hence close to $V(t_0, x_0)$, say $|W_\varepsilon^+(t_0, x_0) - V(t_0, x_0)| < \delta$ (for some $\delta > 0$). This estimate is valid for a (t, x)-neighborhood about (t_0, x_0), so by using compactness arguments, we can get a sequence of vanishing viscosity approximate solutions W_ε^+ which converge to V uniformly on every compact subset of $(0, T) \times \mathbb{R}$.

LEMMA 10. *Fix any* δ *with* $0 < \delta < T$ *and any a and b with* $a < b$. *Then*

$$\lim_{k \to \infty} \int_a^b \int_\delta^T \left| \frac{\partial W^k}{\partial x}(\tau, x) - \frac{\partial W}{\partial x}(\tau, x) \right| d\tau\, dx = 0. \tag{4.13}$$

Proof. Since $W(t, x) \equiv V(T-t, x)$, W is uniformly Lipschitz continuous on the set $S = \{(\tau, x); \delta \leqslant \tau \leqslant T, a-1 \leqslant x \leqslant b+1\}$, say with Lipschitz constant M^* (see [9, Theorem 2.3, p. 10]). By Rademacher's theorem, $\partial W/\partial x$ is defined almost everywhere on S. Since $\partial W/\partial x$ is bounded and measurable on S, it is integrable there. For $j = 1, 2, \ldots$, let $J(j)$ be chosen so that for all $(\tau, x) \in S$ and all $k \geqslant J(j)$ we have $|W^k(\tau, x) - W(\tau, x)| \leqslant 1/j^2$. For $j = 1, 2, \ldots$, and any $(\tau, x) \in S$, define

$$g_j(\tau, x) = \frac{W(\tau, x+1/j) - W(\tau, x)}{1/j}.$$

An easy estimate shows that for $j = 1, 2, \ldots$, $k \geqslant J(j)$, $a \leqslant x \leqslant b$, and $\delta \leqslant \tau \leqslant T$ we have

$$\left| g_j(\tau, x) - \frac{W^k(\tau, x+1/j) - W^k(\tau, x)}{1/j} \right| \leqslant 2/j.$$

Since $\int_{\mathbb{R}} |(\partial^2 W^k/\partial x^2)(\tau, x)|\, dx \leqslant E$ for $0 \leqslant \tau \leqslant T$ (by Lemma 8) we have for all k and j (with x^* below denoting a real number depending on k, j, x, and τ, but with $x < x^* < x + 1/j$) that

$$\begin{aligned}
&\int_a^b \int_\delta^T \left| \frac{W^k(\tau, x+1/j) - W^k(\tau, x)}{1/j} - \frac{\partial W^k}{\partial x}(\tau, x) \right| d\tau\, dx \\
&= \int_a^b \int_\delta^T \left| \frac{\partial W^k}{\partial x}(\tau, x^*) - \frac{\partial W^k}{\partial x}(\tau, x) \right| d\tau\, dx \\
&\leqslant \int_a^b \int_\delta^T \int_x^{x+1/j} \left| \frac{\partial^2 W^k}{\partial x^2}(\tau, \xi) \right| d\xi\, d\tau\, dx \\
&\leqslant \int_\delta^T \int_a^{b+1/j} \left| \frac{\partial^2 W^k}{\partial x^2}(\tau, \xi) \right| \int_{\xi - 1/j}^{\xi} dx\, d\xi\, d\tau \leqslant (T-\delta)\, E(1/j).
\end{aligned}$$

But $\int_a^b \int_\delta^T |g_j(\tau, x) - (\partial W/\partial x)(\tau, x)|\, d\tau\, dx \to 0$ as $j \to \infty$ follows from the bounded convergence theorem, since $g_j(\tau, x) \to (\partial W/\partial x)(\tau, x)$ almost everywhere on S as $j \to \infty$ and since $|g_j(\tau, x) - (\partial W/\partial x)(\tau, x)| \leqslant 2M^*$.

We can now complete the proof. Let $\varepsilon > 0$ be given. Select j so that $\int_a^b \int_\delta^T |g_j(\tau, x) - (\partial W/\partial x)(\tau, x)|\, d\tau\, dx < \varepsilon/3$, so that $(T-\delta)(b-a)\, 2j^{-1} < \varepsilon/3$, and so that $(T-\delta)\, Ej^{-1} < \varepsilon/3$. We clearly have then for $k \geqslant \max(j, J(j))$ that

$$\int_a^b \int_\delta^T \left| \frac{\partial W^k}{\partial x}(\tau, x) - \frac{\partial W}{\partial x}(\tau, x) \right| d\tau\, dx < \varepsilon.$$

LEMMA 11. *For any $a<b$, any h with $0<h<1$, and any δ with $0<\delta<T$ we have*

$$\int_\delta^T \int_a^b \left| \frac{\partial W}{\partial x}(\tau, y+h) - \frac{\partial W}{\partial x}(\tau, y) \right| dy\, d\tau \leqslant (T-\delta)\, Eh. \tag{4.14}$$

Proof. We have for $k=1, 2, \ldots$, that

$$\begin{aligned}
&\int_\delta^T \int_a^b \left| \frac{\partial W^k}{\partial x}(\tau, y+h) - \frac{\partial W^k}{\partial x}(\tau, y) \right| dy\ d\tau \\
&\quad \leqslant \int_\delta^T \int_a^b \int_y^{y+h} \left| \frac{\partial^2 W^k}{\partial x^2}(\tau, \xi) \right| d\xi\, dy\, d\tau \\
&\quad \leqslant \int_\delta^T \int_a^{b+h} \left| \frac{\partial^2 W^k}{\partial x^2}(\tau, \xi) \right| \int_{\xi-h}^{\xi} dy\, d\xi\, d\tau \leqslant (T-\delta)\, Eh.
\end{aligned}$$

Taking $k \to \infty$ and using Lemma 10, we have (4.14).

The sequence of lemmas we have just completed (Lemmas 5, 6, 7, 8, 10, and 11) was suggested by a somewhat similar argument in [12, Sect. 4].

LEMMA 12. *For any $a<b$ and any δ and δ^* with $0<\delta^*<T-\delta<T$ there is a constant C such that*

$$\int_{\delta^*}^{T-\delta} \int_a^b \left| \frac{\partial W}{\partial x}(\tau+h^*, y) - \frac{\partial W}{\partial x}(\tau, y) \right| dy\, d\tau \leqslant Ch^* \tag{4.15}$$

for all h^* with $0<h^*<\delta$.

Proof. Because of (4.12), (2.5), and the fact that W is locally Lipschitz continuous, we know that $\partial W/\partial t = \mathscr{H}(t, x, \partial W/\partial x)$ almost everywhere. Thus for small $\varepsilon>0$ we have for every $t \in (\varepsilon, T-\varepsilon)$ and every $x \in \mathbb{R}$ that $[\partial W/\partial t]_\varepsilon = [\mathscr{H}(t, x, \partial W/\partial x)]_\varepsilon$, where $[\ ]_\varepsilon$ denotes the ε-mollification described in Section 2, and thus

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}[W]_\varepsilon = \frac{\partial}{\partial x}\left[\mathscr{H}\left(t, x, \frac{\partial W}{\partial x}\right)\right]_\varepsilon \tag{4.16}$$

holds for $(t, x) \in (\varepsilon, T-\varepsilon) \times \mathbb{R}$, since $[\partial W/\partial t]_\varepsilon = \partial[W]_\varepsilon/\partial t$ on this same set (see [1, Theorem 1.8, p. 6], e.g.) and since mollified functions are of class C^∞.

It suffices to prove (4.15) with W replaced by $[W]_\varepsilon$ for all sufficiently small $\varepsilon>0$ (where C is independent of ε and where $0<h^*<\delta$ is replaced by $0<h^*<\delta-\varepsilon$), since the fact that $\partial[W]_\varepsilon/\partial x = [\partial W/\partial x]_\varepsilon \to \partial W/\partial x$ in $L^1([\delta^*, T-\delta+h^*] \times [a, b])$ as $\varepsilon \to 0^+$ for each fixed h^* with $0<h^*<\delta$ (see [1, Theorem 1.7, p. 5], e.g.) then clearly allows us to prove (4.15). For convenience in the remainder of this proof, write $W^\varepsilon = [W]_\varepsilon$.

Since for $0 < h^* < \delta - \varepsilon$ we have

$$\begin{aligned}
&\int_{\delta^*}^{T-\delta} \int_a^b \left| \frac{\partial W^\varepsilon}{\partial x}(\tau + h^*, y) - \frac{\partial W^\varepsilon}{\partial x}(\tau, y) \right| dy\, d\tau \\
&\quad \leqslant \int_{\delta^*}^{T-\delta} \int_a^b \int_\tau^{\tau + h^*} \left| \frac{\partial}{\partial t} \frac{\partial W^\varepsilon}{\partial x}(\xi, y) \right| d\xi\, dy\, d\tau \\
&\quad \leqslant \int_{\delta^*}^{T-\delta+h^*} \int_a^b \left| \frac{\partial}{\partial t} \frac{\partial W^\varepsilon}{\partial x}(\xi, y) \right| \int_{\xi - h^*}^{\xi} d\tau\, dy\, d\xi \\
&\quad = h^* \int_{\delta^*}^{T-\delta+h^*} \int_a^b \left| \frac{\partial}{\partial t} \frac{\partial W^\varepsilon}{\partial x}(\xi, y) \right| dy\, d\xi,
\end{aligned}$$

it suffices to show that this last integral is less than some constant C independent of ε and h^*. In view of (4.16), it clearly suffices to show that

$$\begin{aligned}
&\frac{1}{h} \int_{\delta^*}^{T-\delta+h^*} \int_a^b \left| \left[\mathscr{H}\left(t, x, \frac{\partial W}{\partial x}\right) \right]_\varepsilon (\xi, y+h) \right. \\
&\qquad \left. - \left[\mathscr{H}\left(t, x, \frac{\partial W}{\partial x}\right) \right]_\varepsilon (\xi, y) \right| dy\, d\xi \qquad (4.17)
\end{aligned}$$

is bounded by some constant C independent of ε, h^*, and h (for h small and positive). Denoting by $\varphi_\varepsilon(t, x) \equiv l_\varepsilon(t)\, l_\varepsilon(x)$ the mollification kernel, the integrand of (4.17) is equal to

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\varphi_\varepsilon(\xi - \tau, y + h - Y) - \varphi_\varepsilon(\xi - \tau, y - Y) \right] \right. \\
&\qquad \left. \times \mathscr{H}\left(\tau, Y, \frac{\partial W}{\partial x}(\tau, Y)\right) dY\, d\tau \right| \\
&\quad = \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_\varepsilon(\xi - \tau, y - Y) \left[\mathscr{H}\left(\tau, Y + h, \frac{\partial W}{\partial x}(\tau, Y + h)\right) \right. \right. \\
&\qquad \left. \left. - \mathscr{H}\left(\tau, Y, \frac{\partial W}{\partial x}(\tau, Y)\right) \right] dY\, d\tau \right| \\
&\quad = \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_\varepsilon(\xi - \tau, y - Y) \left[\mathscr{H}\left(\tau, Y + h, \frac{\partial W}{\partial x}(\tau, Y + h)\right) \right. \right. \\
&\qquad - \mathscr{H}\left(\tau, Y, \frac{\partial W}{\partial x}(\tau, Y + h)\right) \\
&\qquad \left. \left. + \mathscr{H}\left(\tau, Y, \frac{\partial W}{\partial x}(\tau, Y + h)\right) - \mathscr{H}\left(\tau, Y, \frac{\partial W}{\partial x}(\tau, Y)\right) \right] dY\, d\tau \right|
\end{aligned}$$

$$\leqslant (AK+A)h + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_\varepsilon(\xi-\tau, y-Y) B$$
$$\times \left| \frac{\partial W}{\partial x}(\tau, Y+h) - \frac{\partial W}{\partial x}(\tau, Y) \right| dY\, d\tau,$$

where the $(AK+A)h$ part of this last estimate comes from (2.12) and (2.18), and where the remainder of the estimate uses (2.11). Substituting this last bound into (4.17), it clearly suffices to obtain an upper bound independent of ε, h^*, and h for

$$\frac{1}{h} \int_{\delta^*}^{T-\delta+h^*} \int_a^b \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_\varepsilon(\xi-\tau, y-Y)$$
$$\times B \left| \frac{\partial W}{\partial x}(\tau, Y+h) - \frac{\partial W}{\partial x}(\tau, Y) \right| dY\, d\tau\, dy\, d\xi,$$

which under the substitution $z = y - Y$ and $t = \xi - \tau$ is easily seen to be

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_\varepsilon(t, z)\, B \left[\frac{1}{h} \int_{\delta^*}^{T-\delta+h^*} \int_a^b \left| \frac{\partial W}{\partial x}(\xi-t, y-z+h) \right.\right.$$
$$\left.\left. - \frac{\partial W}{\partial x}(\xi-t, y-z) \right| dy\, d\xi \right] dz\, dt.$$

But for $|t| < \varepsilon$ and $|z| < \varepsilon$ (otherwise $\varphi_\varepsilon(t, z) = 0$) we have the expression in square brackets above bounded by

$$\frac{1}{h} \int_{\delta^*-\varepsilon}^{T-\delta+h^*+\varepsilon} \int_{a-\varepsilon}^{b+\varepsilon} \left| \frac{\partial W}{\partial x}(\tau, x+h) - \frac{\partial W}{\partial x}(\tau, x) \right| dx\, d\tau,$$

which is bounded independently of ε, h^*, and h by Lemma 11.

5. The Main Result

THEOREM. *Make the assumptions of Section 2. Assume that $m = 1$. Assume that H has mild x-variation. Assume that g has locally Hölder continuous second derivatives. Then*

$$\frac{\partial V}{\partial x} \in BV((0, T) \times \mathbb{R}),$$

where V is the (Friedman) value function of the original differential game (before the modifications which made $f = g = h = 0$ for $|x| \geqslant P$).

Proof. Lemmas 11 and 12 show that $\partial V/\partial x$ satisfies the conditions of Lemma 3 for $\Omega = (0, T) \times \mathbb{R}$.

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