A NOTE ON LATIN SQUARES WITH RESTRICTED SUPPORT

Roland HÄGGKVIST
Matematiska Institutionen, Stockholms Universitet, Box 6701, 113 85 Stockholm, Sweden

The purpose of this note is to give a simple theorem which hopefully will inspire some reader to more profound explorations. First we give some definitions. A partial \( n \times n \) column-latin square \( L \) on 1, 2, \ldots, \( n \) is an \( n \times n \) array filled with the symbols 1, 2, \ldots, \( n \) in such a way that every cell contains at most one symbol, and every symbol occurs at most once in every column. The array \( L \) is a latin square if, in addition, every symbol occurs exactly once in every row and column.

**Theorem.** Let \( n = 2^k \) and let \( L \) be a partial \( n \times n \) column-latin square on 1, 2, \ldots, \( n \) with empty last column. Then there exists an \( n \times n \) latin square \( A \) on the same symbols which differs from \( L \) in every cell.

**Proof.** We use induction on \( k \). The theorem is obviously true when \( k = 0 \). Assume that the theorem has been proved for order \( m \) and let \( n = 2m \). By rearranging rows if necessary (and filling in some empty cells perhaps), we may assume that the \( m \)th column of \( L \) has the entries 1, 2, \ldots, 2\( m \) in that order. If we suppress the symbols 1, 2, \ldots, \( m \) in the upper left \( m \times m \) quadrant \( B \) and the lower right \( m \times m \) quadrant \( E \) in \( L \), we find ourselves with a pair of partial column-latin squares \( H \) and \( I \) on \( m + 1, m + 2, \ldots, 2m \) which both have empty last columns. Therefore we can find a pair of latin squares \( F \) and \( G \) on \( m + 1, m + 2, \ldots, 2m \), without any entries in common with \( H \) and \( I \) respectively, and certainly not with \( B \) and \( E \) either. Similarly, by suppressing the symbols \( m + 1, m + 2, \ldots, 2m \) in the upper right \( m \times m \) quadrant \( C \) and lower left \( m \times m \) quadrant \( D \) in \( L \), and applying the theorem, we find a pair of latin squares \( J \) and \( K \) on the symbols 1, 2, \ldots, \( m \), which fit into the upper right and lower left corner of \( L \) respectively, without any entries in common with \( C \) and \( D \). Together \( F, J, G \) and \( K \) make up \( A \). \( \square \)

The theorem is not valid for every \( n \) as seen by example below.

\[
\begin{array}{cc}
1 & 1 * \\
3 & 2 * \\
2 & 3 *
\end{array}
\]

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However, this is likely to be the only exception. In general perhaps the following is true for some positive constant $c$, which could be as large as $\frac{1}{3}$, say.

**Conjecture.** Let $L$ be an $n \times n$ array of $m$-sets from a set of symbols $1, 2, \ldots, n$ where every symbol is used at most $m \leq cn$ times in each row and column. Then there exists an $n \times n$ latin square $A$ on $1, 2, \ldots, n$ with entries in the complement of $L$.

A positive answer could have some impact on the following question.

**Dinitz' problem.**

Given an $m \times m$ array of $m$-sets, is it always possible to choose one element from each set, keeping the chosen elements distinct in every row and column?

For some related material see the references.

**References**