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On the well-posedness of the incompressible density-dependent Euler equations in the L^p framework

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ABSTRACT

The present paper is devoted to the study of the well-posedness issue for the density-dependent Euler equations in the whole space. We establish local-in-time results for the Cauchy problem pertaining to data in the Besov spaces embedded in the set of Lipschitz functions, including the borderline case $B^{\frac{N}{p}+1}_{p,1}(\mathbb{R}^N)$. A continuation criterion in the spirit of the celebrated one by Beale, Kato and Majda (1984) in [2] for the classical Euler equations, is also proved.

In contrast with the previous work dedicated to this system in the whole space, our approach is not restricted to the L^2 framework or to small perturbations of a constant density state: we just need the density to be bounded away from zero. The key to that improvement is a new a priori estimate in Besov spaces for an elliptic equation with nonconstant coefficients.

The evolution of the density $\rho = \rho(t,x) \in \mathbb{R}^+$ and of the velocity field $u = u(t,x) \in \mathbb{R}^N$ of a non-homogeneous incompressible fluid satisfies the following *density-dependent Euler equations*:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla \Pi = \rho f, \\ \operatorname{div} u = 0. \end{cases} \tag{1}$$

Above, f stands for a given body force and the gradient of the pressure $\nabla \Pi$ is the Lagrangian multiplier associated to the divergence free constraint over the velocity. We assume the space variable x to belong to the whole $\mathbb{R}^N$ with $N \geqslant 2$.

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A plethoric number of recent mathematical works have been devoted to the study of the classical incompressible Euler equations

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \Pi = f, \\ \operatorname{div} u = 0, \end{cases} \tag{2}$$

which may be seen as a special case of (1) (just take $\rho \equiv 1$).

In contrast, not so many works have been devoted to the study of (1) in the nonconstant density case. In the situation where the equations are considered in a suitably smooth bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$, the local well-posedness issue has been investigated by H. Beirão da Veiga and A. Valli in [3–5] for data with high enough Hölder regularity. The case of data with $W^{2,p}$ regularity has been studied by A. Valli and W. Zajączkowski in [18] and by S. Itoh and A. Tani in [13]. The whole space case $\mathbb{R}^3$ has been addressed by S. Itoh in [12]. There, the local existence for initial data (ρ_0, u_0) such that ρ_0 is bounded, bounded away from 0 and such that $(\nabla\rho_0, u_0)$ is in $H^2 \times H^3$ has been obtained. In [9], we have generalized [12]'s result to any dimension $N \geqslant 2$ and any Sobolev space H^s with $s > 1 + N/2$ and have studied the inviscid limit in this framework. There, data in the limit Besov space $B^{\frac{N}{2}+1}_{2,1}$ are also considered. Finally, let us mention that in [20], Y. Zhou has established the local well-posedness for (1) in the case where the density is a small perturbation of a positive constant in $B^{\frac{N}{p}+1}_{p,1}$ $(1 < p < \infty)$.

According to the work by J. Marsden in [15], the finite energy solutions to (1) may be interpreted in terms of the action of geodesics. This latter study is motivated by the fact that, as in the homogeneous situation, system (1) has a conserved energy, namely

$$\big\|(\sqrt{\rho}u)(t)\big\|_{L^2}^2 = \|\sqrt{\rho}_0 u_0\|_{L^2}^2 + 2\int_0^t \int_{\mathbb{R}^N} (\rho f \cdot u)(\tau, x)\, d\tau\, dx. \tag{3}$$

Motivated by the fact that, in real life, a fluid is hardly homogeneous, we here want to study whether the classical results for homogeneous fluids remain true in the nonhomogeneous framework. More precisely, we aim at investigating the existence and uniqueness issue in the whole space and in the L^p framework for densities which may be large perturbations of a constant function: we only require the density to be bounded and bounded away from zero and to have enough regularity. We shall also establish blow-up criteria in the spirit of the celebrated one by Beale–Kato–Majda criterion [2] for (2).

The functional framework that we shall adopt – Besov spaces embedded in the set $C^{0,1}$ of bounded globally Lipschitz functions – is motivated by the fact that the density and velocity equations of (1) are transport equations by the velocity field. Hence no gain of smoothness may be expected during the evolution and conserving the initial regularity requires the velocity field to be at least locally Lipschitz with respect to the space variable. In fact, the spaces that we shall use are exactly those that are suitable for (2). We thus believe our results to be optimal in terms of regularity.

Compared to the classical Euler equations, handling the gradient of the pressure is much more involved. To eliminate the pressure, the natural strategy consists in solving the elliptic equation

$$\operatorname{div}(a\nabla\Pi) = \operatorname{div} F \quad \text{with } F := \operatorname{div}(f - u \cdot \nabla u) \text{ and } a := 1/\rho.$$

If a is a small perturbation of a constant function $\bar{a}$ then the above equation may be rewritten

$$\bar{a}\Delta\Pi = \operatorname{div}\big((\bar{a} - a)\nabla\Pi\big) + \operatorname{div} F.$$

Now, if $1 < p < \infty$ then the standard L^p elliptic estimate may be used for absorbing the first term in the right-hand side. Hence we expect to get the same well-posedness results as for (2) in this situation. As a matter of fact, this strategy has been successfully implemented by Y. Zhou in [20,21] to get the well-posedness in $B^{\frac{N}{p}}_{p,1}$ in the case where the density is a small perturbation of a constant.

In the general case of *large* perturbations of a constant density state, solving the above equation in the $\mathbb{R}^N$ framework for $F \in L^p$ may be a problem (unless $p = 2$ of course). In fact, to our knowledge, even if a is smooth, bounded and bounded away from zero, there is no solution operator $\mathcal{H} : F \to \nabla\Pi$ such that

$$\|\nabla\Pi\|_{L^p} \leqslant C\|F\|_{L^p}$$

unless p is "close" to 2 (see the work by N. Meyers in [16]). However that closeness is strongly related to whether a itself is close to a constant hence no result for all p may be obtained by this process.

In the present work, we shall overcome this difficulty by requiring the data to satisfy a finite energy condition so as to ensure that F is in L^2. Indeed, starting from the classical L^2 estimate for elliptic equations, we will be able to get estimates in high order Besov spaces $B^s_{p,r}$.

This is the conducting thread leading to the first two well-posedness results stated in the next section. The rest of the paper unfolds as follows. In Section 2, we introduce the Littlewood–Paley decomposition and recall the definition of the nonhomogeneous Besov spaces $B^s_{p,r}$. Then, we define the paraproduct and remainder operators and state a few classical results in Fourier analysis. Section 3 is devoted to the proof of existence results and a priori estimates for an elliptic equation with non-constant coefficients in the Besov space framework. To our knowledge, most of the results that are presented therein are new. Sections 4, 5 and 6 are dedicated to the proof of our main existence and continuation results. Some technical lemmas have been postponed in Appendices A and B.

Notation. Throughout the paper, C stands for a harmless "constant" whose exact meaning depends on the context.

For all Banach space X and interval I of $\mathbb{R}$, we denote by $\mathcal{C}(I;X)$ (resp. $\mathcal{C}_b(I;X)$) the set of continuous (resp. continuous bounded) functions on I with values in X. If X has predual X^* then we denote by $\mathcal{C}_w(I;X)$ the set of bounded measurable functions $f : I \to X$ such that for any $\phi \in X^*$, the function $t \mapsto \langle f(t), \phi\rangle_{X\times X^*}$ is continuous over I. For $p \in [1,\infty]$, the notation $L^p(I;X)$ stands for the set of measurable functions on I with values in X such that $t \mapsto \|f(t)\|_X$ belongs to $L^p(I)$. We denote by $L^p_{loc}(I)$ the set of those functions defined on I and valued in X which, restricted to any compact subset J of I, are in $L^p(J)$.

Finally, for any real valued function a over $\mathbb{R}^N$, we denote

$$a_* := \inf_{x\in\mathbb{R}^N} a(x) \quad \text{and} \quad a^* := \sup_{x\in\mathbb{R}^N} a(x).$$

1. Main results

As explained in the introduction, we hardly expect to get any well-posedness result if the initial velocity is not in $C^{0,1}$. It is well known (see e.g. [1, Chap. 2]) that the nonhomogeneous Besov space $B^s_{p,r}$ is continuously embedded in $C^{0,1}$ if and only if the triplet $(s,p,r) \in \mathbb{R} \times [1,\infty]^2$ satisfies the following condition:

$$s > 1 + N/p \quad \text{or} \quad s \geqslant 1 + N/p \text{ and } r = 1. \tag{C}$$

This motivates the following statement concerning the existence of smooth solutions with finite energy:

Theorem 1. *Let (s,p,r) satisfy condition (C) with $1 < p < \infty$. Let u_0 be a divergence-free vector-field with coefficients in $L^2 \cap B^s_{p,r}$. Suppose that the body force f has coefficients in $L^1([-T_0,T_0]; B^s_{p,r}) \cap \mathcal{C}([-T_0,T_0]; L^2)$ for some $T_0 > 0$. Assume that ρ_0 is positive, bounded and bounded away from zero and that $\nabla\rho_0 \in B^{s-1}_{p,r}$. If $p < 2$, suppose in addition that $(\rho_0 - \overline{\rho}) \in L^{p^*}$ with $p^* := 2p/(2-p)$ for some positive real number $\overline{\rho}$.*

There exists a time $T \in (0, T_0]$ such that system (1) supplemented with initial data (ρ_0, u_0) has a unique local solution $(\rho, u, \nabla\Pi)$ on $[-T,T] \times \mathbb{R}^N$ with:

- $\rho^{\pm 1} \in \mathcal{C}_b([-T,T]\times\mathbb{R}^N)$, $D\rho \in \mathcal{C}_w([-T,T]; B^{s-1}_{p,r})$ *(and* $(\rho-\overline{\rho}) \in \mathcal{C}([-T,T]; L^{p^*})$ *if* $p<2$*),*
- $u \in \mathcal{C}^1([-T,T]; L^2) \cap \mathcal{C}_w([-T,T]; B^s_{p,r})$*,*
- $\nabla\Pi \in \mathcal{C}([-T,T]; L^2) \cap L^1([-T,T]; B^s_{p,r})$*.*

Besides, the energy equality (3) *is satisfied for all* $t \in [-T,T]$*, and time continuity holds with respect to the strong topology, if* $r<\infty$*.*

A few comments are in order:

- For the classical incompressible Euler equations (2), the above result statement (without the L^2 assumption) belongs to the mathematical folklore. It has been established in e.g. [19] in the case $1<p<\infty$ and in e.g. [1, Chap. 7] in the case $1\leqslant p\leqslant\infty$.
- The above statement covers the borderline case $B^{\frac{N}{p}+1}_{p,1}$ *without any smallness assumption*. Thus, up to the lower order L^2 assumption which is needed to control the low frequencies of the pressure, it extends the result [20] by Y. Zhou mentioned in the introduction.
- If one makes the stronger assumption that $(\rho_0-\overline{\rho}) \in B^s_{p,r}$ for some positive constant $\overline{\rho}$ then we get in addition $(\rho-\overline{\rho}) \in \mathcal{C}([-T,T]; B^s_{p,r})$ (or $\mathcal{C}_w([-T,T]; B^s_{p,r})$ if $r=\infty$).
- If $1<p\leqslant 2$ then $u_0 \in B^s_{p,r}$ implies that $u_0 \in L^2$. Furthermore, in dimension $N\geqslant 3$, the assumption that $(\rho_0-\overline{\rho}) \in L^{p^*}$ may be omitted if $p>N/(N-1)$. Therefore, except if $N=2$ and $p<2$ or if $N\geqslant 3$ and $p\leqslant N/(N-1)$, the density need not to tend to some constant at infinity.
- In contrast with the homogeneous case, in dimension $N=2$, the global well-posedness issue for (1) with nonconstant density is an open (and challenging) problem. Indeed, the vorticity $\omega := \partial_1 u^2 - \partial_2 u^1$ satisfies

$$\partial_t\omega + u\cdot\nabla\omega + \partial_1\left(\frac{1}{\rho}\right)\partial_2\Pi - \partial_2\left(\frac{1}{\rho}\right)\partial_1\Pi = 0,$$

hence is no transported by the flow of u if the density is not a constant.

Under the assumptions of Theorem 1, the solutions to (1) satisfy the following Beale–Kato–Majda type continuation criterion. For simplicity, we state the result for positive times only.

Theorem 2. *Consider a solution* $(\rho,u,\nabla\Pi)$ *to* (1) *on* $[0,T)\times\mathbb{R}^N$ *with the properties described in Theorem* 1*. If in addition*

$$\int_0^T \left(\|\nabla u\|_{L^\infty} + \|\nabla\Pi\|_{B^{s-1}_{p,r}}\right)dt < \infty \tag{4}$$

then $(\rho,u,\nabla\Pi)$ *may be continued beyond* T *into a solution of* (1) *with the same regularity.*

Moreover, in the case $s>1+N/p$*, the term* ∇u *may be replaced by* curl u *in* (4)*.*

Remark 1. The above statement has two important consequences:

- First, as condition (C) implies that $B^{s-1}_{p,r}$ is embedded in L^∞, one can show by means of an easy bootstrap argument that for data in $B^s_{p,r}$, the lifespan of a solution in $B^s_{p,r}$ is *the same* as the lifespan in $B^{\frac{N}{p}+1}_{p,1}$ (which is the larger space in this scale satisfying condition (C)).
- Second, by combining the previous remark with an induction argument, we see that if we start with smooth data (a_0,u_0) such that the derivatives at any order of ∇a_0 and u_0 are in L^p then we get a local-in-time smooth solution with the same properties. In addition, as above, the lifespan

for that smooth solution is only determined by the $B^{\frac{N}{p}+1}_{p,1}$ regularity. This generalizes prior results in the Hölder spaces framework in the case of a bounded domain (see [5]).

In the two-dimensional case, the assumption that $u_0 \in L^2$ is somewhat restrictive since if, say, the initial vorticity is in the Schwartz class then $u_0 \in L^2$ implies that the vorticity has average 0 over $\mathbb{R}^N$. This motivates the following statement which allows for *any* suitably smooth initial vector-field with compactly supported vorticity.

Theorem 3. *Let T_0 be in $]0,\infty[$ and let (s,p,r) satisfy condition (C) with $2 \leqslant p \leqslant 4$. Let u_0 be a divergence-free vector-field with coefficients in $B^s_{p,r}$. Assume that ρ_0 is positive, bounded and bounded away from zero, and that $\nabla\rho_0 \in B^{s-1}_{p,r}$. Finally, suppose that the body force f has coefficients in $L^1([-T_0,T_0];B^s_{p,r})$ and that the potential part $\mathcal{Q}f$ of f is in $\mathcal{C}([-T_0,T_0];L^2)$.*

There exists a time $T \in (0,T_0]$ such that system (1) supplemented with initial data (ρ_0,u_0) has a unique local solution $(\rho,u,\nabla\Pi)$ on $[-T,T]\times\mathbb{R}^N$ with:

- $\rho^{\pm1} \in \mathcal{C}_b([-T,T]\times\mathbb{R}^N)$, $D\rho \in \mathcal{C}_w([-T,T];B^{s-1}_{p,r})$,
- $u \in \mathcal{C}_w([-T,T];B^s_{p,r})$,
- $\nabla\Pi \in \mathcal{C}([-T,T];L^2)\cap L^1([-T,T];B^s_{p,r})$.

Besides, time continuity holds with respect to the strong topology, if $r<\infty$, and the continuation criterion stated in Theorem 2 also holds under the above assumptions.

As regards the well-posedness theory, the study of the limit case $p=\infty$ is of interest for different reasons. First, the Besov space $B^1_{\infty,1}$ is the largest one for which condition (C) holds. Second, the usual Hölder spaces belong to the family $B^s_{\infty,r}$ (take $r=\infty$) and are suitable for the study of the propagation of tangential regularity in (1), and of vortex patches than we plan to do in future works. Here we shall prove the following result.

Theorem 4. *Assume that $u_0 \in B^s_{\infty,r}\cap L^p$, $f \in L^1([-T_0,T_0];B^s_{\infty,r}\cap L^p)$ and $\rho_0 \in B^s_{\infty,r}$ for some $p\in(1,\infty)$ and some $s>1$ (or $s\geqslant 1$ if $r=1$). There exists a constant $\alpha>0$ depending only on s and N such that if, for some positive real number $\overline{\rho}$, we have*

$$\|\rho_0-\overline{\rho}\|_{B^s_{\infty,r}} \leqslant \alpha\overline{\rho}, \tag{5}$$

then there exists some $T>0$ such that system (1) has a unique solution $(\rho,u,\nabla\Pi)$ with

- $\rho \in \mathcal{C}([-T,T];B^s_{\infty,r})$ (or $\mathcal{C}_w([-T,T];B^s_{\infty,r})$ if $r=\infty$),
- $u \in \mathcal{C}([-T,T];L^p\cap B^s_{\infty,r})$, (or $u \in \mathcal{C}([-T,T];L^p)\cap\mathcal{C}_w([-T,T];B^s_{\infty,r})$ if $r=\infty$),
- $\nabla\Pi \in L^1([-T,T];B^s_{\infty,r})$.

Note that our result holds for small perturbations of a constant density state only. The reason why is that, in contrast with the previous statements, here bounding the pressure relies on estimates for the ordinary Laplace operator Δ. In other words, the heterogeneity $(a-\bar{a})\nabla\Pi$ is treated as a small perturbation term. We expect this smallness assumption to be just a technical artifact. However, removing it goes beyond the scope of this paper.

2. Tools

Our results mostly rely on the use of a nonhomogeneous dyadic partition of unity with respect to the Fourier variable, the so-called Littlewood–Paley decomposition. More precisely, fix a smooth radial function χ supported in (say) the ball $B(0,\frac{4}{3})$, equals to 1 in a neighborhood of $B(0,\frac{3}{4})$ and such that $r\mapsto\chi(r\,e_r)$ is nondecreasing over $\mathbb{R}^+$, and set $\varphi(\xi)=\chi(\frac{\xi}{2})-\chi(\xi)$.

The *dyadic blocks* $(\Delta_q)_{q\in\mathbb{Z}}$ are defined by[1]

$$\Delta_q := 0 \quad \text{if } q \leqslant -2, \qquad \Delta_{-1} := \chi(D) \quad \text{and} \quad \Delta_q := \varphi\big(2^{-q}D\big) \quad \text{if } q \geqslant 0.$$

We also introduce the following low frequency cut-off:

$$S_q u := \chi\big(2^{-q}D\big) = \sum_{p\geqslant q-1} \Delta_p \quad \text{for } q \geqslant 0.$$

The following classical properties will be used freely throughout in the paper:

- for any $u\in\mathcal{S}'$, the equality $u=\sum_q \Delta_q u$ makes sense in $\mathcal{S}'$;
- for all u and v in $\mathcal{S}'$, the sequence $(S_{q-1}u\,\Delta_q v)_{q\in\mathbb{N}}$ is spectrally supported in dyadic annuli. Indeed, as $\operatorname{Supp}\chi\subset B(0,\frac{4}{3})$ and $\operatorname{Supp}\varphi\subset\{\xi\in\mathbb{R}^n \,/\, \frac{3}{4}\leqslant|\xi|\leqslant\frac{8}{3}\}$, we have

$$\operatorname{Supp}\big(\mathcal{F}(S_{q-1}u\Delta_q v)\big)\subset\left\{\xi\in\mathbb{R}^N \Big/ \frac{1}{12}\cdot 2^q\leqslant|\xi|\leqslant\frac{10}{3}\cdot 2^q\right\}.$$

One can now define what a Besov space $B^s_{p,r}$ is:

Definition 1. Let u be a tempered distribution, s a real number, and $1\leqslant p,r\leqslant\infty$. We set

$$\|u\|_{B^s_{p,r}} := \left(\sum_q 2^{rqs}\|\Delta_q u\|^r_{L^p}\right)^{\frac{1}{r}} \quad \text{if } r<\infty \quad \text{and} \quad \|u\|_{B^s_{p,\infty}} := \sup_q 2^{qs}\|\Delta_q u\|_{L^p}.$$

We then define the space $B^s_{p,r}$ as the subset of distributions $u\in\mathcal{S}'$ such that $\|u\|_{B^s_{p,r}}$ is finite.

The Besov spaces have many nice properties which will be recalled throughout the paper whenever they are needed. For the time being, let us just recall that if condition (C) holds true then $B^s_{p,r}$ is an algebra continuously embedded in the set $C^{0,1}$ of bounded Lipschitz functions (see e.g. [1, Chap. 2]), and that the gradient operator maps $B^s_{p,r}$ in $B^{s-1}_{p,r}$. The following result will be also needed:

Proposition 1. *Let F be a smooth homogeneous function of degree 0 on $\mathbb{R}^N\setminus\{0\}$. Then for all $p\in(1,\infty)$, operator $F(D)$ is a self-map on L^p. In addition, if $r\in[1,\infty]$ and $s\in\mathbb{R}$ then $F(D)$ is a self-map on $B^s_{p,r}$.*

Proof. The continuity on L^p stems from the Hörmander–Mihlin theorem (see e.g. [11]). The rest of the proposition follows from the fact that if $u\in B^s_{p,r}$ then one may write, owing to $F(2^{-q}\xi)=F(\xi)$ for all $q\geqslant 0$ and $\xi\neq 0$,

$$F(D)u = F(D)\Delta_{-1}u + \sum_{q\geqslant 0}(F\widetilde{\varphi})\big(2^{-q}D\big)\Delta_q u$$

where $\widetilde{\varphi}$ is a smooth function with compact support away from the origin and value 1 on the support of φ. Note that $\mathcal{F}^{-1}(F\widetilde{\varphi})$ is in L^1. Therefore, the standard convolution inequality implies that

$$\big\|(F\widetilde{\varphi})\big(2^{-q}D\big)\Delta_q u\big\|_{L^p}\leqslant C\|\Delta_q u\|_{L^p}$$

[1] Throughout we agree that $f(D)$ stands for the pseudo-differential operator $u\mapsto\mathcal{F}^{-1}(f\mathcal{F}u)$.

while the L^p continuity result implies that

$$\left\|F(D)\Delta_{-1}u\right\|_{L^p} \leqslant \|\Delta_{-1}u\|_{L^p}.$$

Putting these two results together entails that $F(D)$ maps $B^s_{p,r}$ in itself. □

Remark 2. Both the Leray projector $\mathcal{P}$ over divergence free vector-fields and $\mathcal{Q} := \mathrm{Id} - \mathcal{P}$ satisfy the assumptions of the above proposition. Indeed, in Fourier variables, we have for all vector-field u with coefficients in $\mathcal{S}'(\mathbb{R}^N)$,

$$\widehat{\mathcal{Q}u}(\xi) = -\frac{\xi}{|\xi|^2}\xi\cdot\widehat{u}(\xi).$$

The following lemma (referred in what follows as *Bernstein's inequalities*) describes the way derivatives act on spectrally localized functions.

Lemma 1. *Let $0 < r < R$. A constant C exists so that, for any nonnegative integer k, any couple (p, q) in $[1, \infty]^2$ with $q \geqslant p \geqslant 1$ and any function u of L^p, we have for all $\lambda > 0$,*

$$\operatorname{Supp}\widehat{u} \subset B(0,\lambda R) \quad\Longrightarrow\quad \left\|D^k u\right\|_{L^q} \leqslant C^{k+1}\lambda^{k+N(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p};$$
$$\operatorname{Supp}\widehat{u} \subset \left\{\xi\in\mathbb{R}^N \,/\, r\lambda \leqslant |\xi| \leqslant R\lambda\right\} \quad\Longrightarrow\quad C^{-k-1}\lambda^k\|u\|_{L^p} \leqslant \left\|D^k u\right\|_{L^p} \leqslant C^{k+1}\lambda^k\|u\|_{L^p}.$$

The first Bernstein inequality entails the following embedding result:

Proposition 2. *The space $B^{s_1}_{p_1,r}$ is embedded in the space $B^{s_2}_{p_2,r}$ whenever*

$$1 \leqslant p_1 \leqslant p_2 \leqslant \infty \quad \text{and} \quad s_2 \leqslant s_1 - N/p_1 + N/p_2.$$

Remark 3. Recall that for all $s \in \mathbb{R}$, the Besov space $B^s_{2,2}$ coincides with the nonhomogeneous Sobolev space H^s. Furthermore if, for $k \in \mathbb{N}$, we denote by $W^{k,p}$ the set of L^p functions with derivatives up to order k in L^p then we have the following chain of continuous embedding:

$$B^k_{p,1} \hookrightarrow W^{k,p} \hookrightarrow B^k_{p,\infty}.$$

Let us now recall a few nonlinear estimates in Besov spaces. Formally, any product of two tempered distributions u and v, may be decomposed into

$$uv = T_u v + T_v u + R(u, v) \tag{6}$$

with

$$T_u v := \sum_q S_{q-1}u\Delta_q v, \qquad T_v u := \sum_q S_{q-1}v\Delta_q u \quad \text{and} \quad R(u, v) := \sum_q \sum_{|q'-q|\leqslant 1} \Delta_q u\,\Delta_{q'}v.$$

The above operator T is called "paraproduct" whereas R is called "remainder". The decomposition (6) has been introduced by J.-M. Bony in [6]. We shall sometimes use the notation

$$T'_u v := T_u v + R(u, v).$$

The paraproduct and remainder operators have many nice continuity properties. The following ones will be of constant use in this paper (see the proof in e.g. [1, Chap. 2]):

Proposition 3. *For any $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ and $t<0$, the paraproduct operator T maps $L^\infty\times B^s_{p,r}$ in $B^s_{p,r}$, and $B^t_{\infty,\infty}\times B^s_{p,r}$ in $B^{s+t}_{p,r}$. Moreover, the following estimates hold:*

$$\|T_u v\|_{B^s_{p,r}}\leqslant C\|u\|_{L^\infty}\|\nabla v\|_{B^{s-1}_{p,r}}\quad\textit{and}\quad \|T_u v\|_{B^{s+t}_{p,r}}\leqslant C\|u\|_{B^t_{\infty,\infty}}\|\nabla v\|_{B^{s-1}_{p,r}}.$$

For any (s_1,p_1,r_1) and (s_2,p_2,r_2) in $\mathbb{R}\times[1,\infty]^2$ such that $s_1+s_2>0$, $1/p:=1/p_1+1/p_2\leqslant 1$ and $1/r:=1/r_1+1/r_2\leqslant 1$ the remainder operator R maps $B^{s_1}_{p_1,r_1}\times B^{s_2}_{p_2,r_2}$ in $B^{s_1+s_2}_{p,r}$.

Combining the above proposition with Bony's decomposition (6), we easily get the following "tame estimate":

Corollary 1. *Let a be a bounded function such that $\nabla a\in B^{s-1}_{p,r}$ for some $s>0$ and $(p,r)\in[1,\infty]^2$. Then for any $b\in B^s_{p,r}\cap L^\infty$ we have $ab\in B^s_{p,r}\cap L^\infty$ and there exists a constant C depending only on N, p and s such that*

$$\|ab\|_{B^s_{p,r}}\leqslant C\big(\|a\|_{L^\infty}\|b\|_{B^s_{p,r}}+\|b\|_{L^\infty}\|Da\|_{B^{s-1}_{p,r}}\big).$$

The following result pertaining to the composition of functions in Besov spaces will be needed for estimating the reciprocal of the density.

Proposition 4. *Let I be an open interval of $\mathbb{R}$ and $F:I\to\mathbb{R}$, a smooth function. Then for all compact subset $J\subset I$, $s>0$ and $(p,r)\in[1,\infty]^2$ there exists a constant C such that for all function a valued in J and with gradient in $B^{s-1}_{p,r}$, we have $\nabla(F(a))\in B^{s-1}_{p,r}$ and*

$$\big\|\nabla\big(F(a)\big)\big\|_{B^{s-1}_{p,r}}\leqslant C\|\nabla a\|_{B^{s-1}_{p,r}}.$$

Proof. This is a variation on the proof of the classical composition lemma in Besov spaces based on Meyer's paralinearization method. We decompose $F(a)$ into

$$F(a)=F(S_1a)+\sum_{j\geqslant 1}m_j\Delta_j a\quad\text{with } m_j:=\int_0^1 F'(S_ja+\tau\Delta_ja)\,d\tau.$$

Under our assumptions, one may show (see e.g. Lemma 2.63 in [1]) that for all multi-index α, there exists a constant C_α such that

$$\|\partial_\alpha m_j\|_{L^\infty}\leqslant C_\alpha 2^{j|\alpha|}\quad\text{for all } j\in\mathbb{N}.$$

Note that, owing to the localization properties of the Littlewood–Paley decomposition, the function a may be replaced by $a-S_0a$ in every term Δ_ja with $j\geqslant 1$. Now, as $\nabla a\in B^{s-1}_{p,r}$, the function $a-S_0a$ belongs to $B^s_{p,r}$ and there exists a constant C such that

$$\|a-S_0a\|_{B^s_{p,r}}\leqslant C\|\nabla a\|_{B^{s-1}_{p,r}}.$$

Therefore, mimicking the proof of the classical composition lemma, we gather that $F(a)-F(S_1a)$ is in $B^s_{p,r}$ and satisfies

$$\big\|F(a)-F(S_1a)\big\|_{B^s_{p,r}}\leqslant C\|\nabla a\|_{B^{s-1}_{p,r}}.$$

Now, we notice that

$$\nabla\big(F(S_1a)\big) = F'(S_1a)\nabla S_1a.$$

Bersntein's inequality ensures that all the derivatives of ∇S_1a belong to L^p. So combining the chain rule and the Leibniz formula, one may conclude that all the derivatives of $\nabla(F(S_1a))$ are in L^p. In particular, $\nabla(F(S_1a))$ is in $B^{s-1}_{p,r}$ and satisfies the desired inequality. □

Our results concerning Eq. (1) rely strongly on a priori estimates in Besov spaces for the transport equation

$$\begin{cases} \partial_t f + v\cdot\nabla f = g, \\ f_{|t=0} = f_0. \end{cases} \tag{T}$$

We shall often use the following result, the proof of which may be found in e.g. [1, Chap. 3], or in the appendix of [7].

Proposition 5. *Let $1 \leqslant p, r \leqslant \infty$ and $\sigma > 0$. Let $f_0 \in B^\sigma_{p,r}$, $g \in L^1([0,T]; B^\sigma_{p,r})$ and v be a time dependent vector-field in $\mathcal{C}_b([0,T]\times\mathbb{R}^N)$ such that for some $p_1 \geqslant p$, we have*

$$\nabla v \in L^1\big([0,T]; B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty\big) \quad \text{if } \sigma < 1+\frac{N}{p_1},$$
$$\nabla v \in L^1\big([0,T]; B^{\sigma-1}_{p_1,r}\big) \quad \text{if } \sigma > 1+\frac{N}{p_1},\ \text{or } \sigma = 1+\frac{N}{p_1} \text{ and } r=1.$$

Then Eq. (T) has a unique solution f in

- *the space $\mathcal{C}([0,T]; B^\sigma_{p,r})$ if $r<\infty$,*
- *the space $(\bigcap_{\sigma'<\sigma}\mathcal{C}([0,T]; B^{\sigma'}_{p,\infty}))\cap\mathcal{C}_w([0,T]; B^\sigma_{p,\infty})$ if $r=\infty$.*

Moreover, for all $t\in[0,T]$, we have

$$e^{-CV(t)}\big\|f(t)\big\|_{B^\sigma_{p,r}} \leqslant \|f_0\|_{B^\sigma_{p,r}} + \int_0^t e^{-CV(t')}\big\|g(t')\big\|_{B^\sigma_{p,r}}\,dt' \tag{7}$$

with

$$V'(t) := \begin{cases} \|\nabla v(t)\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty} & \text{if } \sigma < 1+\frac{N}{p_1}, \\ \|\nabla v(t)\|_{B^{\sigma-1}_{p_1,r}} & \text{if } \sigma > 1+\frac{N}{p_1},\ \text{or } \sigma = 1+\frac{N}{p_1} \text{ and } r=1. \end{cases}$$

If $f=v$ then, for all $\sigma>0$, estimate (7) holds with $V'(t) := \|\nabla v(t)\|_{L^\infty}$.

3. Elliptic estimates

In this section, we want to prove high regularity estimates in Besov spaces for the following elliptic equation

$$-\operatorname{div}(a\nabla\Pi) = \operatorname{div} F \quad \text{in } \mathbb{R}^N \tag{8}$$

where $a = a(x)$ is a given suitably smooth bounded function satisfying

$$a_* := \inf_{x\in\mathbb{R}^N} a(x) > 0. \tag{9}$$

Let us recall that in the case $a \equiv 1$ the following result is available:

Proposition 6. *If $a \equiv 1$ and $p \in (1,\infty)$ then there exists a solution map $F \mapsto \nabla\Pi$ continuous on L^p.*

Proof. We set $\nabla\Pi = \nabla(-\Delta)^{-1}\operatorname{div} F$. Obviously the pseudo-differential operator $\nabla(-\Delta)^{-1}\operatorname{div}$ satisfies the conditions of Proposition 1. Hence $F \mapsto \nabla\Pi$ is a continuous self-map on L^p. □

We now turn to the study of (8) for nonconstant coefficients. For the convenience of the reader let us first establish the following classical result pertaining to the L^2 case.

Lemma 2. *For all vector-field F with coefficients in L^2, there exists a tempered distribution Π, unique up to constant functions, such that $\nabla\Pi \in L^2$ and Eq. (8) is satisfied. In addition, we have*

$$a_*\|\nabla\Pi\|_{L^2} \leqslant \|F\|_{L^2}. \tag{10}$$

Proof. The existence part of the statement is a consequence of the Lax–Milgram theorem. Indeed, for $\lambda > 0$, consider the following bilinear map:

$$b_\lambda(u,v) = (a\nabla u \mid \nabla v)_{L^2} + \lambda(u \mid v)_{L^2} \quad \text{for } u \text{ and } v \text{ in } H^1(\mathbb{R}^N).$$

Obviously b_λ is continuous and coercive, hence, given $F \in (L^2(\mathbb{R}^N))^N$, there exists a unique $\Pi_\lambda \in H^1(\mathbb{R}^N)$ so that

$$b_\lambda(u,\Pi_\lambda) = (u \mid F)_{L^2} \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

Taking $u = \Pi_\lambda$ and using the Cauchy–Schwarz inequality, we see that (10) is satisfied by Π_λ. Hence $(\nabla\Pi_\lambda)_{\lambda>0}$ is bounded in L^2 and there exist some $Q \in (L^2(\mathbb{R}^N))^N$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to 0, such that $\nabla\Pi_{\lambda_n} \rightharpoonup Q$ weakly in L^2. Note that this implies that Q satisfies $\operatorname{div}(aQ) = \operatorname{div} F$ in the distributional sense, and also that Q is the gradient of some tempered distribution Π. Besides, we have

$$\|\nabla\Pi\|_{L^2} = \|Q\|_{L^2} \leqslant \liminf \|\nabla\Pi_{\lambda_n}\|_{L^2} \leqslant a_*^{-1}\|F\|_{L^2}.$$

As regards uniqueness, it suffices to check that the constant functions are the only tempered solutions with gradient in L^2 which satisfy (8) with $F \equiv 0$. So let us consider $\Pi \in \mathcal{S}'$ with $\nabla\Pi \in L^2$ and $\operatorname{div}(a\nabla\Pi) = 0$. We thus have

$$\int a\nabla u \cdot \nabla\Pi \, dx = 0 \quad \text{for all } u \in H^1. \tag{11}$$

By taking advantage of the Fourier transform and of Parseval equality, it is easy to check that for $n > 0$, the tempered distribution $\Pi_n := (\mathrm{Id} - \chi(nD))\Pi$ (where the cut-off function χ has been defined in Section 2) belongs to H^1. Hence one may take $u = \Pi_n$ in (11) and we get

$$\int a\nabla\Pi \cdot \nabla\Pi_n \, dx = 0 \quad \text{for all } n > 0.$$

As $\nabla\Pi_n$ tends to $\nabla\Pi$ in L^2 and $a \geqslant a_* > 0$, this readily implies that $\nabla\Pi = 0$. □

Let us now establish higher order estimates.

Proposition 7. *Let $1 < p < \infty$ and $1 \leqslant r \leqslant \infty$. Let a be a bounded function satisfying* (9) *and such that $Da \in B^{s-1}_{p,r}$ for some $s > 1 + N/p$ or $s \geqslant 1 + N/p$ if $r = 1$.*

- *If $\sigma \in (1, s]$ and $\nabla\Pi \in B^{\sigma}_{p,r}$ satisfies* (8) *for some function F such that* $\operatorname{div} F \in B^{\sigma-1}_{p,r}$ *then we have for some constant C depending only on s, σ, p, N,*

$$a_*\|\nabla\Pi\|_{B^{\sigma}_{p,r}} \leqslant C\big(\|\mathrm{div}F\|_{B^{\sigma-1}_{p,r}} + a_*\big(1 + a_*^{-1}\|Da\|_{B^{s-1}_{p,r}}\big)^{\sigma}\|\nabla\Pi\|_{L^p}\big).$$

- *If $2 \leqslant p < \infty$ and F is in L^2 and satisfies* $\mathrm{div}F \in B^{\sigma-1}_{p,r}$ *for some $\sigma \in (1 + N/p - N/2, s]$ then Eq.* (8) *has a unique solution Π* (*up to constant functions*) *such that $\nabla\Pi \in L^2 \cap B^{\sigma}_{p,r}$. Furthermore, inequality* (10) *is satisfied and there exists a positive exponent γ depending only on σ, p, N and a positive constant C depending only on s, σ, p, N such that*

$$a_*\|\nabla\Pi\|_{B^{\sigma}_{p,r}} \leqslant C\big(\|\mathrm{div}F\|_{B^{\sigma-1}_{p,r}} + \big(1 + a_*^{-1}\|Da\|_{B^{s-1}_{p,r}}\big)^{\gamma}\|F\|_{L^2}\big).$$

- *If $\sigma > 1$ and $1 < p < \infty$ then the following inequality holds*:

$$a_*\|\nabla\Pi\|_{B^{\sigma}_{p,r}} \leqslant C\big(\|\mathrm{div}F\|_{B^{\sigma-1}_{p,r}} + \|\nabla a\|_{L^\infty}\|\nabla\Pi\|_{B^{\sigma-1}_{p,r}} + \|\nabla\Pi\|_{L^\infty}\|\nabla a\|_{B^{\sigma-1}_{p,r}}\big).$$

Proof. Throughout, $(c_q)_{q\geqslant -1}$ denotes a sequence in the unit sphere of ℓ^r.

The proof relies on two ingredients:

(i) the following commutator estimates (see Lemmas 6 and 7 in Appendix A)

$$\big\|\mathrm{div}[a, \Delta_q]\nabla\Pi\big\|_{L^p} \leqslant Cc_q 2^{-q(\sigma-1)}\|\nabla a\|_{B^{s-1}_{p,r}}\|\nabla\Pi\|_{B^{\sigma-1}_{p,r}}, \tag{12}$$

$$\big\|\mathrm{div}[a, \Delta_q]\nabla\Pi\big\|_{L^p} \leqslant Cc_q 2^{-q(\sigma-1)}\big(\|\nabla\Pi\|_{L^\infty}\|\nabla a\|_{B^{\sigma-1}_{p,r}} + \|\nabla a\|_{L^\infty}\|\nabla\Pi\|_{B^{\sigma-1}_{p,r}}\big) \tag{13}$$

which hold true whenever $\sigma \in (0, s]$ and (s, p, r) satisfies condition (C) (as regards (12)) and whenever $\sigma > 1$ (as concerns (13));

(ii) a Bernstein type inequality (see Lemma 8 in Appendix B).

For proving the first part of the lemma, apply the spectral cut-off operator Δ_q to (8). We get

$$-\operatorname{div}(a\Delta_q\nabla\Pi) = \operatorname{div}\Delta_q F + \operatorname{div}\big([\Delta_q, a]\nabla\Pi\big) \quad \text{for all } q \geqslant 0.$$

Hence, multiplying both sides by $|\Delta_q\Pi|^{p-2}\Delta_q\Pi$ and integrating over $\mathbb{R}^N$, we get

$$-\int |\Delta_q\Pi|^{p-2}\Delta_q\Pi\ \operatorname{div}(a\Delta_q\nabla\Pi)\,dx = \int |\Delta_q\Pi|^{p-2}\Delta_q\Pi\ \operatorname{div}\Delta_q F\,dx + \int |\Delta_q\Pi|^{p-2}\Delta_q\Pi\,\operatorname{div}\big([\Delta_q, a]\nabla\Pi\big)\,dx.$$

Apply Lemma 8 to bound by below the left-hand side of the above inequality. Using Hölder's inequality to handle the right-hand side, we get for all $q \geqslant 0$,

$$a_* 2^{2q}\|\Delta_q\Pi\|^p_{L^p} \leqslant C\|\Delta_q\Pi\|^{p-1}_{L^p}\big(\|\operatorname{div}\Delta_q F\|_{L^p} + \|\mathrm{div}[\Delta_q, a]\nabla\Pi\|_{L^p}\big). \tag{14}$$

To deal with the last term, one may now take advantage of inequality (12). Since, for $q \geqslant 0$, we have $\|\Delta_q \nabla \Pi\|_{L^p} \approx 2^q \|\Delta_q \Pi\|_{L^p}$ according to Lemma 1, we get after our multiplying inequality (14) by $2^{q(\sigma-1)}$:

$$a_* 2^{q\sigma} \|\Delta_q \nabla \Pi\|_{L^p} \leqslant C\big(2^{q(\sigma-1)} \|\Delta_q \operatorname{div} F\|_{L^p} + c_q \|\nabla a\|_{B^{s-1}_{p,r}} \|\nabla \Pi\|_{B^{\sigma-1}_{p,r}}\big) \quad \text{for all } q \in \mathbb{N}.$$

Taking the ℓ^r norm of both sides and adding up the low frequency block pertaining to $\Delta_{-1}\nabla \Pi$, we get

$$a_* \|\nabla \Pi\|_{B^\sigma_{p,r}} \leqslant C\big(\|\operatorname{div} F\|_{B^{\sigma-1}_{p,r}} + \|\nabla a\|_{B^{s-1}_{p,r}} \|\nabla \Pi\|_{B^{\sigma-1}_{p,r}} + a_* \|\Delta_{-1} \nabla \Pi\|_{L^p}\big). \tag{15}$$

Observe that $\|\Delta_{-1}\nabla \Pi\|_{L^p} \leqslant C\|\nabla \Pi\|_{L^p}$, and that the following interpolation inequality is available (recall that $0 < \sigma - 1$):

$$\|\nabla \Pi\|_{B^{\sigma-1}_{p,r}} \leqslant C\|\nabla \Pi\|_{L^p}^{\frac{1}{\sigma}} \|\nabla \Pi\|_{B^\sigma_{p,r}}^{1-\frac{1}{\sigma}}.$$

Then, applying a suitable Young inequality completes the proof of the first part of the proposition.

Let us now tackle the proof of the second part of the proposition. As $F \in L^2$, the existence of a solution $\nabla \Pi$ in L^2 is ensured by Lemma 2. Let us admit for a while that $\nabla \Pi \in B^\sigma_{p,r}$ and let us prove the desired inequality. As $p \geqslant 2$, we have

$$L^2 \hookrightarrow B^{N(\frac{1}{p}-\frac{1}{2})}_{p,\infty}.$$

Hence, as $B^{\sigma-1}_{p,r}$ is an interpolation space between $B^{N(\frac{1}{p}-\frac{1}{2})}_{p,\infty}$ and $B^\sigma_{p,r}$ (here comes the assumption that $\sigma - 1 > N/p - N/2$), one may write for some convenient exponent $\theta = \theta(p, \sigma, N) \in (0, 1)$,

$$\|\nabla \Pi\|_{B^{\sigma-1}_{p,r}} \leqslant C\|\nabla \Pi\|^\theta_{L^2} \|\nabla \Pi\|^{1-\theta}_{B^\sigma_{p,r}}.$$

In addition, as $p \geqslant 2$, Bernstein's inequality implies that

$$\|\Delta_{-1}\nabla \Pi\|_{L^p} \leqslant C\|\nabla \Pi\|_{L^2}.$$

Hence, plugging the last two inequalities in (15) and using (10) yields

$$a_* \|\nabla \Pi\|_{B^\sigma_{p,r}} \leqslant C\Big(\|\operatorname{div} F\|_{B^{\sigma-1}_{p,r}} + \|F\|_{L^2} + a_*^{-1} \|F\|^\theta_{L^2} \|\nabla a\|_{B^{s-1}_{p,r}} \big(a_* \|\nabla \Pi\|_{B^\sigma_{p,r}}\big)^{1-\theta}\Big).$$

Then applying Young's inequality completes the proof.

Remark that inequality (15) remains valid whenever $\nabla \Pi$ is in $B^{\sigma-1}_{p,r}$. Starting from the fact that the constructed solution $\nabla \Pi$ is in $B^{N(\frac{1}{p}-\frac{1}{2})}_{p,\infty}$, a straightforward induction argument allows to conclude that $\nabla \Pi$ is indeed in $B^\sigma_{p,r}$. This completes the second part of the proof.

For proving the last part of the proposition, the starting point is inequality (14) which implies that

$$a_* 2^{q\sigma} \|\nabla \Delta_q \Pi\|_{L^p} \leqslant C 2^{q(\sigma-1)}\big(\|\Delta_q \operatorname{div} F\|_{L^p} + \big\|\operatorname{div}[\Delta_q, a]\nabla \Pi\big\|_{L^p}\big).$$

Now, taking advantage of inequality (13) then summing up over $q \geqslant -1$, we readily obtain the desired result. □

4. Proof of the first local well-posedness result

As a preliminary step, let us observe that system (1) is *time reversible*. That is, changing (t,x) in $(-t,-x)$ restricts the study of the Cauchy problem to the evolution for *positive* times. To simplify the presentation, we shall thus focus on the unique solvability of the system for positive times only.

In the first part of this section, we establish the uniqueness part of Theorem 1. When proving existence, it is convenient to treat the two cases $p \geqslant 2$ and $p < 2$ separately. The reason why is that the proof strongly relies on Proposition 7 which enables to compute the pressure *only if* $p \geqslant 2$ (if $p < 2$ then only an a priori estimate is available).

So, we shall first prove the existence part of Theorem 1 in the case $p \geqslant 2$. The third subsection is devoted to the proof of Theorem 2 in the case $p \geqslant 2$. It will be needed for proving Theorem 1 in the case $p < 2$. The following part of this section is devoted to the proof of Theorems 1 and 2 in the case $p < 2$. In the last paragraph, we justify the claim pertaining to the case $p > N/(N-1)$ (see just after the statement of Theorem 1).

For expository purpose, we shall assume in this section and in the rest of the paper that $r < \infty$. For treating the case $r = \infty$, it is only a matter of replacing the strong topology by weak topology whenever regularity *up to index s* is involved.

4.1. Uniqueness

Uniqueness in Theorem 1 is a consequence of the following general stability result for solutions to (1).

Proposition 8. *Let $(\rho_1, u_1, \nabla\Pi_1)$ and $(\rho_2, u_2, \nabla\Pi_2)$ satisfy (1) with exterior forces f_1 and f_2. Assume in addition that ρ_1 and ρ_2 are bounded and bounded away from zero, that $\delta u := u_2 - u_1$ and $\delta\rho := \rho_2 - \rho_1$ belong to $\mathcal{C}^1([0,T];L^2)$, that $\delta f := f_2 - f_1$ is in $\mathcal{C}([0,T];L^2)$ and that $\nabla\Pi_1$, $\nabla\rho_1$ and ∇u_1 belong to $L^1([0,T];L^\infty)$. Then for all $t \in [0,T]$, we have*

$$\begin{aligned}&\big\|\delta\rho(t)\big\|_{L^2} + \big\|(\sqrt{\rho_2}\delta u)(t)\big\|_{L^2}\\ &\quad\leqslant e^{A(t)}\left(\big\|\delta\rho(0)\big\|_{L^2} + \big\|(\sqrt{\rho_2}\delta u)(0)\big\|_{L^2} + \int_0^t e^{-A(\tau)}\|\sqrt{\rho_2}\delta f\|_{L^2}\right)\end{aligned} \tag{16}$$

with $A(t) := \int_0^t (\|\frac{\nabla\rho_1}{\sqrt{\rho_2}}\|_{L^\infty} + \|\frac{\nabla\Pi_1}{\rho_1\sqrt{\rho_2}}\|_{L^\infty} + \|\nabla u_1\|_{L^\infty})\,d\tau.$

Proof. On the one hand, as

$$\partial_t\delta\rho + u_2\cdot\nabla\delta\rho = -\delta u\cdot\nabla\rho_1,$$

taking the L^2 inner product with $\delta\rho$ and integrating by parts in the second term of the left-hand side yields

$$\big\|\delta\rho(t)\big\|_{L^2} \leqslant \big\|\delta\rho(0)\big\|_{L^2} + \int_0^t \big\|(\sqrt{\rho_2}\delta u)\big\|_{L^2}\left\|\frac{\nabla\rho_1}{\sqrt{\rho_2}}\right\|_{L^\infty} d\tau. \tag{17}$$

On the other hand, denoting $\nabla\delta\Pi := \nabla\Pi_2 - \nabla\Pi_1$, we notice that

$$\rho_2(\partial_t\delta u + u_2\cdot\nabla\delta u) + \nabla\delta\Pi = \rho_2\left(\delta f + \frac{\delta\rho}{\rho_1\rho_2}\nabla\Pi_1 - \delta u\cdot\nabla u_1\right).$$

So taking the L^2 inner product of the second equation with δu, integrating by parts and using the fact that $\operatorname{div}\delta u = 0$ and that

$$\partial_t \rho_2 + u_2 \cdot \nabla \rho_2 = 0,$$

we eventually get

$$\begin{aligned}\big\|(\sqrt{\rho}_2\delta u)(t)\big\|_{L^2} \leqslant{}& \big\|(\sqrt{\rho}_2\delta u)(0)\big\|_{L^2} \\ &+ \int_0^t \left(\|\sqrt{\rho}_2 \delta f\|_{L^2} + \|\delta\rho\|_{L^2} \left\| \frac{\nabla \Pi_1}{\rho_1\sqrt{\rho}_2}\right\|_{L^\infty} + \|\nabla u_1\|_{L^\infty} \|\sqrt{\rho}_2\delta u\|_{L^2}\right) d\tau.\end{aligned}$$

Adding up inequality (17) to the above inequality and applying Gronwall lemma completes the proof of the proposition. □

Proof of uniqueness in Theorem 1. Consider two solutions $(\rho_1, u_1, \nabla\Pi_1)$ and $(\rho_2, u_2, \nabla\Pi_2)$ of (1) with the same data. Under the assumptions of Theorem 1, it is clear that the velocity and pressure fields satisfy the assumptions of the above proposition. As concerns the density, we notice that $u_i \in \mathcal{C}([0,T]; L^2)$ and $\nabla\rho_i \in \mathcal{C}([0,T]; L^\infty)$ for $i = 1, 2$ implies that $\partial_t\rho_i \in \mathcal{C}([0,T]; L^2)$. Hence we have $\delta\rho \in \mathcal{C}^1([0,T]; L^2)$. Therefore inequality (16) implies that $(\rho_1, u_1, \nabla\Pi_1) \equiv (\rho_2, u_2, \nabla\Pi_2)$ on $[0,T] \times \mathbb{R}^N$.

4.2. *The proof of existence in Theorem 1: the case* $2 \leqslant p < \infty$

We notice that, formally, the density-dependent incompressible Euler equations are equivalent to[2]

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0 \quad \text{with } a := 1/\rho, \\ \partial_t u + u \cdot \nabla u + a\nabla\Pi = f, \\ -\operatorname{div}(a\nabla\Pi) = \operatorname{div}(u \cdot \nabla\mathcal{P}u) - \operatorname{div} f. \end{cases} \tag{18}$$

Let us give conditions under which this equivalence is rigorous.

Lemma 3. *Let u be a time-dependent vector-field with coefficients in $\mathcal{C}^1([0,T] \times \mathbb{R}^N)$ and such that $\mathcal{Q}u \in \mathcal{C}^1([0,T]; L^2)$. Assume that $\nabla\Pi \in \mathcal{C}([0,T]; L^2)$. Let ρ be a continuous bounded function on $[0,T] \times \mathbb{R}^N$ which is positive and bounded away from* 0.

If in addition $\operatorname{div} u(0,\cdot) \equiv 0$ *in $\mathbb{R}^N$ then $(\rho, u, \nabla\Pi)$ is a solution to* (1) *if and only if $(a, u, \nabla\Pi)$ satisfies* (18).

Proof. If $(\rho, u, \nabla\Pi)$ satisfies (1) then, owing to $\rho > 0$, we see that $a := 1/\rho$ verifies the first equation of (18). Next, applying operator div to the velocity equation of (1) divided by ρ, and using that $\mathcal{P}u = u$ yields the third equation of (18).

Conversely, if $(a, u, \nabla\Pi)$ satisfies (18), it is obvious, owing to positivity, that $\rho := 1/a$ satisfies the density equation of (1). In order to justify that the other two equations are satisfied, it is only a matter of proving that $\operatorname{div} u \equiv 0$. For that, one may apply $\mathcal{Q}$ to the second equation. Then, using the third equation, we discover that

$$\partial_t \mathcal{Q}u + \mathcal{Q}(u \cdot \nabla\mathcal{Q}u) = 0.$$

[2] Recall that $\mathcal{P}$ stands for the Leray projector over divergence free vector-fields and that $\mathcal{Q} := \operatorname{Id} - \mathcal{P}$.

Recall that $\mathcal{Q}u \in \mathcal{C}^1([0,T]; L^2)$. Therefore, taking the L^2 inner product with $\mathcal{Q}u$, we get

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{Q}u\|_{L^2}^2 + \big(\mathcal{Q}(u\cdot\nabla\mathcal{Q}u) \,\big|\, \mathcal{Q}u\big)_{L^2} = 0.$$

As $\mathcal{Q}^T = \mathcal{Q}$ and $\mathcal{Q}^2 = \mathcal{Q}$, we thus get after integrating by parts in the second term:

$$\frac{d}{dt}\|\mathcal{Q}u\|_{L^2}^2 = \int |\mathcal{Q}u|^2 \operatorname{div} u\, dx,$$

and, as $\mathcal{Q}u(0,\cdot) = 0$, Gronwall lemma entails that $\mathcal{Q}u \equiv 0$. Hence $\operatorname{div} u = 0$. □

As explained in Lemma 3, it suffices to solve system (18). So, for $T > 0$, let us introduce the set E_T of functions $(a, u, \nabla\Pi)$ such that

$$a \in \mathcal{C}_b\big([0,T]\times\mathbb{R}^N\big), \qquad \nabla a \in \mathcal{C}\big([0,T]; B^{s-1}_{p,r}\big),$$
$$u \in \mathcal{C}^1\big([0,T]; L^2\big)\cap\mathcal{C}\big([0,T]; B^s_{p,r}\big), \qquad \nabla\Pi \in \mathcal{C}\big([0,T]; L^2\big)\cap L^1\big([0,T]; B^s_{p,r}\big).$$

We denote

$$a_* := \inf_{x\in\mathbb{R}^N} a_0(x), \qquad a^* := \sup_{x\in\mathbb{R}^N} a_0(x), \qquad \rho_* := \inf_{x\in\mathbb{R}^N} \rho_0(x) \quad\text{and}\quad \rho^* := \sup_{x\in\mathbb{R}^N} \rho_0(x).$$

Note that if ρ is bounded and bounded away from zero, and satisfies $\nabla\rho \in B^{s-1}_{p,r}$ then the same properties hold for a (and conversely). This may be easily shown by combining Propositions 2 and 4. Moreover, there exists some constant C depending only on a_*, a^*, N and on the regularity parameters such that

$$C^{-1}\|\nabla\rho\|_{B^{s-1}_{p,r}} \leqslant \|\nabla a\|_{B^{s-1}_{p,r}} \leqslant C\|\nabla\rho\|_{B^{s-1}_{p,r}}.$$

This fact will be used repeatedly in the rest of the paper.

Step 1. Construction of a sequence of approximate solutions. As a first step for solving (18), we construct a sequence $(a^n, u^n, \nabla\Pi^n)_{n\in\mathbb{N}}$ of global approximate solutions which belong to E_T for all $T > 0$.

For doing so, one may argue by induction. We first set $(a^0, u^0, \nabla\Pi^0) := (a_0, u_0, 0)$. Next, we assume that $(a^n, u^n, \nabla\Pi^n)$ has been constructed over $\mathbb{R}^+$, belongs to the space E_T for all $T > 0$ and that there exists a positive time T^* such that for all $t \in [0, T^*]$,

$$a_* \leqslant a^n(t,x) \leqslant a^*, \tag{19}$$

$$\big\|\nabla a^n(t)\big\|_{B^{s-1}_{p,r}} \leqslant 2\|\nabla a_0\|_{B^{s-1}_{p,r}}, \tag{20}$$

$$\big\|\sqrt{\rho^n(t)}u^n(t)\big\|_{L^2} \leqslant \sqrt{\rho^* a^*}\big(4\|\sqrt{\rho_0}\,u_0\|_{L^2} + 8\sqrt{\rho^*}\|f\|_{L^1_t(L^2)}\big) \quad \text{with } \rho^n := 1/a^n, \tag{21}$$

$$U^n(t) \leqslant 4U_0(t) + C_0\rho^* A_0\|\operatorname{div} f\|_{L^1_t(B^{s-1}_{p,r})} + C_0(\rho^* A_0)^{\gamma+1}\big(\|u_0\|_{L^2} + \|f\|_{L^1_t(L^2)}\big), \tag{22}$$

$$a_*\big\|\nabla\Pi^n\big\|_{L^1_t(B^s_{p,r})} \leqslant C\left(\int_0^t \big(U^n(\tau)\big)^2\, d\tau + \|\operatorname{div} f\|_{L^1_t(B^{s-1}_{p,r})} + (\rho^* A_0)^\gamma\big(\|u_0\|_{L^2} + \|f\|_{L^1_t(L^2)}\big)\right), \tag{23}$$

$$\big\|\nabla\Pi^n\big\|_{L^1_t(L^2)}\, d\tau \leqslant \sqrt{\rho^*}\|\sqrt{\rho_0}\,u_0\|_{L^2} + 3\rho^*\|f\|_{L^1_t(L^2)} \tag{24}$$

with $A_0 := a^* + \|\nabla a_0\|_{B^{s-1}_{p,r}}$, $U_0(t) := \|u_0\|_{B^s_{p,r}} + \|f\|_{L^1_t(B^s_{p,r})}$ and $U^n(t) := \|u^n(t)\|_{B^s_{p,r}}$. The positive exponent γ is given by Proposition 7. The constants C_0 and C depend only on (s,p,r) and N, and may be made explicit from the following computations (in fact one can take $C_0 = 2C^2$ with C large enough).

Denoting by ψ^n the flow of u^n, (which belongs to $\mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R}^N)$ owing to $u^n \in \mathcal{C}(\mathbb{R}^+; B^s_{p,r})$ and to $B^{s-1}_{p,r} \hookrightarrow \mathcal{C}_b$), we set

$$a^{n+1}(t,x) := a_0\big(\big(\psi^n_t\big)^{-1}(x)\big) \quad \text{and} \quad \rho^{n+1}(t,x) := \rho_0\big(\big(\psi^n_t\big)^{-1}(x)\big).$$

As ψ^n_t is a diffeomorphism over $\mathbb{R}^N$ for all $t \geqslant 0$, we have

$$\big\|a^{n+1}(t)\big\|_{L^\infty} = \|a_0\|_{L^\infty} = a^* \quad \text{and} \quad \big\|\rho^{n+1}(t)\big\|_{L^\infty} = \|\rho_0\|_{L^\infty} = \rho^*.$$

Hence (19) is satisfied by a^{n+1}. In addition, we have

$$\partial_t a^{n+1} + u^n \cdot \nabla a^{n+1} = 0$$

so that for all $i \in \{1,\dots,N\}$,

$$\partial_t \partial_i a^{n+1} + u^n \cdot \nabla \partial_i a^{n+1} = -\partial_i u^n \cdot \nabla a^{n+1}.$$

As $\partial_i a^{n+1}_{|t=0} = \partial_i a_0 \in B^{s-1}_{p,r}$ by assumption, (a slight generalization of) Proposition 5 combined with Gronwall lemma guarantees that $\nabla a^{n+1} \in \mathcal{C}(\mathbb{R}^+; B^{s-1}_{p,r})$ and that

$$\big\|\nabla a^{n+1}(t)\big\|_{B^{s-1}_{p,r}} \leqslant e^{C\int_0^t U^n(\tau)\,d\tau} \|\nabla a_0\|_{B^{s-1}_{p,r}}. \tag{25}$$

So if we assume that T^* has been chosen so that

$$C\int_0^{T^*} U^n(t)\,dt \leqslant \log 2 \tag{26}$$

then a^{n+1} satisfies (20). Next, we want to define u^{n+1} as the unique solution in $\mathcal{C}(\mathbb{R}^+; B^s_{p,r})$ of the transport equation:

$$\partial_t u^{n+1} + u^n \cdot \nabla u^{n+1} = -a^{n+1}\nabla \Pi^n + f, \qquad u^{n+1}_{|t=0} = u_0. \tag{27}$$

That the right-hand side belongs to $L^1_{loc}(\mathbb{R}^+; B^s_{p,r})$ is a consequence of Corollary 1 and of the embedding $B^{s-1}_{p,r} \hookrightarrow L^\infty$. In addition, we have for a.e. positive time

$$\big\|a^{n+1}\nabla\Pi^n\big\|_{B^s_{p,r}} \leqslant C\big(\big\|a^{n+1}\big\|_{L^\infty} + \big\|\nabla a^{n+1}\big\|_{B^{s-1}_{p,r}}\big)\big\|\nabla\Pi^n\big\|_{B^s_{p,r}}. \tag{28}$$

So finally, the existence of $u^{n+1} \in \mathcal{C}(\mathbb{R}^+; B^s_{p,r})$ is ensured by Proposition 5, and we have

$$\begin{aligned}\big\|u^{n+1}(t)\big\|_{B^s_{p,r}} \leqslant e^{C\int_0^t U^n(\tau)\,d\tau}\Bigg(&\|u_0\|_{B^s_{p,r}} + \int_0^t e^{-C\int_0^\tau U^n(\tau')\,d\tau'} \\ &\times \Big(\big(\big\|a^{n+1}\big\|_{L^\infty} + \big\|\nabla a^{n+1}\big\|_{B^{s-1}_{p,r}}\big)\big\|\nabla\Pi^n\big\|_{B^s_{p,r}} + \|f\|_{B^s_{p,r}}\Big)\,d\tau\Bigg).\end{aligned} \tag{29}$$

Therefore, if we restrict our attention to those t that are in $[0,T^*]$ with T^* satisfying (26), we see that for all $t\in[0,T^*]$,

$$U^{n+1}(t)\leqslant 2U_0(t)+CA_0\int_0^t \|\nabla\Pi^n\|_{B^s_{p,r}}\,d\tau\quad\text{with } A_0:=a^*+\|\nabla a_0\|_{B^{s-1}_{p,r}}.$$

So if we assume that T^* and C_0 have been chosen so that

$$2C^2\rho^*A_0\int_0^{T^*}U^n(t)\,dt\leqslant 1\quad\text{and}\quad C_0=2C^2 \tag{30}$$

then taking advantage of inequalities (22) and (23), we see that u^{n+1} satisfies (22) on $[0,T^*]$.

Let us now prove (21) for u^{n+1}. First, we notice that the right-hand side of (27) belongs to $\mathcal{C}(\mathbb{R}^+;L^2)$ so that u^{n+1} is in $\mathcal{C}^1(\mathbb{R}^+;L^2)$. As ρ^{n+1} is bounded and $\mathcal{C}^1$ with respect to the time and space variables, this allows us to take the L^2 inner product of the equation for u^{n+1} with $\rho^{n+1}u^{n+1}$. We readily get

$$\frac12\frac{d}{dt}\big\|\sqrt{\rho^{n+1}}\,u^{n+1}\big\|_{L^2}^2-\int\rho^{n+1}u^{n+1}\cdot f\,dx=\frac12\int\rho^{n+1}\big|u^{n+1}\big|^2\operatorname{div}u^n\,dx-\big(\nabla\Pi^n\,\big|\,u^{n+1}\big)_{L^2}. \tag{31}$$

Let us point out that u^n and u^{n+1} *need not* be divergence-free, so that the right-hand side may be nonzero. However, from the above inequality, it is easy to get

$$\begin{aligned}\big\|\big(\sqrt{\rho^{n+1}}\,u^{n+1}\big)(t)\big\|_{L^2}&\leqslant\|\sqrt{\rho}_0\,u_0\|_{L^2}\\&\quad+\int_0^t\Big(\sqrt{a^*}\big\|\nabla\Pi^n\big\|_{L^2}+\sqrt{\rho^*}\|f\|_{L^2}+\frac12\big\|\sqrt{\rho^{n+1}}u^{n+1}\big\|_{L^2}\big\|\operatorname{div}u^n\big\|_{L^\infty}\Big)\,dx.\end{aligned}$$

So, if we assume that C has been taken large enough in (26) then Gronwall's lemma implies that

$$\big\|\big(\sqrt{\rho^{n+1}}\,u^{n+1}\big)(t)\big\|_{L^2}\leqslant 2\big(\|\sqrt{\rho}_0\,u_0\|_{L^2}+\sqrt{\rho^*}\|f\|_{L^1_t(L^2)}+\sqrt{a^*}\big\|\nabla\Pi^n\big\|_{L^1_t(L^2)}\big). \tag{32}$$

Now, putting the above inequality together with inequality (24) ensures that inequality (21) is also satisfied by u^{n+1} on $[0,T^*]$.

To finish with, we have to construct the approximate pressure Π^{n+1}. For that, we aim at solving the elliptic equation

$$\operatorname{div}\big(a^{n+1}\nabla\Pi^{n+1}\big)=\operatorname{div}\big(f-u^{n+1}\cdot\nabla\mathcal{P}u^{n+1}\big) \tag{33}$$

for every positive time.

We have already proved that a^{n+1} satisfies the required ellipticity condition through (19). Moreover, as $u^{n+1}\in\mathcal{C}(\mathbb{R}^+;B^s_{p,r})$, Remark 2 ensures that $\nabla\mathcal{P}u^{n+1}$ is in $\mathcal{C}(\mathbb{R}^+;B^{s-1}_{p,r})$. As $B^{s-1}_{p,r}\hookrightarrow L^\infty$ and $u^{n+1}\in\mathcal{C}(\mathbb{R}^+;L^2)$, we thus have $u^{n+1}\cdot\nabla\mathcal{P}u^{n+1}\in\mathcal{C}(\mathbb{R}^+;L^2)$ and

$$\begin{aligned}\big\|u^{n+1}\cdot\nabla\mathcal{P}u^{n+1}\big\|_{L^2}&\leqslant\sqrt{a^*}\big\|\sqrt{\rho^{n+1}}u^{n+1}\big\|_{L^2}\big\|\nabla\mathcal{P}u^{n+1}\big\|_{L^\infty},\\&\leqslant C\sqrt{a^*}\big\|\sqrt{\rho^{n+1}}\,u^{n+1}\big\|_{L^2}\big\|\nabla u^{n+1}\big\|_{B^{s-1}_{p,r}}.\end{aligned}$$

Therefore Lemma 2 guarantees that (33) has a solution $\nabla\Pi^{n+1}$ in $\mathcal{C}(\mathbb{R}^+;L^2)$ which satisfies

$$a_*\|\nabla\Pi^{n+1}\|_{L^1_t(L^2)}\leqslant\|f\|_{L^1_t(L^2)}+C\sqrt{a^*}\int_0^t U^{n+1}\big\|\sqrt{\rho^{n+1}}u^{n+1}\big\|_{L^2}\,d\tau. \tag{34}$$

Let us insert inequality (32) in the above inequality. We see that if T^* has been chosen so that

$$4Ca^*\rho^*\int_0^{T^*}U^{n+1}\,d\tau\leqslant 1 \tag{35}$$

then inequality (34) implies that

$$\|\nabla\Pi^{n+1}\|_{L^1_t(L^2)}\leqslant\frac{3}{2}\rho^*\|f\|_{L^1_t(L^2)}+\frac{1}{2}\sqrt{\rho^*}\|\sqrt{\rho_0}u_0\|_{L^2}+\frac{1}{2}\|\nabla\Pi^n\|_{L^1_t(L^2)},$$

hence inequality (24) is satisfied by $\nabla\Pi^{n+1}$ on $[0,T^*]$.

In order to prove that $\nabla\Pi^{n+1}$ belongs to $L^1_{loc}(\mathbb{R}^+;B^s_{p,r})$, one may apply the second part of Proposition 7. Indeed, because, owing to $\operatorname{div}\mathcal{P}u^{n+1}=0$, we have

$$\operatorname{div}\big(u^{n+1}\cdot\nabla\mathcal{P}u^{n+1}\big)=\nabla u^{n+1}:\nabla\mathcal{P}u^{n+1}$$

and as $B^{s-1}_{p,r}$ is an algebra, the term $\operatorname{div}(u^{n+1}\cdot\nabla\mathcal{P}u^{n+1})$ is in $B^{s-1}_{p,r}$ and

$$\big\|\operatorname{div}\big(u^{n+1}\cdot\nabla\mathcal{P}u^{n+1}\big)\big\|_{B^{s-1}_{p,r}}\leqslant C\big(U^{n+1}\big)^2.$$

Hence Proposition 7 implies that for all $t\in\mathbb{R}^+$,

$$\begin{aligned}a_*\|\nabla\Pi^{n+1}\|_{L^1_t(B^s_{p,r})}\leqslant C\bigg(&\int_0^t\big(U^{n+1}\big)^2\,d\tau+\|\operatorname{div}f\|_{L^1_t(B^{s-1}_{p,r})}\\&+\big(1+\rho*\|Da^{n+1}\|_{L^\infty_t(B^{s-1}_{p,r})}\big)^\gamma\big(\|f\|_{L^1_t(L^2)}+\|u^{n+1}\cdot\nabla\mathcal{P}u^{n+1}\|_{L^1_t(L^2)}\big)\bigg),\end{aligned}$$

whence, using (20) at rank $n+1$ and Hölder inequality, we get

$$\begin{aligned}a_*\|\nabla\Pi^{n+1}\|_{L^1_t(B^s_{p,r})}\leqslant C\Bigg(&\int_0^t\big(U^{n+1}\big)^2\,d\tau+\|\operatorname{div}f\|_{L^1_t(B^{s-1}_{p,r})}\\&+(\rho^*A_0)^\gamma\bigg(\|f\|_{L^1_t(L^2)}+\sqrt{a^*}\big\|\sqrt{\rho^{n+1}}u^{n+1}\big\|_{L^\infty_t(L^2)}\int_0^t U^{n+1}\,d\tau\bigg)\Bigg).\end{aligned}$$

Taking advantage of inequality (21) at rank $n+1$ one can now conclude that if (35) holds then $\nabla\Pi^{n+1}$ satisfies (23).

At this stage we have proved that if inequalities (19) to (24) hold for $(a^n, u^n, \nabla\Pi^n)$ then they also hold for $(a^{n+1}, u^{n+1}, \nabla\Pi^{n+1})$ *provided* T^* *satisfies inequalities* (26), (30) *and* (35). Note that (30) is the strongest condition. Obviously it is satisfied if we take for T^* the supremum of

$$\left\{t>0 \big/ \rho^* t A_0\big(U_0(t)+\rho^* A_0\|\mathrm{div} f\|_{L^1_t(B^{s-1}_{p,r})}+(\rho^* A_0)^{\gamma+1}\big(\|u_0\|_{L^2}+\|f\|_{L^1_t(L^2)}\big)\big)\leqslant c\right\} \tag{36}$$

for a small enough constant c depending only on s, p and N.

Step 2. Convergence of the sequence. Let $\widetilde{a}^n := a^n - a_0$. In this step, we shall establish that $(\widetilde{a}^n, u^n, \nabla\Pi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T^*];L^2)$.

Let $\delta a^n := \widetilde{a}^{n+1} - \widetilde{a}^n$, $\delta u^n := u^{n+1} - u^n$ and $\delta\Pi^n := \Pi^{n+1} - \Pi^n$. We have for $n \geqslant 2$,

$$\begin{cases} \partial_t \delta a^n + u^n\cdot\nabla\delta a^n = -\delta u^{n-1}\cdot\nabla a^n,\\ \partial_t \delta u^n + u^n\cdot\nabla\delta u^n = -\delta u^{n-1}\cdot\nabla u^n - a^n\nabla\delta\Pi^{n-1} - \delta a^n\nabla\Pi^n,\\ \mathrm{div}\big(a^{n-1}\nabla\delta\Pi^{n-1}\big) = -\mathrm{div}\big(\delta u^{n-1}\cdot\nabla\mathcal{P}u^n + u^{n-1}\cdot\nabla\mathcal{P}\delta u^{n-1} + \delta a^{n-1}\nabla\Pi^n\big). \end{cases} \tag{37}$$

For all $n\in\mathbb{N}$, we have $\partial_t\widetilde{a}^{n+1} = -u^n\cdot\nabla a^{n+1}$. So, given that, according to the previous step, $u^n \in \mathcal{C}([0,T^*];L^2)$ and $\nabla a^{n+1}\in\mathcal{C}_b([0,T^*]\times\mathbb{R}^N)$, and that $\widetilde{a}^n_{|t=0} = \widetilde{a}^{n+1}_{|t=0} = 0$, we discover that $\widetilde{a}^n$, $\widetilde{a}^{n+1}$, and thus also δa^n, are in $\mathcal{C}^1([0,T^*];L^2)$. Taking the L^2 inner product of the equation for δa^n with δa^n, we thus get

$$\frac{1}{2}\frac{d}{dt}\|\delta a^n\|_{L^2}^2 = \frac{1}{2}\int\big(\delta a^n\big)^2\,\mathrm{div}\,u^n\,dx - \int\delta u^{n-1}\cdot\nabla a^n\delta a^n\,dx,$$

whence for all $t\in[0,T^*]$,

$$\|\delta a^n(t)\|_{L^2} \leqslant \frac{1}{2}\int_0^t\|\mathrm{div}\,u^n\|_{L^\infty}\|\delta a^n\|_{L^2}\,d\tau + \int_0^t\|\nabla a^n\|_{L^\infty}\|\delta u^{n-1}\|_{L^2}\,d\tau. \tag{38}$$

Next, taking the L^2 inner product of the equation for δu^n with $\rho^{n+1}\delta u^n$, performing integration by parts and using the equation for ρ^{n+1}, we get

$$\frac{1}{2}\frac{d}{dt}\int\rho^{n+1}|\delta u^n|^2\,dx = \frac{1}{2}\int\rho^{n+1}|\delta u^n|^2\,\mathrm{div}\,u^n\,dx - \int\rho^{n+1}\delta u^n\cdot\big(\delta u^{n-1}\cdot\nabla u^n + a^n\nabla\delta\Pi^{n-1} + \delta a^n\nabla\Pi^n\big)\,dx.$$

Hence

$$\Big\|\sqrt{\rho^{n+1}(t)}\delta u^n(t)\Big\|_{L^2} \leqslant \frac{1}{2}\int_0^t\|\mathrm{div}\,u^n\|_{L^\infty}\Big\|\sqrt{\rho^{n+1}}\delta u^n\Big\|_{L^2}\,d\tau + \int_0^t\big(\|\nabla u^n\|_{L^\infty}\|\delta u^{n-1}\|_{L^2} + \|a^n\|_{L^\infty}\|\nabla\delta\Pi^{n-1}\|_{L^2} + \|\nabla\Pi^n\|_{L^\infty}\|\delta a^n\|_{L^2}\big)\,d\tau. \tag{39}$$

Adding up inequalities (38) and (39), applying Gronwall lemma and using the fact that $\rho^{n+1} \geqslant \rho_*$ and the bounds stated in the first step, we thus get for all $t \in [0, T^*]$,

$$\big\|\big(\delta a^n, \delta u^n\big)(t)\big\|_{L^2} \leqslant C_{T^*}\left(\int_0^t \big\|\big(\delta a^{n-1}, \delta u^{n-1}\big)\big\|_{L^2}\, d\tau + \int_0^t \big\|\nabla \delta \Pi^{n-1}\big\|_{L^2}\, d\tau\right), \tag{40}$$

where the constant C_{T^*} depends only on T^* and on the initial data.

In order to bound $\nabla\delta\Pi^{n-1}$, we shall use that for any $\mathcal{C}^1$ vector-fields a and b, we have

$$\operatorname{div}(a\cdot\nabla b) = \operatorname{div}(b\cdot\nabla a) + \operatorname{div}(a \operatorname{div} b) - \operatorname{div}(b \operatorname{div} a).$$

Applying this to $a = u^{n-1}$ and $b = \mathcal{P}\delta u^{n-1}$ and bearing in mind that $\operatorname{div}\mathcal{P}\delta u^{n-1} = 0$, we deduce from the third equation of (37) that

$$\operatorname{div}\big(a^{n-1}\nabla\delta\Pi^{n-1}\big) = \operatorname{div}\big(\mathcal{P}\delta u^{n-1} \operatorname{div} u^{n-1} - \mathcal{P}\delta u^{n-1}\cdot\nabla u^{n-1} - \delta u^{n-1}\cdot\nabla\mathcal{P}u^n - \delta a^{n-1}\nabla\Pi^n\big).$$

Therefore, Lemma 2 and the fact that $\|\mathcal{P}\|_{\mathcal{L}(L^2;L^2)} = 1$ guarantee that

$$a_*\big\|\nabla\delta\Pi^{n-1}\big\|_{L^2} \leqslant \big\|\delta u^{n-1}\big\|_{L^2}\big(\big\|\operatorname{div} u^{n-1}\big\|_{L^\infty} + \big\|\nabla u^{n-1}\big\|_{L^\infty} + \big\|\nabla\mathcal{P}u^n\big\|_{L^\infty}\big) + \big\|\delta a^{n-1}\big\|_{L^2}\big\|\nabla\Pi^n\big\|_{L^\infty}.$$

Using the uniform bounds of the previous step, we thus get for all $t \in [0, T^*]$,

$$\big\|\nabla\delta\Pi^{n-1}\big\|_{L^2} \leqslant C_{T^*}\big(\big\|\delta u^{n-1}\big\|_{L^2} + \big\|\delta a^{n-1}\big\|_{L^2}\big). \tag{41}$$

Plugging inequality (41) in inequality (40), we end up with (up to a change of C_{T^*}),

$$\big\|\big(\delta a^n, \delta u^n\big)(t)\big\|_{L^2} \leqslant C_{T^*}\int_0^t \big\|\big(\delta a^{n-1}, \delta u^{n-1}\big)(\tau)\big\|_{L^2}\, d\tau.$$

Arguing by induction, one may conclude that

$$\sup_{t\in[0,T^*]}\big\|\big(\delta a^n, \delta u^n\big)(t)\big\|_{L^2} \leqslant \frac{(C_{T^*}T^*)^n}{n!}\sup_{t\in[0,T^*]}\big\|\big(\delta a^0, \delta u^0\big)(t)\big\|_{L^2}.$$

It is now obvious that both $(\widetilde{a}^n)_{n\in\mathbb{N}}$ and $(u^n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathcal{C}([0, T^*]; L^2)$, hence converge to some functions $\widetilde{a}$ and u in $\mathcal{C}([0, T^*]; L^2)$. Taking advantage of (41), it is also clear that $(\nabla\Pi^n)_{n\in\mathbb{N}}$ converges to some function $\nabla\Pi$ in $\mathcal{C}([0, T^*]; L^2)$.

Step 3. Final checking. Let $a := a_0 + \widetilde{a}$. We now have to check that $(a, u, \nabla\Pi)$ is indeed a solution to (1) and that it has the properties stated in Theorem 1. From the previous step, we already know that $(a - a_0)$, u and $\nabla\Pi$ are in $\mathcal{C}([0, T^*]; L^2)$. Moreover:

- As $(\nabla a^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0, T^*]; B^{s-1}_{p,r})$ and as Besov spaces have the Fatou property, we deduce that ∇a belongs to $L^\infty([0, T^*]; B^{s-1}_{p,r})$. Since $(a^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0, T^*] \times \mathbb{R}^N)$, we also have $a \in L^\infty([0, T^*] \times \mathbb{R}^N)$.
- As $(u^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0, T^*]; B^s_{p,r})$, we deduce that $u \in L^\infty([0, T^*]; B^s_{p,r})$.
- Finally, as $(\nabla\Pi^n)_{n\in\mathbb{N}}$ is bounded in $L^1([0, T^*]; B^s_{p,r})$ we deduce that $\nabla\Pi$ belongs to $L^1([0, T^*]; B^s_{p,r})$.

Arguing by interpolation, we see that the above sequences converge strongly in every intermediate space between $\mathcal{C}([0,T^*];L^2)$ and $\mathcal{C}([0,T^*];B^s_{p,r})$ which is more than enough to pass to the limit in the equations satisfied by $(a^n,u^n,\nabla\Pi^n)$. Hence $(a,u,\nabla\Pi)$ satisfies (18).

Passing to the limit in (31), we see that, in addition, (ρ,u) satisfies the energy equality (3).

Finally, the continuity properties of the solution with respect to the time may be recovered by using the equations satisfied by $(a,u,\nabla\Pi)$, and Proposition 5.

4.3. A continuation criterion

The key to the proof of Theorem 2 is the following lemma:

Lemma 4. *Let (s,p,r) satisfy condition (C) with $1<p<\infty$. Consider a solution $(\rho,u,\nabla\Pi)$ to (1) on $[0,T[\times\mathbb{R}^N$ such that*[3] *$u\in\mathcal{C}([0,T);B^s_{p,r})$ and*

$$\rho_*\leqslant\rho\leqslant\rho^*,\quad \rho\in\mathcal{C}\big([0,T)\times\mathbb{R}^N\big)\quad\textit{and}\quad \nabla\rho\in\mathcal{C}\big([0,T);B^{s-1}_{p,r}\big).$$

If in addition

$$\int_0^T\big(\|\nabla u\|_{L^\infty}+\|\nabla\Pi\|_{B^{s-1}_{p,r}}\big)\,dt<\infty \tag{42}$$

then

$$\int_0^t\|\nabla\Pi\|_{B^s_{p,r}}\,d\tau+\sup_{0\leqslant t<T}\big(\big\|u(t)\big\|_{B^s_{p,r}}+\big\|\nabla\rho(t)\big\|_{B^{s-1}_{p,r}}\big)<\infty.$$

Proof. By virtue of Proposition 4, the function $a:=1/\rho$ satisfies the same assumptions as ρ. Therefore we shall rather work with a, for convenience. Recall that

$$\partial_t\partial_k a+u\cdot\nabla\partial_k a=-\partial_k u\cdot\nabla a\quad\text{for }k=1,\dots,N. \tag{43}$$

So, applying operator Δ_q to the above equality and using that $\operatorname{div}u=0$, one may write (with the summation convention)

$$\partial_t\Delta_q\partial_k a+u\cdot\nabla\Delta_q\partial_k a=-\Delta_q(\partial_k u\cdot\nabla a)+\partial_j\big[u^j,\Delta_q\big]\partial_k a.$$

Therefore for all $t\in[0,T)$,

$$\big\|\Delta_q\partial_k a(t)\big\|_{L^p}\leqslant\|\Delta_q\partial_k a_0\|_{L^p}+\int_0^t\big\|\Delta_q(\partial_k u\cdot\nabla a)\big\|_{L^p}\,d\tau+\int_0^t\big\|\partial_j\big[u^j,\Delta_q\big]\partial_k a\big\|_{L^p}\,d\tau. \tag{44}$$

According to Proposition 3, the term $\partial_k u\cdot\nabla a$ belongs to $B^{s-1}_{p,r}$ and satisfies

$$\|\partial_k u\cdot\nabla a\|_{B^{s-1}_{p,r}}\leqslant C\big(\|\partial_k u\|_{L^\infty}\|\nabla a\|_{B^{s-1}_{p,r}}+\|\nabla a\|_{L^\infty}\|\partial_k u\|_{B^{s-1}_{p,r}}\big)$$

[3] With the usual convention if $r=\infty$.

while Lemma 7 ensures that for all $q \geqslant -1$,

$$\left\| \partial_j\left[u^j, \Delta_q\right]\partial_k a \right\|_{L^p} \leqslant C c_q 2^{q(s-1)}\left(\|\partial_k a\|_{L^\infty}\|\nabla u\|_{B^{s-1}_{p,r}} + \|\nabla u\|_{L^\infty}\|\partial_k a\|_{B^{s-1}_{p,r}}\right).$$

Using the definition of the norm in $B^{s-1}_{p,r}$, we thus get after summation in (44) that

$$\left\|\nabla a(t)\right\|_{B^{s-1}_{p,r}} \leqslant \|\nabla a_0\|_{B^{s-1}_{p,r}} + C\int_0^t \left(\|\nabla u\|_{L^\infty}\|\nabla a\|_{B^{s-1}_{p,r}} + \|\nabla a\|_{L^\infty}\|\nabla u\|_{B^{s-1}_{p,r}}\right) d\tau. \tag{45}$$

In order to bound the velocity, let us apply the last part of Proposition 5 to the velocity equation, and the following inequality (which stems from Corollary 1):

$$\|a\nabla\Pi\|_{B^s_{p,r}} \leqslant C\left(a^*\|\nabla\Pi\|_{B^s_{p,r}} + \|\nabla\Pi\|_{L^\infty}\|\nabla a\|_{B^{s-1}_{p,r}}\right).$$

We get for all $t \in [0, T)$,

$$\begin{aligned}\left\|u(t)\right\|_{B^s_{p,r}} \leqslant e^{C\int_0^t \|\nabla u\|_{L^\infty}\,d\tau}\Bigg(&\|u_0\|_{B^s_{p,r}} + \int_0^t e^{-C\int_0^\tau \|\nabla u\|_{L^\infty}\,d\tau'}\left(\|f\|_{B^s_{p,r}} + Ca^*\|\nabla\Pi\|_{B^s_{p,r}}\right.\\ &\left.+ C\|\nabla\Pi\|_{L^\infty}\|\nabla a\|_{B^{s-1}_{p,r}}\right)d\tau\Bigg).\end{aligned} \tag{46}$$

In order to bound the pressure term, one may use the fact that

$$\operatorname{div}(a\nabla\Pi) = \operatorname{div} f - \operatorname{div}(u\cdot\nabla u)$$

and apply the last part of Proposition 7. Performing a time integration and using that

$$\left\|\operatorname{div}(u\cdot\nabla u)\right\|_{B^{s-1}_{p,r}} = \|\nabla u : \nabla u\|_{B^{s-1}_{p,r}} \leqslant C\|\nabla u\|_{L^\infty}\|\nabla u\|_{B^{s-1}_{p,r}},$$

we get

$$\begin{aligned}a_*\|\nabla\Pi\|_{L^1_t(B^s_{p,r})} \leqslant C\Bigg(&\|\operatorname{div} f\|_{L^1_t(B^{s-1}_{p,r})} + \int_0^t \left(\|\nabla u\|_{L^\infty}\|\nabla u\|_{B^{s-1}_{p,r}} + \|\nabla a\|_{L^\infty}\|\nabla\Pi\|_{B^{s-1}_{p,r}}\right.\\ &\left.+ \|\nabla\Pi\|_{L^\infty}\|\nabla a\|_{B^{s-1}_{p,r}}\right)\Bigg)d\tau.\end{aligned}$$

Let us insert this latter inequality in (46). Then adding up inequality (45) and applying Gronwall lemma we end up with

$$\|\nabla a(t)\|_{B^{s-1}_{p,r}} + \|u(t)\|_{B^s_{p,r}} \leqslant C \exp\left(\int_0^t \|(\nabla a, \nabla u, \nabla\Pi)\|_{L^\infty}\,d\tau\right) \times \left(\|\nabla a_0\|_{B^{s-1}_{p,r}} + \|u_0\|_{B^s_{p,r}} + \|f\|_{L^1_t(B^s_{p,r})} + \int_0^t \|\nabla a\|_{L^\infty}\|\nabla\Pi\|_{B^{s-1}_{p,r}}\,d\tau\right) \tag{47}$$

for some constant C depending only on the regularity parameters and on N, a_* and a^*.

Now, let us notice that ∇a is bounded on $[0,T)\times\mathbb{R}^N$. Indeed, from Eq. (43) and Gronwall lemma, we see that

$$\|\nabla a(t)\|_{L^\infty} \leqslant e^{\int_0^t \|\nabla u\|_{L^\infty}}\|\nabla a_0\|_{L^\infty}.$$

As $\nabla\Pi$ is in $L^1([0,T); B^{s-1}_{p,r})$ and ∇u is in $L^1([0,T); L^\infty)$ by assumption and as $B^{s-1}_{p,r} \hookrightarrow L^\infty$, we discover that both the last term in (47) and the exponential term are bounded on $[0,T)$. This completes the proof of the lemma. □

The following lemma implies the first part of Theorem 2 in the case $p \geqslant 2$.

Lemma 5. *Let (s,p,r) satisfy condition (C) with $2 \leqslant p < \infty$. Consider a solution $(\rho, u, \nabla\Pi)$ to (1) on $[0,T[\,\times\mathbb{R}^N$ such that*[4]

- $\rho_* \leqslant \rho \leqslant \rho^*$, $\rho \in \mathcal{C}([0,T)\times\mathbb{R}^N)$ *and* $\nabla\rho \in \mathcal{C}([0,T); B^{s-1}_{p,r})$,
- $u \in \mathcal{C}([0,T); B^s_{p,r}) \cap \mathcal{C}^1([0,T]; L^2)$,
- $\nabla\Pi \in \mathcal{C}([0,T); L^2) \cap L^1([0,T); B^s_{p,r})$.

If in addition condition (42) is satisfied then $(\rho, u, \nabla\Pi)$ may be continued beyond T into a solution of (1) with the above regularity.

Proof. Lemma 4 ensures that $\|u\|_{L^\infty_T(B^s_{p,r})}$ and $\|Da\|_{L^\infty_T(B^{s-1}_{p,r})}$ are finite. So one may set

$$\varepsilon := c(\rho^* A_0)^{-1}\big(U_0(T) + \rho^* A_0 \|\operatorname{div} f\|_{L^1_T(B^{s-1}_{p,r})} + (\rho^* A_0)^{\gamma+1}\big(\|u_0\|_{L^2} + \|f\|_{L^1_T(L^2)}\big)\big)^{-1}$$

where c is the small constant (depending only on N and (s,p,r)) defined in (36).

Then we know from the proof of Theorem 1 in the case $p \geqslant 2$ that for any $T' < T$, system (1) with data $(\rho(T'), u(T'), f(T'+\cdot))$ has a unique solution up to time ε. Taking $T' = T - \varepsilon/2$ we thus get a continuation of $(\rho, u, \nabla\Pi)$ up to time $T + \varepsilon/2$. □

Let us now justify the last part of Theorem 2. It stems from the following logarithmic interpolation inequality (see e.g. [14]):

$$\|\nabla u\|_{L^\infty} \leqslant C\big(1 + \|\nabla u\|_{\dot B^0_{\infty,\infty}} \log\big(e + \|\nabla u\|_{B^{s-1}_{p,r}}\big)\big) \quad \text{with } \|\nabla u\|_{\dot B^0_{\infty,\infty}} := \sup_{q\in\mathbb{Z}} \big\|\varphi\big(2^{-q}D\big)\nabla u\big\|_{L^\infty}$$

which holds true whenever the embedding of $B^{s-1}_{p,r}$ is not critical (that is $s > 1 + N/p$).

[4] With the usual convention if $r = \infty$.

Then, arguing exactly as in Proposition 5.3 of [9], we discover that condition (42) may be replaced by the following *weaker* condition:

$$\int_0^T \left(\|\nabla u\|_{\dot B^0_{\infty,\infty}} + \|\nabla\Pi\|_{B^{s-1}_{p,r}}\right) dt < \infty. \tag{48}$$

Now, it is classical (see e.g. [1, Chap. 7]) that there exists some constant C such that

$$\|\nabla u\|_{\dot B^0_{\infty,\infty}} \leqslant C\|\operatorname{curl} u\|_{L^\infty}.$$

This completes the proof of Theorem 2 in the case $p \geqslant 2$.

4.4. The case $1 < p < 2$

Note that by virtue of Proposition 2, the data satisfy the assumptions of the theorem for the triplet $(s - N/p + N/2, 2, r)$. Hence, applying the theorem in the case $p = 2$ supplies a local solution with the $B^{s-N/p+N/2}_{2,r}$ regularity. However, proving that the $B^s_{p,r}$ regularity is also preserved, is not utterly obvious. We shall proceed as follows:

i) first, we smooth out the data so as to get a solution in $H^\infty := \cap_\sigma H^\sigma$ for which the $B^s_{p,r}$ regularity is also preserved;
ii) second, we establish uniform bounds in $B^s_{p,r}$ on a fixed suitably small time interval;
iii) third, we show the convergence of the sequence of smooth solutions and that the limit has the required properties.

Step 1. Smooth solutions. Set

$$a_0^n := S_n a_0, \qquad u_0^n := S_n u_0 \quad \text{and} \quad f^n := S_n f$$

where S_n is the low frequency cut-off introduced in Section 2.

Note that for all large enough $n \in \mathbb{N}$ and $t \in [0, T_0]$, we have

$$a_*/2 \leqslant a_0^n \leqslant 2a^*, \qquad \|Da_0^n\|_{B^{s-1}_{p,r}} \leqslant C\|Da_0\|_{B^{s-1}_{p,r}}, \tag{49}$$

$$\|u_0^n\|_{B^s_{p,r}} \leqslant C\|u_0\|_{B^s_{p,r}}, \tag{50}$$

$$\|f^n\|_{L^1_t(B^s_{p,r})} \leqslant C\|f\|_{L^1_t(B^s_{p,r})} \quad \text{and} \quad \|f^n(t)\|_{L^2} \leqslant \|f(t)\|_{L^2}. \tag{51}$$

It is also clear that (with obvious notation) ∇a_0^n and u_0^n are in $B^\infty_{p,r}$ (hence also in H^∞) and that $f^n \in \mathcal{C}([0, T_0]; H^\infty) \cap L^1([0, T_0]; B^\infty_{p,r})$.

Finally, taking advantage of Lebesgue's dominated convergence theorem one may prove, if $r < \infty$, that[5]

$$Da_0^n \longrightarrow Da_0 \quad \text{in } B^{s-1}_{p,r} \quad \text{and} \quad \left(a_0^n - \bar a\right) \to (a_0 - \bar a) \quad \text{in } L^{p^*} \quad \text{with } \bar a := 1/\bar\rho,$$

$$u_0^n \longrightarrow u_0 \quad \text{in } B^{s-1}_{p,r},$$

$$f^n \longrightarrow f \quad \text{in } L^1\left([0, T_0]; B^s_{p,r}\right) \cap \mathcal{C}\left([0, T_0]; L^2\right).$$

As usual, the strong convergence has to be replaced by the weak convergence if $r = \infty$.

[5] Recall that $(\rho_0 - \bar\rho) \in L^{p^*}$ by assumption.

Applying Theorem 1 in the case $p=2$ and using the fact that the lifespan does not depend on the index of regularity (see Remark 1), we get a local maximal solution $(a^n, u^n, \nabla\Pi^n)$ with Da^n, u^n and $\nabla\Pi^n$ in $\mathcal{C}([0,T_n^*); H^\infty)$, and

$$a_*/2 \leqslant a^n \leqslant 2a^*. \tag{52}$$

Note that as a^n and ρ^n are just transported by the (smooth) flow of u^n, we also have

$$\big\|\big(\rho^n(t)-\bar\rho\big)\big\|_{L^{p^*}} = \big\|\rho_0^n-\bar\rho\big\|_{L^{p^*}} \leqslant C\|\rho_0-\bar\rho\|_{L^{p^*}} \quad \text{for all } t\in\big[0,T_n^*\big) \tag{53}$$

(and similarly for a^n) and ∇a^n and $\nabla\rho^n$ belong to $\mathcal{C}([0,T_n^*); B^{s-1}_{p,r})$.

Let us now establish that $\nabla\Pi^n$ is in $L^1([0,T]; B^s_{p,r})$ for all $T\in[0,T_n^*)$. Fix some $T\in[0,T_n^*)$. Applying operator div to the momentum equation of (1) and using that $\operatorname{div} u^n=0$ yields

$$\Delta\Pi^n = \operatorname{div}\big(\rho^n f^n\big) - \operatorname{div}\big(\rho^n u^n\cdot\nabla u^n\big) - \nabla\rho^n\cdot\partial_t u^n. \tag{54}$$

According to Proposition 1, $F\mapsto D^2\Pi$ is a self-map on $B^s_{p,r}$. Hence, in order to show that $\nabla\Pi^n \in L^1([0,T]; B^s_{p,r})$, it suffices to establish that $\nabla\Pi^n\in L^1([0,T]; L^p)$ and that $\Delta\Pi^n\in L^1([0,T]; B^{s-1}_{p,r})$.

Let us first show that all the terms of the right-hand side of (54) are in $L^1([0,T]; B^{s-1}_{p,r})$. Since, by assumption, $f^n\in L^1([0,T]; B^s_{p,r})$ and as it as been established that $\rho^n\in L^\infty$ and $\nabla\rho^n\in B^{s-1}_{p,r}$, Corollary 1 implies that $\operatorname{div}(\rho^n f^n)\in L^1([0,T]; B^{s-1}_{p,r})$. For the next term, we use that for all $i\in\{1,\dots,N\}$,

$$\big(\rho^n u^n\cdot\nabla u^n\big)^i = \sum_j T'_{\rho^n(u^n)^j}\partial_j u^n + T_{\partial_j(u^n)^i}\rho^n\big(u^n\big)^j.$$

For fixed time, by embedding, $\rho^n u^n$ and ∇u^n are in L^{p^*} (recall that $p^*>2>p$) and, arguing as for $\rho^n f^n$, one can check that $\rho^n u^n$ is in H^∞. Of course, ∇u^n is also in H^∞. Given that $1/p = 1/p^* + 1/2$, continuity results for the paraproduct and remainder in the spirit of Proposition 3 (see [17]) ensure that $\rho^n u^n\cdot\nabla u^n$ is in $B^{s-1}_{p,r}$.

For the last term in (54), one may write that

$$\nabla\rho^n\cdot\partial_t u^n = T'_{\partial_t u^n}\cdot\nabla\rho^n + T_{\nabla\rho^n}\cdot\partial_t u^n.$$

As, by embedding, $\partial_t u^n\in L^1([0,T]; L^\infty)$, and as $\nabla\rho^n\in L^\infty([0,T]; B^{s-1}_{p,r})$, continuity results for the paraproduct ensure that the first term in the right-hand side is in $L^1([0,T]; B^{s-1}_{p,r})$. Concerning the second term, one may use that $\nabla\rho^n\in L^\infty([0,T]; L^{p^*})$ (by embedding) and that $\partial_t u^n\in L^1([0,T]; H^\infty)$ (from the equation). Hence $\Delta\Pi^n$ is indeed in $L^1([0,T]; B^{s-1}_{p,r})$, as claimed above.

In order to establish that $\nabla\Pi^n\in L^1([0,T]; L^p)$, we use the fact that, owing to $\operatorname{div}\partial_t u^n=0$, one may write

$$\Delta\Pi^n = \operatorname{div}\big(\rho^n f^n - \rho^n u^n\cdot\nabla u^n - \big(\rho^n-\bar\rho\big)\partial_t u^n\big). \tag{55}$$

Hence, it suffices to check that $\rho^n f^n$, $\rho^n u^n\cdot\nabla u^n$ and $(\rho^n-\bar\rho)\partial_t u^n$ are in $L^1([0,T]; L^p)$. For $\rho^n f^n$ this is obvious as, by embedding, $f^n\in L^1([0,T]; L^p)$ and $u^n\in\mathcal{C}_b([0,T]\times\mathbb{R}^N)$. By embedding, we also have $\nabla u^n\in\mathcal{C}([0,T]; L^2)$ and $u^n\in\mathcal{C}([0,T]; L^{p^*})$, hence $\rho^n u^n\cdot\nabla u^n$ is in $L^1([0,T]; L^p)$.

To deal with the last term in (55) the property that $(\rho^n-\bar\rho)\in\mathcal{C}([0,T]; L^{p^*})$ comes into play. Indeed, from the velocity equation, as the solution is in H^∞, one easily gathers that $\partial_t u^n$ belongs to $\mathcal{C}([0,T]; L^2)$. Hence Hölder's inequality ensures that $(\rho^n-\bar\rho)\partial_t u^n\in\mathcal{C}([0,T]; L^p)$.

To finish this step, one has to prove that u^n is in $\mathcal{C}([0,T_n^*);B^s_{p,r})$. In fact, from the product laws in Besov spaces and the properties of regularity that have been just established for the pressure and the density, we get

$$\partial_t u^n + u^n\cdot\nabla u^n = f^n - a^n\nabla\Pi^n \in L^1_{loc}\big([0,T_n^*);B^s_{p,r}\big).$$

As $u_0^n\in B^s_{p,r}$, Proposition 5 ensures that $u^n\in\mathcal{C}([0,T_n^*);B^s_{p,r})$.

Step 2. Uniform estimates. Let us remark that, by Sobolev embedding and owing to (49), (50), (51), one may find some index $\sigma>N/2+1$ such that $(Da_0^n)_{n\in\mathbb{N}}$, $(u_0^n)_{n\in\mathbb{N}}$ and $(f^n)_{n\in\mathbb{N}}$ are bounded in $H^{\sigma-1}$, H^σ and $\mathcal{C}([0,T_0];L^2)\cap L^1([0,T_0];H^\sigma)$, respectively. Taking advantage of Theorem 1 in the case $p=2$ and of the lower bound provided by (36) we thus deduce that there exists some time $T>0$ and some $M>0$ such that for all $n\in\mathbb{N}$, we have $T_n^*>T$ and

$$\|\nabla a^n\|_{L^\infty_T(H^{\sigma-1})}+\|u^n\|_{L^\infty_T(H^\sigma)}+\|\nabla\Pi^n\|_{L^1_T(H^\sigma)}\leqslant M. \tag{56}$$

Of course the energy equality (3) is satisfied on $[0,T]$ by any solution $(a^n,u^n,\nabla\Pi^n)$. Recall that in addition, according to the previous step of the proof, (53) is satisfied and

$$\nabla a^n\in\mathcal{C}\big([0,T];B^{s-1}_{p,r}\big),\qquad u^n\in\mathcal{C}\big([0,T];B^s_{p,r}\big)\quad\text{and}\quad\nabla\Pi^n\in L^1\big([0,T];B^s_{p,r}\big).$$

We claim that, up to a change of T, the norm of the solution may be bounded *independently of n* in the space E_T defined in Section 4.2. In all that follows, we denote by C_M a "constant" depending only on (s,p,r,N,a_*,a^*) and on M.

From Proposition 5, we have

$$\|\nabla a^n(t)\|_{B^s_{p,r}}\leqslant\|\nabla a_0^n\|_{B^s_{p,r}}e^{C\int_0^t\|\nabla u^n\|_{B^{s-1}_{p,r}}\,d\tau} \tag{57}$$

and, arguing as for proving inequality (28),

$$\|u^n(t)\|_{B^s_{p,r}}\leqslant e^{C\int_0^t\|\nabla u^n\|_{B^{s-1}_{p,r}}\,d\tau}\Bigg(\|u_0^n\|_{B^s_{p,r}} + \int_0^t e^{-C\int_0^\tau\|\nabla u^n\|_{B^{s-1}_{p,r}}\,d\tau'}\big(\|f^n\|_{B^s_{p,r}}+\big(a^*+\|\nabla a^n\|_{B^{s-1}_{p,r}}\big)\|\nabla\Pi^n\|_{B^s_{p,r}}\big)\,d\tau\Bigg). \tag{58}$$

In order to bound $\nabla\Pi^n$, we apply the first part of Proposition 7 to the following equation:

$$\operatorname{div}\big(a^n\nabla\Pi^n\big)=\operatorname{div}\big(f^n-u^n\cdot\nabla u^n\big).$$

Using that $B^{s-1}_{p,r}$ is an algebra and the relation $\operatorname{div}(u^n\cdot\nabla u^n)=\nabla u^n:\nabla u^n$, we end up with

$$a_*\|\nabla\Pi^n\|_{L^1_t(B^s_{p,r})}\leqslant C\Bigg(\|f^n\|_{L^1_t(B^s_{p,r})} + \int_0^t\|u^n\|^2_{B^s_{p,r}}\,d\tau + a_*\bigg(1+\frac{\|Da^n\|_{L^\infty_t(B^{s-1}_{p,r})}}{a_*}\bigg)^s\|\nabla\Pi^n\|_{L^1_t(L^p)}\Bigg). \tag{59}$$

In order to "close the estimate", we have to bound $\nabla\Pi^n$ in L^p. For that, we apply the standard L^p elliptic estimate stated in Proposition 6 to (55), and Hölder inequality so as to get

$$\|\nabla\Pi^n\|_{L^1_t(L^p)} \leqslant C\left(\rho^*\left(\|f^n\|_{L^1_t(L^p)} + \int_0^t \|u^n\|_{L^{p^*}}\|\nabla u^n\|_{L^2}\,d\tau\right) + \|\rho^n-\overline{\rho}\|_{L^\infty_t(L^{p^*})}\|\partial_t u^n\|_{L^1_t(L^2)}\right).$$

Note that, by Sobolev embedding, we have

$$\|u^n\|_{L^{p^*}} \leqslant C\|u^n\|_{H^\sigma}.$$

So finally, there exists some constant C_M such that

$$\|\nabla\Pi^n\|_{L^1_T(L^p)} \leqslant C_M.$$

Plugging this latter inequality in (59), we thus get

$$a_*\|\nabla\Pi^n\|_{L^1_t(B^s_{p,r})} \leqslant C\left(\|f^n\|_{L^1_t(B^s_{p,r})} + \int_0^t \|\nabla u^n\|^2_{B^{s-1}_{p,r}}\,d\tau + a_*C_M\left(1+\frac{\|Da^n\|_{L^\infty_t(B^{s-1}_{p,r})}}{a_*}\right)^s\right).$$

It is now easy to conclude this step: denoting

$$U^n(t) := \|u^n(t)\|_{B^s_{p,r}} \quad \text{and} \quad A^n(t) := a^* + \|Da^n(t)\|_{B^{s-1}_{p,r}},$$

and assuming that $\overline{T} \leqslant T$ has been chosen so that

$$C\int_0^{\overline{T}} \|\nabla u^n\|_{B^{s-1}_{p,r}}\,d\tau \leqslant \log 2, \tag{60}$$

the above inequalities and (49), (50), (51) imply that for all $t \in [0, \overline{T}]$ we have

$$A^n(t) \leqslant 2A_0 \quad \text{with } A_0 := a^* + \|Da_0\|_{B^{s-1}_{p,r}}$$

and

$$U^n(t) \leqslant 2\left(U_0(t) + C\rho^* A_0 \int_0^t \left(\|f\|_{B^s_{p,r}} + \left(U^n\right)^2 + C_M a_* A_0^s\right)d\tau\right).$$

So finally, there exists a nondecreasing function F depending only on the norm of the data and such that for all $t \in [0, \overline{T}]$, we have

$$U^n(t) \leqslant 2F(t) + C\rho^* A_0 \int_0^t \left(U^n(\tau)\right)^2 d\tau.$$

Therefore, if in addition

$$2CA_0 \int_0^{\bar{T}} U^n(\tau)\,d\tau \leqslant a_* \tag{61}$$

then we have $U^n \leqslant 4F$ on $[0, \bar{T}]$.

By arguing exactly as in the case $p \geqslant 2$, it is easy to see that condition (61) is satisfied if $\bar{T}$ is small enough (an explicit lower bound may be obtained in terms of the data). So finally, we have found a positive time T so that $(a^n, u^n, \nabla\Pi^n)_{n\in\mathbb{N}}$ is bounded in the space E_T.

Step 3. Convergence of the sequence. Let $\delta u^n := u^{n+1} - u^n$, $\delta\rho^n := \rho^{n+1} - \rho^n$ and $\delta\Pi^n = \Pi^{n+1} - \Pi^n$. Applying inequality (16) to the solutions $(\rho^n, u^n, \nabla\Pi^n)$ and $(\rho^{n+1}, u^{n+1}, \nabla\Pi^{n+1})$ and using the uniform bounds that have been established in the previous step, and (52) ensures that there exists some $M > 0$ such that for all $t \in [0, T]$ and $n \in \mathbb{N}$, we have

$$\|\delta\rho^n(t)\|_{L^2} + \|\delta u^n(t)\|_{L^2} \leqslant M\left(\|\delta\rho^n(0)\|_{L^2} + \|\delta u^n(0)\|_{L^2} + \int_0^t \|\delta f^n\|_{L^2}\,d\tau\right). \tag{62}$$

Now, from the definition of a_0^n and the mean value theorem, we get for large enough n,

$$\|\delta\rho^n(0)\|_{L^2} \leqslant C2^{-n}\|Da_0\|_{L^2}.$$

Similarly, we have

$$\|\delta u^n(0)\|_{L^2} + \|\delta f^n\|_{L^1_T(L^2)} \leqslant C2^{-n}\big(\|\delta u_0\|_{L^2} + \|\delta f\|_{L^1_T(L^2)}\big).$$

So inequality (62) entails that $(\rho^n - \rho^0)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T]; L^2)$ and that $(u^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T]; L^2)$. Then, using for instance (55), we see that $(\nabla\Pi^n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $\mathcal{C}([0,T]; L^2)$.

Finally, from the bounds in large norm that have been stated in the previous step, and the Fatou property for the Besov space, one may conclude that the limit $(a, u, \nabla\Pi)$ to $(a^n, u^n, \nabla\Pi^n)_{n\in\mathbb{N}}$ converges to some solution of (1) and has the desired properties of regularity. As similar arguments have been used for handling the case $p \geqslant 2$, the details are left to the reader.

Let us now establish Theorem 2 in the case $p < 2$. Let $(\rho, u, \nabla\Pi)$ be a solution with the properties described in Theorem 1. Note that Lemma 4 is also true if $p < 2$. So the only change lies in the proof of Lemma 5 which now uses the (new) lower bound for the lifespan that may be obtained from the computations of step 2, instead of (36). This gives the first part of Theorem 2. As in the case $p \geqslant 2$, the last part of the proof of the theorem is a mere consequence of the logarithmic interpolation inequality stated in [14].

4.5. Removing the assumptions on the low frequency of the data

We claim that in dimension $N \geqslant 3$, the supplementary assumption $(\rho_0 - \bar{\rho}) \in L^{p^*}$ is not needed if $p > N/(N-1)$.

In order to see that, one may repeat the proof of the theorem in the case $1 < p \leqslant 2$. As before, bounding $\nabla\Pi^n$ in $L^1_T(L^p)$ is the main difficulty. For that, one may decompose $\nabla\Pi^n$ into two terms $\nabla\Pi_1^n$ and $\nabla\Pi_2^n$ such that

$$\Delta\Pi_1^n = \operatorname{div}\big(\rho^n f^n - \rho^n u^n \cdot \nabla u^n\big) \quad \text{and} \quad \Delta\Pi_2^n = -\nabla\rho^n \cdot \partial_t u^n.$$

On the one hand, as before, one may write that

$$\|\nabla\Pi_1^n\|_{L^p} \leqslant C\rho^*\big(\|f^n\|_{L^p} + \|u^n\|_{L^{p^*}}\|\nabla u^n\|_{L^2}\big).$$

On the other hand, we have

$$\nabla\Pi_2^n = (-\Delta)^{-1}\nabla\big(\nabla\rho^n\cdot\partial_t u^n\big).$$

Recall that in dimension $N \geqslant 2$, the kernel of operator $(-\Delta)^{-1}\nabla$ behaves as $|x|^{1-N}$. Hence, according to the Hardy–Littlewood–Sobolev inequality, if $1/p + 1/N < 1$ then we have

$$\|\nabla\Pi_2^n\|_{L^p} \leqslant C\|\nabla\rho^n\cdot\partial_t u^n\|_{L^q} \quad \text{with } \frac{1}{q} = \frac{1}{p} + \frac{1}{N}.$$

As $(\nabla\rho^n)_{n\in\mathbb{N}}$ may be bounded in $\mathcal{C}([0,T];L^p)$ and, by embedding, $(\partial_t u^n)_{n\in\mathbb{N}}$ may be bounded in $L^1([0,T];L^N)$, *in terms of Sobolev norms only*, it is thus possible to bound $\nabla\Pi_2^n$ in $L^1([0,T];L^p)$ in terms of the initial data. The rest of the proof goes by the steps that we used before.

5. The proof of Theorem 3

For $T>0$, let us introduce the set F_T of functions $(a,u,\nabla\Pi)$ such that

$$\begin{aligned} &a\in\mathcal{C}_b\big([0,T]\times\mathbb{R}^N\big), && Da\in\mathcal{C}\big([0,T];B^{s-1}_{p,r}\big),\\ &u\in\mathcal{C}\big([0,T];B^s_{p,r}\big), && \nabla\Pi\in\mathcal{C}\big([0,T];L^2\big)\cap L^1\big([0,T];B^s_{p,r}\big).\end{aligned}$$

Uniqueness in Theorem 3 stems from Proposition 8. Indeed, we see that, as $p\leqslant 4$, any solution $(\rho,u,\nabla\Pi)$ in F_T satisfies $u\in\mathcal{C}([0,T];W^{1,4})$ (according to Proposition 2 and to the remark that follows) and $\nabla\Pi\in\mathcal{C}([0,T];L^2)$. Therefore, using the velocity equation and Hölder's inequality, we get

$$(\partial_t u - f) = -(u\cdot\nabla u + a\nabla\Pi) \in \mathcal{C}\big([0,T];L^2\big). \tag{63}$$

Note that, as u and $\nabla\rho$ are in $\mathcal{C}([0,T];L^4)$, we have

$$\partial_t\rho = -u\cdot\nabla\rho\in\mathcal{C}\big([0,T];L^2\big). \tag{64}$$

Now, consider two solutions $(\rho_1,u_1,\nabla\Pi_1)$ and $(\rho_2,u_2,\nabla\Pi_2)$ in F_T, corresponding to the same data. Then (63) implies that $\delta u := u_2 - u_1$ belongs to $\mathcal{C}^1([0,T];L^2)$ while (64) guarantees that $\delta\rho := \rho_2-\rho_1$ is in $\mathcal{C}^1([0,T];L^2)$. So Proposition 8 applies and yields uniqueness.

Let us now tackle the proof of the existence part of the theorem. We claim that if we restrict our attention to solutions which are F_T then the assumptions of Lemma 3 are fulfilled so that it suffices to solve system (18). Indeed, it is only a matter of checking whether $\mathcal{Q}u$ is in $\mathcal{C}([0,T];L^2)$. Applying $\mathcal{Q}$ to the velocity equation of (18), we get

$$\partial_t\mathcal{Q}u = \mathcal{Q}f - \mathcal{Q}(a\nabla\Pi) - \mathcal{Q}(u\cdot\nabla u).$$

From the assumptions on f, the definition of F_T and the fact that $\mathcal{Q}$ maps L^2 in L^2, we see that the first two terms in the right-hand side are in $\mathcal{C}([0,T];L^2)$. Concerning the last term, we just use the fact that, as pointed out above, $u\cdot\nabla u$ belongs to $\mathcal{C}([0,T];L^2)$ so $\mathcal{Q}(u\cdot\nabla u)$, too.

Let us now go to the proof of the existence of a local-in-time solution for (18) under the assumptions of Theorem 3. Compared to Theorem 1, the main change is that we do not expect to have $u \in \mathcal{C}([0,T];L^2)$ any longer (i.e. the energy may be infinite). However, as the pressure satisfies

$$-\operatorname{div}(a\nabla\Pi) = \operatorname{div}(u\cdot\nabla\mathcal{P}u) - \operatorname{div} f,$$

Lemma 2 will ensure that $\nabla\Pi \in \mathcal{C}([0,T];L^2)$ anyway *if* $u\cdot\nabla\mathcal{P}u$ *belongs to* $\mathcal{C}([0,T];L^2)$. In view of Proposition 2, Remark 2 and Hölder's inequality, this latter property is guaranteed by the fact that $u \in \mathcal{C}([0,T];B^s_{p,r})$ *for some* $p \leqslant 4$.

Once this has been noticed, one may use the same approximation scheme as in Theorem 1: we first set $(a^0,u^0,\nabla\Pi^0) :\equiv (a_0,u_0,0)$. Next, we assume that $(a^n,u^n,\nabla\Pi^n)$ has been constructed over $\mathbb{R}^+$, belongs to the space F_T for all $T>0$ and that there exists a positive time T^* such that (19) is satisfied for all $t\in[0,T^*]$ and, for suitable constants C_0 and C (one can take $C_0 = 2C^2$),

$$\|\nabla a^{n+1}(t)\|_{B^{s-1}_{p,r}} \leqslant 2\|\nabla a_0\|_{B^{s-1}_{p,r}} \quad \text{for all } t\in[0,T^*], \tag{65}$$

$$U^n(t) \leqslant 4U_0(t) + C_0\rho^*A_0\big(\|\operatorname{div} f\|_{L^1_t(B^{s-1}_{p,r})} + (\rho^*A_0)^\gamma\|\mathcal{Q}f\|_{L^1_t(L^2)}\big), \tag{66}$$

$$a_*\|\nabla\Pi^n\|_{L^1_t(B^s_{p,r})} \leqslant C\left(\|\operatorname{div} f\|_{L^1_t(B^{s-1}_{p,r})} + (\rho^*A_0)^\gamma\int_0^t\big(\big(U^n\big)^2 + \|\mathcal{Q}f\|_{L^2}\big)\,d\tau\right) \tag{67}$$

with $A_0 := a^* + \|Da_0\|_{B^{s-1}_{p,r}}$, $U_0(t) := \|u_0\|_{B^s_{p,r}} + \|f\|_{L^1_t(B^s_{p,r})}$ and $U^n(t) := \|u^n(t)\|_{B^s_{p,r}}$.

Arguing exactly as in the proof of Theorem 1, we see that if we define a^{n+1} as the solution to

$$\partial_t a^{n+1} + u^n\cdot\nabla a^{n+1} = 0, \qquad a^{n+1}_{|t=0} = a_0$$

then (19) is satisfied for all time, $\nabla a^{n+1} \in \mathcal{C}(\mathbb{R}^+;B^{s-1}_{p,r})$ and

$$\|\nabla a^{n+1}(t)\|_{B^{s-1}_{p,r}} \leqslant e^{C\int_0^t U^n(\tau)\,d\tau}\|\nabla a_0\|_{B^{s-1}_{p,r}}. \tag{68}$$

So if we assume that T^* has been chosen so that

$$C\int_0^{T^*} U^n(t)\,dt \leqslant \log 2 \tag{69}$$

then a^{n+1} satisfies (65). Next, we take u^{n+1} to be the unique solution in $\mathcal{C}(\mathbb{R}^+;B^s_{p,r})$ of the transport equation (27). As before, the right-hand side of (27) belongs to $\mathcal{C}(\mathbb{R}^+;B^s_{p,r})$ and one may use (28). So finally, the existence of $u^{n+1}\in\mathcal{C}(\mathbb{R}^+;B^s_{p,r})$ is ensured by Proposition 5, and we have

$$\|u^{n+1}(t)\|_{B^s_{p,r}} \leqslant e^{C\int_0^t U^n(\tau)}\Bigg(\|u_0\|_{B^s_{p,r}} + \int_0^t e^{-C\int_0^\tau U^n(\tau')\,d\tau'}\Big(\big(\|a^{n+1}\|_{L^\infty} + \|\nabla a^{n+1}\|_{B^{s-1}_{p,r}}\big)\|\nabla\Pi^n\|_{B^s_{p,r}} + \|f\|_{B^s_{p,r}}\Big)\,d\tau\Bigg).$$

Therefore, if we restrict our attention to those t that are in $[0,T^*]$ with T^* satisfying (69), and use inequality (68), we see that for all $t\in[0,T^*]$,

$$U^{n+1}(t)\leqslant 2U_0(t)+CA_0\int_0^t \|\nabla\Pi^n\|_{B^s_{p,r}}\,d\tau\quad\text{with } A_0:=a^*+\|\nabla a_0\|_{B^{s-1}_{p,r}}.$$

So if we assume that $C_0=2C^2$ and that T^* has been chosen so that

$$C^2\rho^*A_0\int_0^{T^*}U^n(t)\,dt\leqslant\frac12 \tag{70}$$

then taking advantage of inequality (67), we see that u^{n+1} satisfies (66) on $[0,T^*]$.

To finish with, in order to construct the approximate pressure Π^{n+1}, we solve the elliptic equation (33) for every positive time. Recall that we have $\operatorname{div}(u^{n+1}\cdot\nabla\mathcal{P}u^{n+1})\in B^{s-1}_{p,r}$ and that

$$\big\|\operatorname{div}\big(u^{n+1}\cdot\nabla\mathcal{P}u^{n+1}\big)\big\|_{B^{s-1}_{p,r}}\leqslant C\big(U^{n+1}\big)^2.$$

Next, given our assumptions on (s,p,r) we have $B^s_{p,r}\hookrightarrow W^{1,4}$. Therefore, since $\mathcal{P}$ maps L^4 in L^4, one may write

$$\begin{aligned}\big\|u^{n+1}\cdot\nabla\mathcal{P}u^{n+1}\big\|_{L^2}&\leqslant\big\|u^{n+1}\big\|_{L^4}\big\|\nabla\mathcal{P}u^{n+1}\big\|_{L^4}\\&\leqslant C\big\|u^{n+1}\big\|_{L^4}\big\|u^{n+1}\big\|_{W^{1,4}}\\&\leqslant C\big(U^{n+1}\big)^2.\end{aligned}$$

Therefore, the second part of Proposition 7 ensures that $\nabla\Pi^{n+1}$ is well defined in $\mathcal{C}(\mathbb{R}^+;L^2)\cap L^1_{loc}(\mathbb{R}^+;B^s_{p,r})$ and that

$$a_*\big\|\nabla\Pi^{n+1}\big\|_{L^1_t(B^s_{p,r})}\leqslant C\left(\|\operatorname{div}f\|_{L^1_t(B^{s-1}_{p,r})}+\big(1+\rho^*\big\|Da^{n+1}\big\|_{L^\infty_t(B^{s-1}_{p,r})}\big)^\gamma\int_0^t\Big(\big(U^{n+1}\big)^2+\|\mathcal{Q}f\|_{L^2}\Big)\,d\tau\right).$$

Taking advantage of inequality (65) at rank $n+1$, one can now conclude that $\nabla\Pi^{n+1}$ satisfies (67).

At this stage we have proved that if inequalities (65), (66) and (67) hold for $(a^n,u^n,\nabla\Pi^n)$ then they also hold for $(a^{n+1},u^{n+1},\nabla\Pi^{n+1})$ provided T^* satisfies inequality (70). One may easily check that this is indeed the case if we set

$$T^*:=\sup\Big\{t>0\big/\rho^*A_0t\big(U_0(t)+\rho^*A_0\|\operatorname{div}f\|_{L^1_t(B^{s-1}_{p,r})}+(\rho^*A_0)^{\gamma+1}\|\mathcal{Q}f\|_{L^1_t(L^2)}\big)\leqslant c\Big\} \tag{71}$$

for a small enough constant c depending only on s, p and N.

Once the bounds in F_{T^*} have been established, the last steps of the proof are almost identical to those of Theorem 1. Indeed, introducing

$$\widetilde{a}^n(t,x):=a^n(t,x)-a_0(x)\quad\text{and}\quad\widetilde{u}^n(t,x):=u^n(t,x)-u_0(x)-\int_0^t f(\tau,x)\,d\tau$$

and observing that $\delta u^n := u^{n+1} - u^n = \widetilde{u}^{n+1} - \widetilde{u}^n$, one can use exactly the same computations as before for bounding δa^n, δu^n and $\nabla\delta\Pi^n$. As a consequence $(\widetilde{a}^n, \widetilde{u}^n, \nabla\Pi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T^*]; L^2)$. Next, the bounds (65), (66) and (67) enable us to show that the limit is indeed in F_T and satisfies (18). The details are left to the reader.

Let us finally establish the continuation criterion. Note that Lemma 4 still applies in the context of infinite energy solutions. Hence, repeating the proof of Lemma 5 and using the logarithmic interpolation inequality of [14] yields the result. This completes the proof of Theorem 3.

6. The proof of Theorem 4

In this section, we aim at investigating the well-posedness issue of system (1) in Hölder spaces C^s (which coincide with the Besov spaces $B^s_{\infty,\infty}$ if s is not an integer), and, more generally, in Besov spaces of type $B^s_{\infty,r}$. Of particular interest is the case of the Besov space $B^1_{\infty,1}$ which is the largest one for which condition (C) holds.

The main difficulty is that the previous proofs where based on the elliptic estimate stated in Proposition 7 which fails in the limit case $p=\infty$. In this section, we shall see that the case of a small perturbation of a constant density state may be handled by a different approach, similar to that used in [20].

6.1. The proof of uniqueness

Note that in the case $p>4$ the solution provided by Theorem 4 need not satisfy $\nabla\Pi \in L^1([0,T]; L^2)$. Hence $\partial_t u$ need not be in $L^1([0,T]; L^2)$ and the assumptions of Proposition 8 are not satisfied.

So, in order to prove uniqueness, we shall prove stability estimates in L^p rather than in L^2. These estimates will be also needed in the last step of the proof of the existence part of Theorem 4.

Consider two solutions $(\rho_1, u_1, \nabla\Pi_1)$ and $(\rho_2, u_2, \nabla\Pi_2)$ of (1). Let $a_1 := 1/\rho_1$ and $a_2 := 1/\rho_2$. As usual, denote $\delta a := a_2 - a_1$, $\delta u := u_2 - u_1$, $\nabla\delta\Pi := \nabla\Pi_2 - \nabla\Pi_1$ and $\delta f := f_2 - f_1$. First, as $\operatorname{div} u_2 = 0$ and

$$\partial_t \delta a + u_2 \cdot \nabla\delta a = -\delta u \cdot \nabla a_1,$$

one may write

$$\|\delta a(t)\|_{L^p} \leqslant \|\delta a(0)\|_{L^p} + \int_0^t \|\nabla a_1\|_{L^\infty} \|\delta u\|_{L^p}\, d\tau. \tag{72}$$

Next, as

$$\partial_t \delta u + u_2 \cdot \nabla\delta u = \delta f - \delta u \cdot \nabla u_1 - \delta a \nabla\Pi_1 - a_2 \nabla\delta\Pi,$$

we have

$$\begin{aligned}\|\delta u(t)\|_{L^p} \leqslant \|\delta u(0)\|_{L^p} + \int_0^t \big(&\|\delta f\|_{L^p} + \|\nabla u_1\|_{L^\infty}\|\delta u\|_{L^p} + \|\nabla\Pi_1\|_{L^\infty}\|\delta a\|_{L^p} \\ &+ a^* \|\nabla\delta\Pi\|_{L^p}\big)\, d\tau.\end{aligned} \tag{73}$$

Finally, we notice that $\nabla\delta\Pi$ satisfies the elliptic equation

$$\operatorname{div}(a_2\nabla\delta\Pi) = \operatorname{div}\delta f - \operatorname{div}(\delta a \nabla\Pi_1) - \operatorname{div}(\delta u \cdot \nabla u_1) - \operatorname{div}(u_2 \cdot \nabla\delta u). \tag{74}$$

The key point here is that, owing to $\operatorname{div} u_2 = \operatorname{div} \delta u = 0$, we have

$$\operatorname{div}(u_2 \cdot \nabla \delta u) = \operatorname{div}(\delta u \cdot \nabla u_2).$$

Hence equality (74) rewrites

$$a_* \Delta \delta \Pi = \operatorname{div} \delta f + \operatorname{div}\big((a_* - a_2)\nabla \delta \Pi\big) - \operatorname{div}(\delta a \nabla \Pi_1) - \operatorname{div}\big(\delta u \cdot \nabla(u_1 + u_2)\big)$$

so that the L^p elliptic estimate stated in Proposition 6 implies that

$$a_* \|\nabla \delta \Pi\|_{L^p} \leqslant C\big(\|\delta f\|_{L^p} + (a^* - a_*)\|\nabla \delta \Pi\|_{L^p} + \|\nabla \Pi_1\|_{L^\infty}\|\delta a\|_{L^p} + \big\|\nabla(u_1 + u_2)\big\|_{L^\infty}\|\delta u\|_{L^p}\big).$$

If the quantity $a^*/a_* - 1$ is small enough then we thus have, up to a change of C,

$$a_* \|\nabla \delta \Pi\|_{L^p} \leqslant C\big(\|\delta f\|_{L^p} + \|\nabla \Pi_1\|_{L^\infty}\|\delta a\|_{L^p} + \big\|\nabla(u_1 + u_2)\big\|_{L^\infty}\|\delta u\|_{L^p}\big).$$

Plugging this latter inequality in (73) then adding up inequality (72), we get

$$\big\|(\delta a, \delta u)(t)\big\|_{L^p} \leqslant \big\|(\delta a, \delta u)(0)\big\|_{L^p} + C\int_0^t \Big(\|\delta f\|_{L^p} + \big(\|\nabla u_1\|_{L^\infty} + \|\nabla u_2\|_{L^\infty} + \|\nabla a_1\|_{L^\infty} + \|\nabla \Pi_1\|_{L^\infty}\big)\big\|(\delta a, \delta u)\big\|_{L^p}\Big)\, d\tau.$$

Applying Gronwall's lemma yields the following result which obviously implies the uniqueness part of Theorem 4:

Proposition 9. *Let $(\rho_1, u_1, \nabla \Pi_1)$ and $(\rho_2, u_2, \nabla \Pi_2)$ be two solutions of* (1) *on* $[0, T] \times \mathbb{R}^N$ *such that for some $p \in (1, \infty)$,*

- *$\delta a := a_2 - a_1$ and $\delta u := u_2 - u_1$ are in $\mathcal{C}([0, T]; L^p)$,*
- *$\nabla \delta \Pi := \nabla \Pi_2 - \nabla \Pi_1$ is in $L^1([0, T]; L^p)$,*

and for some positive real numbers a_* and a^* such that $a_* \leqslant a^*$,

$$a_* \leqslant a_1, a_2 \leqslant a^*.$$

There exists a constant c depending only on p and on N such that if

$$a^* - a_* \leqslant c a_*$$

and if for all $t \in [0, T]$,

$$V(t) := \int_0^t \big(\|\nabla u_1\|_{L^\infty} + \|\nabla u_2\|_{L^\infty} + \|\nabla a_1\|_{L^\infty} + \|\nabla \Pi_1\|_{L^\infty}\big)\, d\tau < \infty$$

then the following inequality is satisfied:

$$\big\|(\delta a, \delta u)(t)\big\|_{L^p} \leqslant e^{CV(t)}\left(\big\|(\delta a, \delta u)(0)\big\|_{L^p} + C\int_0^t e^{-CV(\tau)}\big\|\delta f(\tau)\big\|_{L^p}\, d\tau\right).$$

6.2. A priori estimates

Here we assume that $(\rho, u, \nabla\Pi)$ is a solution to (1) on the time interval $[0,T]$ with the $B^s_{\infty,r}$ regularity. We want to show that if T has been chosen small enough then the size of the solution at time $t \leqslant T$ is of the same order as the size of the data.

First, it is clear that we have

$$a_* \leqslant a \leqslant a^* \quad \text{on } [0,T]\times\mathbb{R}^N.$$

Moreover, one may write thanks to Proposition 5:

$$\big\|\nabla a(t)\big\|_{B^{s-1}_{\infty,r}} \leqslant e^{C\int_0^t \|u\|_{B^s_{\infty,r}}\,d\tau}\|\nabla a_0\|_{B^{s-1}_{\infty,r}}, \tag{75}$$

and for the velocity, we have, as in the case $p<\infty$,

$$\begin{aligned}\big\|u(t)\big\|_{B^s_{\infty,r}} \leqslant e^{\|C\int_0^t \|u\|_{B^s_{\infty,r}}\,d\tau}\bigg(\|u_0\|_{B^s_{\infty,r}}\\ + \int_0^t e^{-C\int_0^\tau \|u\|_{B^s_{\infty,r}}\,d\tau'}\big(\|f\|_{B^s_{\infty,r}} + \|a\|_{B^s_{\infty,r}}\|\nabla\Pi\|_{B^s_{\infty,r}}\big)\,d\tau\bigg).\end{aligned} \tag{76}$$

Note that applying standard L^p estimates for the transport equation yields

$$\big\|u(t)\big\|_{L^p} \leqslant \|u_0\|_{L^p} + \int_0^t \|f\|_{L^p}\,d\tau + a^*\int_0^t \|\nabla\Pi\|_{L^p}\,d\tau. \tag{77}$$

As Propositions 6 and 7 fail in the limit case $p=\infty$, in order to bound the pressure, we have to resort to other arguments. Now, dividing the velocity equation of (1) by ρ and applying div, we get

$$\bar a\Delta\Pi = \operatorname{div} f - \operatorname{div}(u\cdot\nabla u) + \operatorname{div}\big((\bar a - a)\nabla\Pi\big) \quad \text{with } \bar a := 1/\bar\rho \tag{78}$$

and, by virtue of the Bernstein inequality, we have

$$\begin{aligned}\|\nabla\Pi\|_{B^s_{\infty,r}} &\leqslant \|\Delta_{-1}\nabla\Pi\|_{B^s_{\infty,r}} + \big\|(\mathrm{Id}-\Delta_{-1})\nabla\Pi\big\|_{B^s_{\infty,r}}\\ &\leqslant C\|\nabla\Pi\|_{L^p} + \big\|(\mathrm{Id}-\Delta_{-1})\nabla\Pi\big\|_{B^s_{\infty,r}}.\end{aligned}$$

On the one hand, in order to bound the L^p norm of $\nabla\Pi$, we simply apply the standard L^p elliptic estimate (see Proposition 6) to (78). We get

$$\bar a\|\nabla\Pi\|_{L^p} \leqslant C\big(\|\mathcal{Q}f\|_{L^p} + \|u\|_{L^p}\|\nabla u\|_{L^\infty} + \|\bar a - a\|_{L^\infty}\|\nabla\Pi\|_{L^p}\big).$$

Hence, if $a^*/\bar a - 1$ is small enough then

$$\bar a\|\nabla\Pi\|_{L^p} \leqslant C\big(\|\mathcal{Q}f\|_{L^p} + \|u\|_{L^p}\|\nabla u\|_{L^\infty}\big). \tag{79}$$

On the other hand, for bounding the high frequency part of the pressure, one can use the fact that operator $\nabla(-\Delta)^{-1}(\mathrm{Id}-\Delta_{-1})$ is homogeneous of degree -1 away from a ball centered at the origin, hence maps $B^{s-1}_{p,r}$ in $B^s_{p,r}$ (see e.g. [1, Chap. 2]). Therefore we have

$$\bar a\big\|(\mathrm{Id}-\Delta_{-1})\nabla\Pi\big\|_{B^s_{\infty,r}}\leqslant C\bar a\|\Delta\Pi\|_{B^{s-1}_{\infty,r}},$$
$$\leqslant C\big(\|\operatorname{div} f\|_{B^{s-1}_{\infty,r}}+\big\|\operatorname{div}(u\cdot\nabla u)\big\|_{B^{s-1}_{\infty,r}}+\big\|\operatorname{div}\big((\bar a-a)\nabla\Pi\big)\big\|_{B^{s-1}_{\infty,r}}\big).$$

In order to bound the second term, one may combine the Bony decomposition and the fact that $\operatorname{div} u=0$. This gives

$$\operatorname{div}(u\cdot\nabla u)=\sum_{i,j}\big(2T_{\partial_i u^j}\partial_j u^i+\partial_i R\big(u^j,\partial_j u^i\big)\big).$$

Thus applying Proposition 3, we may write

$$\big\|\operatorname{div}(u\cdot\nabla u)\big\|_{B^{s-1}_{\infty,r}}\leqslant C\|\nabla u\|_{L^\infty}\|u\|_{B^s_{\infty,r}}.$$

Finally, as $B^s_{\infty,r}$ is a Banach algebra, we have

$$\big\|\operatorname{div}\big((\bar a-a)\nabla\Pi\big)\big\|_{B^{s-1}_{\infty,r}}\leqslant C\|a-\bar a\|_{B^s_{\infty,r}}\|\nabla\Pi\|_{B^s_{\infty,r}}.$$

Putting this together with (79), one may conclude that there exists a constant c such that if

$$\|a-\bar a\|_{L^\infty_T(B^s_{\infty,r})}\leqslant c\bar a \tag{80}$$

then

$$\bar a\big(\|\nabla\Pi\|_{L^p}+\|\nabla\Pi\|_{B^s_{\infty,r}}\big)\leqslant C\big(\|\mathcal{Q}f\|_{L^p}+\|\operatorname{div} f\|_{B^{s-1}_{p,r}}+\|u\|_{L^p\cap B^s_{\infty,r}}\|\nabla u\|_{L^\infty}\big). \tag{81}$$

Let us assume that T has been chosen so that

$$C\int_0^T\|\nabla u\|_{B^{s-1}_{\infty,r}}\leqslant\log 2 \tag{82}$$

and that the initial density is such that

$$\|a_0-\bar a\|_{B^{s-1}_{\infty,r}}\leqslant\frac{c}{2}\bar a.$$

Then (80) is fulfilled and, combining inequalities (76), (77) and (81), we get

$$U(t)\leqslant 2U_0(t)+C\bar\rho\|a_0\|_{B^s_{\infty,r}}\int_0^t\big(\|\mathcal{Q}f\|_{L^p}+\|\operatorname{div} f\|_{B^{s-1}_{p,r}}+U^2\big)\,d\tau$$

with

$$U(t):=\big\|u(t)\big\|_{L^p\cap B^s_{\infty,r}}\quad\text{and}\quad U_0(t):=\|u_0\|_{L^p\cap B^s_{\infty,r}}+\int_0^t\|f\|_{L^p\cap B^s_{\infty,r}}\,d\tau.$$

It is now easy to find a time $T > 0$ depending only on the data and such that both condition (82) and

$$U(t) \leqslant 4U_0(t) \quad \text{for all } t \in [0, T]$$

are satisfied.

6.3. The proof of existence

This is mainly a matter of making the above estimates rigorous. We have to be a bit careful though since the data which are considered here do not enter in the framework of Theorems 1 and 3.

As a first step, we construct a sequence of smooth solutions. In order to enter in the Sobolev spaces framework, one may proceed as follows.

For the density, one may consider $\rho_0^n := \overline{\rho} + S_n(\phi(n^{-1}\cdot)(\rho_0 - \overline{\rho}))$ where ϕ is a smooth compactly supported cut-off function with value 1 on the unit ball of $\mathbb{R}^N$. Obviously, $\rho_0^n - \overline{\rho}$ is in H^∞ and converges weakly to $\rho_0 - \overline{\rho}$ when n goes to infinity. In addition, by using the fact that ϕ is smooth and that $B^s_{\infty,r}$ is an algebra, one may establish that there exists some constant C such that for all $n \in \mathbb{N}$,

$$\left\|\rho_0^n - \overline{\rho}\right\|_{B^s_{\infty,r}} \leqslant C\|\rho_0 - \overline{\rho}\|_{B^s_{\infty,r}}.$$

Similarly, for the velocity, one may set $u_0^n := S_n(\phi(n^{-1}\cdot)u_0)$ and for the source term, $f^n := \alpha_n \star_t (S_n(\phi(n^{-1}\cdot)f))$ where the convolution is taken with respect to the time variable only and $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of mollifiers on $\mathbb{R}$.

Applying Theorem 1 thus provides a sequence of continuous-in-time solutions with values in H^∞, defined on a fixed time interval. Then applying the above a priori estimates, it is easy to find a time T independent of n for which the sequence $(\rho^n, u^n, \nabla\Pi^n)_{n\in\mathbb{N}}$ is bounded in the desired space.

For proving convergence, one may take advantage of the stability estimates in L^p. The proof is similar to that of Theorem 1 in the case $1 < p \leqslant 2$ and is thus omitted.

6.4. A continuation criterion

This paragraph is dedicated to the proof of the following continuation criterion:

Proposition 10. *Assume that $s > 1$ (or that $s \geqslant 1$ if $r = 1$). Consider a solution $(\rho, u, \nabla\Pi)$ to (1) on $[0, T[\, \times \mathbb{R}^N$ such that for some $p \in (1, \infty)$ we have*

- $\rho \in \mathcal{C}([0, T); B^s_{\infty,r})$,
- $u \in \mathcal{C}([0, T); B^s_{\infty,r} \cap L^p)$,
- $\nabla\Pi \in L^1([0, T); B^s_{\infty,r} \cap L^p)$.

There exists a constant c depending only on N and s such that if for some $\overline{\rho} > 0$ we have

$$\sup_{0\leqslant t<T} \left\|\rho(t) - \overline{\rho}\right\|_{B^s_{\infty,r}} \leqslant c\overline{\rho} \quad \text{and} \quad \int_0^T \|\nabla u\|_{L^\infty}\, dt < \infty$$

then $(\rho, u, \nabla\Pi)$ may be continued beyond T into a $B^s_{\infty,r}$ solution of (1).

Proof. Applying the last part of Proposition 5 and product estimates to the velocity equation of (1) yields for all $t \in [0, T)$,

$$\left\|u(t)\right\|_{B^s_{\infty,r}} \leqslant e^{C\int_0^t \|\nabla u\|_{L^\infty}\, d\tau}\left(\|u_0\|_{B^s_{\infty,r}} + \int_0^t e^{-C\int_0^\tau \|\nabla u\|_{L^\infty}\, d\tau'}\left(\|f\|_{B^s_{\infty,r}} + \|a\|_{B^s_{\infty,r}}\|\nabla\Pi\|_{B^s_{\infty,r}}\right) d\tau\right).$$

Let us bound the pressure term according to inequality (81). Combining with (77) and (79), we eventually get

$$\big\|u(t)\big\|_{L^p\cap B^s_{\infty,r}} \leqslant e^{C\int_0^t \|\nabla u\|_{L^\infty}\,d\tau}\Bigg(\|u_0\|_{B^s_{\infty,r}} + \bar\rho\int_0^t e^{-C\int_0^\tau \|\nabla u\|_{L^\infty}\,d\tau'}\|a\|_{B^s_{\infty,r}}\big(\|f\|_{L^p\cap B^s_{\infty,r}} + \|u\|_{L^p\cap B^s_{\infty,r}}\|\nabla u\|_{L^\infty}\big)\,d\tau\Bigg).$$

So applying Gronwall's lemma ensures that u belongs to $L^\infty([0,T); L^p\cap B^s_{\infty,r})$. From this point, completing the proof is similar as for the previous continuation criteria. □

Remark 4. As in the $B^s_{p,r}$ framework, an improved continuation criterion involving $\|\nabla u\|_{\dot B^0_{\infty,\infty}}$ instead of $\|\nabla u\|_{L^\infty}$ may be proved for the $B^s_{\infty,r}$ regularity, if $s>1$. The details are left to the reader.

Acknowledgments

After completing this work, the author has been informed of the preprint [10] where results similar to ours are proved in the periodic case.

Appendix A. Commutator estimates

Here we prove two inequalities that have been used for bounding the pressure. The first result reads:

Lemma 6. *Let (s,p,r) satisfy condition (C). Let ς be in $(-1, s-1]$. There exists a constant C depending only on s, p, r, ς and N such that for all $k\in\{1,\dots,N\}$, we have*

$$\big\|\partial_k[a,\Delta_q]w\big\|_{L^p} \leqslant Cc_q 2^{-q\varsigma}\|\nabla a\|_{B^{s-1}_{p,r}}\|w\|_{B^\varsigma_{p,r}} \quad \textit{for all } q\geqslant -1,$$

with $\|(c_q)_{q\geqslant -1}\|_{\ell^r}=1$.

Proof. We follow the proof of Lemma 8.8 in [9]. Let $\widetilde a := a-\Delta_{-1}a$. Taking advantage of the Bony decomposition (6), we rewrite the commutator as[6]

$$\partial_k\big([a,\Delta_q]w\big) = \underbrace{\partial_k\big([T_{\widetilde a},\Delta_q]w\big)}_{R^1_q} + \underbrace{\partial_k T'_{\Delta_q w}\widetilde a}_{R^2_q} - \underbrace{\partial_k\Delta_q T'_w\widetilde a}_{R^3_q} + \underbrace{\partial_k[\Delta_{-1}a,\Delta_q]w}_{R^4_q}. \tag{83}$$

From the localization properties of the Littlewood–Paley decomposition, we gather that

$$R^1_q = \sum_{|q'-q|\leqslant 4}\partial_k\big([S_{q'-1}\widetilde a,\Delta_q]\Delta_{q'}w\big).$$

Note that R^1_q is spectrally supported in an annulus of size 2^q. Hence, combining Bernstein's inequality and Lemma 2.97 in [1], we get

$$\big\|R^1_q\big\|_{L^p} \leqslant C\sum_{|q'-q|\leqslant 4}\|\nabla S_{q'-1}\widetilde a\|_{L^\infty}\|\Delta_{q'}w\|_{L^p},$$

[6] Recall the notation $T'_u v := T_u v + R(u,v)$.

whence for some sequence $(c_q)_{q\geqslant -1}$ in the unit sphere of ℓ^r,

$$\left\|R_q^1\right\|_{L^p} \leqslant C c_q 2^{-q\varsigma} \|\nabla a\|_{L^\infty} \|w\|_{B^\varsigma_{p,r}}. \tag{84}$$

To deal with R_q^2, we use the fact that, owing to the localization properties of the Littlewood–Paley decomposition, we have

$$R_q^2 = \sum_{q'\geqslant q-2} \partial_k\left(S_{q'+2}\Delta_q w \Delta_{q'}\widetilde{a}\right).$$

Hence, using the Bernstein and Hölder inequalities and the fact that $\widetilde{a}$ has no low frequencies,

$$\begin{aligned}
\left\|R_q^2\right\|_{L^p} &\leqslant C \sum_{q'\geqslant q-2} \|S_{q'+2}\Delta_q w\|_{L^\infty} \|\Delta_{q'}\nabla\widetilde{a}\|_{L^p}, \\
&\leqslant C 2^{-q\varsigma} 2^{q(\frac{N}{p}+1-s)} \sum_{q'\geqslant q-2} 2^{(q-q')(s-1)} \left(2^{q(\varsigma-\frac{N}{p})}\|\Delta_q w\|_{L^\infty}\right)\left(2^{q'(s-1)}\|\Delta_{q'}\nabla\widetilde{a}\|_{L^p}\right).
\end{aligned}$$

Therefore, by virtue of convolution inequalities for series and because $N/p+1-s\leqslant 0$,

$$\left\|R_q^2\right\|_{L^p} \leqslant C c_q 2^{-q\varsigma} \|\nabla a\|_{B^{s-1}_{p,r}} \|w\|_{B^{\varsigma-\frac{N}{p}}_{\infty,r}}. \tag{85}$$

Next, Proposition 3 ensures that, under the assumptions of the lemma, the paraproduct and the remainder map $B^\varsigma_{p,r}\times B^s_{p,r}$ in $B^\varsigma_{p,r}$. As moreover we have

$$\|\widetilde{a}\|_{B^s_{p,r}} \leqslant C\|\nabla a\|_{B^{s-1}_{p,r}}, \tag{86}$$

one may conclude that

$$\left\|R_q^3\right\|_{L^p} \leqslant C c_q 2^{-q\varsigma} \|\nabla a\|_{B^{s-1}_{p,r}} \|w\|_{B^\varsigma_{p,r}}. \tag{87}$$

Finally, as the last term R_q^4 is spectrally localized in a ball of size 2^q, Bernstein's inequality ensures that

$$\left\|R_q^4\right\|_{L^p} \leqslant C 2^q \left\|[\Delta_{-1}a, \Delta_q]w\right\|_{L^p}.$$

Then, resorting again to Lemma 2.97 in [1], we get

$$\left\|R_q^4\right\|_{L^p} \leqslant C c_q 2^{-q\varsigma} \|\nabla a\|_{L^\infty} \|w\|_{B^\varsigma_{p,r}}. \tag{88}$$

Putting inequalities (84), (85), (87) and (88) together and using Proposition 2 completes the proof of the lemma. □

Lemma 7. *Let $\varsigma>0$ and $1\leqslant p,r\leqslant\infty$. There exists a constant C such that*

$$\left\|\partial_k[a,\Delta_q]w\right\|_{L^p} \leqslant C c_q 2^{-q\varsigma}\left(\|\nabla a\|_{L^\infty}\|w\|_{B^\varsigma_{p,r}} + \|w\|_{L^\infty}\|\nabla a\|_{B^\varsigma_{p,r}}\right) \quad \textit{for all } q\geqslant -1,$$

with $\|(c_q)_{q\geqslant -1}\|_{\ell^r}=1$.

Proof. We use again decomposition (83). We have already proved in (84) and (88) that R_q^1 and R_q^4 satisfy the desired inequality. Concerning R_q^2, recall that

$$\left\| R_q^2 \right\|_{L^p} \leqslant C \sum_{q' \geqslant q-2} \| S_{q'-1} \Delta_q w \|_{L^\infty} \| \Delta_{q'} \nabla a \|_{L^p},$$

whence

$$\left\| R_q^2 \right\|_{L^p} \leqslant C 2^{-q\varsigma} \sum_{q' \geqslant q-2} 2^{(q-q')\varsigma} \| w \|_{L^\infty} 2^{q'\varsigma} \| \Delta_{q'} \nabla \widetilde{a} \|_{L^p}.$$

As $\varsigma > 0$, convolution inequalities for series yield the desired inequality for R_q^2.

According to Proposition 3, we have

$$\left\| T'_w \widetilde{a} \right\|_{B^{\varsigma+1}_{p,r}} \leqslant C \| w \|_{L^\infty} \| \widetilde{a} \|_{B^{\varsigma+1}_{p,r}}.$$

Hence, as $\|\widetilde{a}\|_{B^{\varsigma+1}_{p,r}} \leqslant C \|\nabla a\|_{B^{\varsigma}_{p,r}}$, the term R_q^3 satisfies the required inequality. □

Appendix B. A Bernstein-type inequality

Lemma 8. *Let $1 < p < \infty$ and $u \in L^p$ such that $\operatorname{Supp} \widehat{u} \subset \{\xi \in \mathbb{R}^N / R_1 \leqslant |\xi| \leqslant R_2\}$ for some real numbers R_1 and R_2 such that $0 < R_1 < R_2$. Let a be a bounded measurable function over $\mathbb{R}^N$ such that $a \geqslant a_* > 0$ a.e. There exists a constant c depending only on N and R_2/R_1, and such that*

$$c a_* \left(\frac{p-1}{p^2} \right) R_1^2 \int |u|^p \, dx \leqslant (p-1) \int a |\nabla u|^2 |u|^{p-2} \, dx = - \int \operatorname{div}(a \nabla u) |u|^{p-2} u \, dx. \tag{89}$$

Proof. The case $a \equiv 1$ has been treated in [8] and readily entails the left inequality in (89) for one may write, owing to the case $a \equiv 1$,

$$c a_* R_1^2 \int |u|^p \, dx \leqslant p^2 \int a_* |\nabla u|^2 |u|^{p-2} \, dx \leqslant p^2 \int a |\nabla u|^2 |u|^{p-2} \, dx.$$

Let us now justify the right equality in (89). In the case $p \geqslant 2$, it stems from a straightforward integration by parts.

Let us focus on the case $1 < p < 2$ which is more involved. Smoothing out a if needed, one may assume with no loss of generality that a is in $C^{0,1}$. Let $T_\varepsilon(x) = \sqrt{x^2 + \varepsilon^2}$ for $x \in \mathbb{R}$ and $\varepsilon > 0$. We have

$$- \int_{\mathbb{R}^N} \operatorname{div}(a \nabla u) \big(T_\varepsilon(u)\big)^{p-1} T'_\varepsilon(u) \, dx = (p-1) \int_{\mathbb{R}^N} a |\nabla u|^2 \big|T'_\varepsilon(u)\big|^2 \big(T_\varepsilon(u)\big)^{p-2} \, dx + \int_{\mathbb{R}^N} a |\nabla u|^2 T''_\varepsilon(u) \big(T_\varepsilon(u)\big)^{p-1} \, dx. \tag{90}$$

In view of the monotonous convergence theorem,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} a |\nabla u|^2 \big|T'_\varepsilon(u)\big|^2 \big(T_\varepsilon(u)\big)^{p-2} \, dx = \int_{\mathbb{R}^N} a |\nabla u|^2 |u|^{p-2} \, dx \in \overline{\mathbb{R}}^+.$$

Next, we notice that

$$\left|\operatorname{div}(a\nabla u)\big(T_\varepsilon(u)\big)^{p-1}T'_\varepsilon(u)\right| \leqslant |u|^{p-1}\left|\operatorname{div}(a\nabla u)\right|. \tag{91}$$

Now, as $a \in C^{0,1}$ and u is a smooth function with all derivatives in L^p (owing to the spectral localization), one may write

$$\operatorname{div}(a\nabla u) = a\Delta u + \nabla a \cdot \nabla u,$$

hence $\operatorname{div}(a\nabla u)$ is in L^p and the right-hand side of (91) is an integrable function. So finally Lebesgue's dominated convergence theorem entails that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \operatorname{div}(a\nabla u)\big(T_\varepsilon(u)\big)^{p-1}T'_\varepsilon(u)\,dx = \int_{\mathbb{R}^N} u|u|^{p-2}\operatorname{div}(a\nabla u)\,dx.$$

Therefore, as the last term of (90) is nonnegative,

$$(p-1)\int_{\mathbb{R}^N} a|\nabla u|^2\,|u|^{p-2}\,dx \leqslant -\int_{\mathbb{R}^N} u|u|^{p-2}\operatorname{div}(a\nabla u)\,dx < \infty. \tag{92}$$

In fact, equality does hold. Indeed, whenever $x \neq 0$, the term $T''_\varepsilon(x)T_\varepsilon(x)^{p-1}$ tends to 0 when ε goes to 0 and

$$T''_\varepsilon(x)T_\varepsilon(x)^{p-1} = |x|^{p-2}\frac{(\varepsilon/x)^2}{(1+(\varepsilon/x)^2)^{2-\frac{p}{2}}} \leqslant |x|^{p-2}.$$

Therefore, as, according to (92), the function $|\nabla u|^2\,|u|^{p-2}$ is integrable over $\mathbb{R}^N$, we get

$$\lim_{\varepsilon\to 0}\int_{u\neq 0} a|\nabla u|^2 T''_\varepsilon(u)\big(T_\varepsilon(u)\big)^{p-1}\,dx = 0.$$

On the other hand, as u is real analytic,

$$\int_{u=0} a|\nabla u|^2 T''_\varepsilon(u)\big(T_\varepsilon(u)\big)^{p-1}\,dx = \varepsilon^{p-2}\int_{u=0} a|\nabla u|^2 = 0.$$

This completes the proof of the lemma. □

References

[1] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Springer, in press.
[2] J. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, Comm. Math. Phys. 94 (1) (1984) 61–66.
[3] H. Beirão da Veiga, A. Valli, On the Euler equations for nonhomogeneous fluids. I, Rend. Sem. Mat. Univ. Padova 63 (1980) 151–168.
[4] H. Beirão da Veiga, A. Valli, On the Euler equations for nonhomogeneous fluids. II, J. Math. Anal. Appl. 73 (2) (1980) 338–350.
[5] H. Beirão da Veiga, A. Valli, Existence of C^∞ solutions of the Euler equations for nonhomogeneous fluids, Comm. Partial Differential Equations 5 (2) (1980) 95–107.
[6] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. Sci. École Norm. Sup. 14 (4) (1981) 209–246.

[7] R. Danchin, A few remarks on the Camassa–Holm equation, Differential Integral Equations 14 (2001) 953–988.
[8] R. Danchin, Local theory in critical spaces for compressible viscous and heat-conductive gases, Comm. Partial Differential Equations 26 (2001) 1183–1233, Erratum, Comm. Partial Differential Equations 27 (2002) 2531–2532.
[9] R. Danchin, The inviscid limit for density dependent incompressible fluids, Ann. Fac. Sci. Toulouse 15 (2006) 637–688.
[10] J. Fan, Z. Xin, Y. Zhou, On the density dependent incompressible Euler equations in the critical Besov spaces, preprint, 2009.
[11] L. Grafakos, Classical and Modern Fourier Analysis, Prentice Hall, 2006.
[12] S. Itoh, Cauchy problem for the Euler equations of a nonhomogeneous ideal incompressible fluid. II, J. Korean Math. Soc. 32 (1) (1995) 1, 41–50.
[13] S. Itoh, A. Tani, Solvability of nonstationary problems for nonhomogeneous incompressible fluids and the convergence with vanishing viscosity, Tokyo J. Math. 22 (1) (1999) 17–42.
[14] H. Kozono, T. Ogawa, Y. Taniuchi, The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations, Math. Z. 242 (2002) 251–278.
[15] J. Marsden, Well-posedness of the equations of a non-homogeneous perfect fluid, Comm. Partial Differential Equations 1 (3) (1976) 215–230.
[16] N. Meyers, An L^p-estimate for the gradient of solutions of second order elliptic divergence equations, Ann. Sc. Norm. Super. Pisa 17 (1963) 189–206.
[17] T. Runst, W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, Nonlinear Anal. Appl. 3., Walter de Gruyter & Co., Berlin, 1996.
[18] A. Valli, W. Zajączkowski, About the motion of nonhomogeneous ideal incompressible fluids, Nonlinear Anal. 12 (1) (1988) 43–50.
[19] Y. Zhou, Local well-posedness for the incompressible Euler equations in the critical Besov spaces, Ann. Inst. Fourier 54 (3) (2004) 773–786.
[20] Y. Zhou, Local well-posedness for the density dependent incompressible Euler equations, preprint, 2003.
[21] Y. Zhou, Local well-posedness and regularity criterion for the density dependent incompressible Euler equations, preprint, 2003.