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Discrete Mathematics 157 (1996) 127–146

DISCRETE MATHEMATICS

Reduced words in affine Coxeter groups

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Received 2 July 1994; revised 11 November 1994

Abstract

Let $r(w)$ denote the number of reduced words for an element w in a Coxeter group W. Stanley proved a formula for $r(w)$ when W is the symmetric group A_n, and he suggested looking at $r(w)$ for the affine group $\tilde{A}_n$. We prove that for any affine Coxeter group $\tilde{X}_n$ there is a finite number of *types* of elements in $\tilde{X}_n$, such that to every element w can be associated (1) a type t, (2) an element v in the finite group X_n, and (3) an n-tuple $(m_1, m_2, \ldots, m_n)$ of integers $m_i \geqslant 0$. Then $r(w) = r_t^v(m_1, \ldots, m_n)$, and for every r_t^v and for large enough m_i, a homogeneous linear n-dimensional recurrence holds. For $\tilde{A}_n$, this takes a nice combinatorial form. We also discuss a canonical reduced word for w associated to its n-tuple.

Résumé

Soit $r(w)$ le nombre de mots réduits pour un élément w d'un groupe de Coxeter. Stanley a démontré une formule pour $r(w)$ dans le cas du groupe symétrique A_n, et il a posé le problème d'étudier $r(w)$ pour le groupe affine $\tilde{A}_n$. Nous montrons qu'il y a, pour tout groupe de Coxeter affine $\tilde{X}_n$, un nombre fini de *types* d'éléments tels qu'on peut associer à chaque élément w (1) un type t, (2) un élément v du groupe fini X_n, et (3) une suite $(m_1, m_2, \ldots, m_n)$ d'entiers $m_i \geqslant 0$. Alors, $r(w) = r_t^v(m_1, \ldots, m_n)$ et pour m_i assez grands, r_t^v satisfait à une récurrence homogène linéaire n-dimensionelle. Pour $\tilde{A}_n$, cela prend une forme combinatoire agréable. Nous présentons aussi une décomposition réduite canonique pour w, associé à la suite des m_i.

1. Introduction

For an element w of a Coxeter group (W, S), a *reduced word* for w is obtained by writing w as a minimal product of generators. Let $r(w)$ denote the number of reduced words for w.

Stanley [6] and Greene and Edelman [1] studied the number of reduced words for elements in A_n, showing an intimate relationship with standard tableaux of the corresponding shape. Haiman [3] generalized their work to include the finite Coxeter group

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SSDI 0012-365X(95)00259-6

B_n as well. Thus, for most finite Coxeter groups, the combinatorics of $r(w)$ is very well understood. In his paper [6], Stanley also suggested that one should study the number of reduced words in the affine group $\tilde{A}_n$. This is our purpose here.

We will mainly work in the Coxeter complex, where reduced words correspond to minimal galleries. We shall show that for any affine Coxeter group $\tilde{X}_n$ corresponding to a finite Coxeter group X_n, there is a finite number of *types* of elements in $\tilde{X}_n$, such that to every element w can be described by an index of the form ${}^x_t(m_1, m_2, \ldots, m_n)$, where t is a type, x is an element of the finite group X_n, and $m_1, m_2, \ldots, m_n$ are nonnegative integers. Thus, for any type t and element $x \in X_n$, we can define the easier-to-handle function $r^x_t(m_1, \ldots, m_n) \stackrel{\text{def}}{=} r({}^x_t(m_1, m_2, \ldots, m_n))$. Every r^x_t is then shown to satisfy a homogeneous linear n-dimensional recurrence. For any fixed type t, the same recurrence will hold for every x but with different start values depending on the element x. An obvious application of the recurrences would be to find asymptotics for r^x_t. To the author's knowledge, no work in this area has been done as yet.

The description of the recurrence is geometrical in nature. However, for the affine groups $\tilde{A}_n$, a combinatorial form of the recurrence can be obtained by digging into the geometry. This is presented in Section 6.

We will also discuss a canonical reduced word for w related to v, t and $(m_1, \ldots, m_n)$.

2. The alcove complex and weak order of an affine Coxeter group

A Coxeter group (or, more precisely, a Coxeter system) (W, S) is a group W together with a distinguished set $\{\sigma_1, \sigma_2, \ldots\}$ of generators, and integers m_{ij} where $m_{ii} = 1$ and $m_{ij} \geqslant 2$ for $i \neq j$, such the group is defined by the relations $(\sigma_i \sigma_j)^{m_{ij}} = e$ (the identity of the group). m_{ij} may be ∞, in which case $\sigma_i \sigma_j$ has infinite order in W. We refer to Humphreys's book [4] for details on Coxeter group theory.

2.1. *The alcove complex*

In this paper we will mainly take the hyperplane arrangement approach to Coxeter groups, since this provides the best intuition for the affine groups. To begin with, let X_n denote an arbitrary finite irreducible crystallographic reflection group in $\mathbb{R}^n$, generated by n reflections $\sigma_1, \ldots, \sigma_n$, and let $\mathscr{H}$ be the arrangement of reflecting hyperplanes, all going through the origin, splitting $\mathbb{R}^n$ into cones. Every cone is bounded by n walls, and they can be canonically labeled by σ_1 through σ_n, such that when one cone is mapped to another via a sequence of reflections in the hyperplanes, the labeling of the walls is invariant. The cones correspond bijectively to the group elements of X_n. Let e denote the identity element of X_n and associate with it one of the cones, denoted V^e. Then for every group element $x \in X_n$, there is a unique cone V^x that you get to by walking from V^e through any sequence of walls labeled $\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_k}$ such that the corresponding product of Coxeter generators is equal to x.

For a cone V^x, let e_i^x be a unit vector in the direction of the ray that is the intersection of all the bounding walls of V^x, except for the one labeled σ_i. Thus, V^x is the positive span of the vectors e_i^x:

$$V^x = \{\lambda_1 e_1^x + \cdots + \lambda_n e_n^x : \lambda_1, \ldots, \lambda_n \geqslant 0\}.$$

Example. We will present a running example with C_2 as the Coxeter group 'X_n'. C_2 is the group of 8 elements generated by reflections in two lines with an angle between them of 45° (see Fig. 1).

The *affine* group $\tilde{X}_n$ corresponding to the finite group X_n is obtained by adding to the set of generators a reflection in an affine hyperplane parallel to one of the hyperplanes in $\mathscr{H}$. Let $\tilde{\mathscr{H}}$ denote the infinite affine hyperplane arrangement; thus $\mathscr{H} \subset \tilde{\mathscr{H}}$ (see Fig. 2).

$\tilde{\mathscr{H}}$ refines the cones of $\mathscr{H}$ into finite *alcoves*, each bounded by $n + 1$ walls, and every collection of n such walls has one common *vertex*. Choose the alcove C_e at the apex of the cone V_e to be the *fundamental* alcove. Let the n walls of C_e that coincides

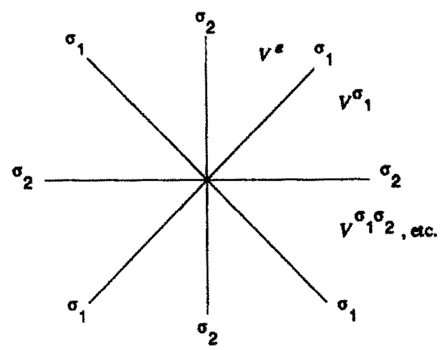


Fig. 1. The hyperplane arrangement of Coxeter group C_2.

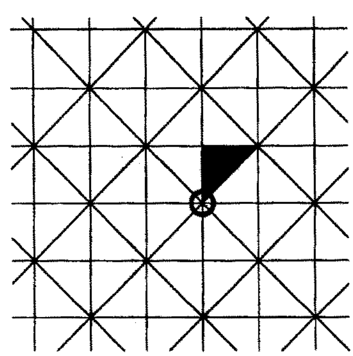


Fig. 2. The affine hyperplane arrangement of affine group $\tilde{C}_2$, with the fundamental alcove C_e painted and the origin encircled.

with the walls of the cone V_e inherit the labels of the coinciding walls, and let the final wall be labeled σ_{n+1}. In analogy to what we did with the cones, label the walls of every alcove by σ_1 through σ_{n+1} such that the labels are invariant under reflections in hyperplanes in $\tilde{\mathscr{H}}$. Let $\mathscr{C}$ be the alcove complex defined by $\tilde{\mathscr{H}}$. A *gallery* is a walk in the complex, and a *minimal* gallery is a shortest possible gallery between two alcoves. Like before, each group element $w \in \tilde{X}_n$ can be associated with an alcove $C_w \in \mathscr{C}$ such that C_w is the alcove reached from the fundamental alcove C_e by a gallery whose labelings correspond to a product of generators equal to w.

The alcove complex of $\tilde{X}_n$ and the parabolic subgroup X_n induces a tesselation of $\mathbb{R}^n$ in the following way. The set of all alcoves with one vertex in the origin corresponds to the group elements of the parabolic subgroup X_n, generated by $\{\sigma_1,\dots,\sigma_n\}$. The shape of the union of these alcoves is an X_n*-block*. Thus, the partitioning of $\tilde{X}_n$ into cosets of X_n gives a tesselation of $\mathbb{R}^n$ in X_n-blocks (see Fig. 3).

We say that two alcoves have the same *orientation* if one can be mapped to the other by pure translation. The alcoves are of course all congruent (since they are all reflection images of each other), but there may be several different orientations. For example, $\tilde{C}_2$ has four different orientations of its alcoves, as can be seen in Fig. 3 How many orientations are there in general? Since hyperplanes of all orientations meet at the origin, all kinds of reflections can be carried out within the X_n-block, and hence all possible orientations of alcoves are represented among the $|X_n|$ alcoves that touch the origin. However, the X_n-block may contain more than one alcove of each orientation.

Lemma 1. *Let $p(\tilde{X}_n)$ be the number of parabolic subgroups of $\tilde{X}_n$ that are isomorphic to X_n. Let k be the number of possible orientations of alcoves in $\tilde{X}_n$. Then*

$$k = \frac{|X_n|}{p(\tilde{X}_n)}.$$

Proof. Number the vertices of the fundamental alcove such that vertex i is the vertex opposite to the wall labeled σ_i. Let S be the generator set $\sigma_1,\dots,\sigma_{n+1}$. The maximal parabolic subgroup generated by $S - \{\sigma_i\}$ can be identified with vertex i of the fundamental alcove, since the n walls of the alcove that contain the vertex correspond to the n generators. This parabolic subgroup is isomorphic to X_n if and only if vertex i is equivalent to the origin, in the sense that the alcove complex (ignoring all labels)

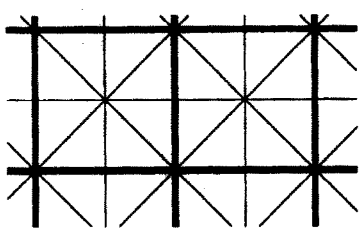


Fig. 3. The C_2-blocks in the complex of $\tilde{C}_2$.

is invariant under translation of vertex i to the origin. Hence, there are $p(\tilde{X}_n)$ such origin-type vertices of the fundamental alcove.

Now, since the complex is invariant under the translation of an origin-type vertex to the origin, in particular the fundamental alcove is translated to another alcove of the same orientation. Conversely, when translating the fundamental alcove to any alcove with the same orientation and touching the origin, then the vertex translated to the origin must be of origin-type. Hence, among the alcoves touching the origin, there are $p(\tilde{X}_n)$ oriented as the fundamental alcove. By symmetry, there are $p(\tilde{X}_n)$ alcoves of each orientation among the $|X_n|$. □

Remark 1. Parabolic subgroups of $\tilde{X}_n$ isomorphic to X_n correspond to subgraphs of the Coxeter graph of $\tilde{X}_n$ isomorphic to the Coxeter graph of X_n. By examining the Coxeter graphs of all affine groups (see tables in Humphreys [4]) one obtains the following table:

X_n	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$p(\tilde{X}_n)$	$n+1$	2	2	4	3	2	1	1	1

2.2. *The weak order*

The *weak order* of a Coxeter group is a partial ordering defined by the following covering relations: if σ is a generator, then w is covered by $w\sigma$ in the weak order if $l(w\sigma) = l(w) + 1$, where the length function $l(w)$ returns the length of a shortest reduced word for w.

In the alcove complex of the affine group $\tilde{X}_n$, reduced words are equivalent to minimal galleries. We might as well regard the weak order as a partial order on the alcoves. In this ordering, any alcove $C \in \mathscr{C}$ defines an interval $[C_e, C]$, which is the subset of $\mathscr{C}$ consisting of all alcoves that you can visit by walking minimal galleries from C_e to C, which is equivalent to walks that cross only hyperplanes that separate C_e from C (see Fig. 4).

At one point we will need the following well-known property of the weak order, see for example Section 5.11 in Humphreys's book [4].

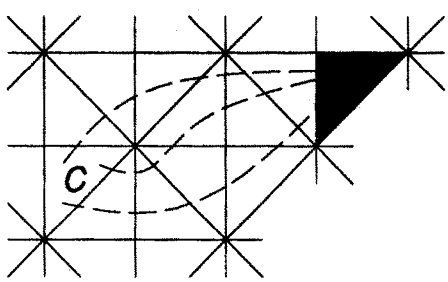


Fig. 4. The interval $[C_e, C]$ with the three minimal galleries indicated.

Lemma 2. *Let w be an element and σ, σ' two generators, and let $>$ denote comparison in the weak order. If $w\sigma > w$ and $w\sigma' > w$, then $w\sigma\sigma' > w\sigma$. Dually, if $w\sigma < w$ and $w\sigma' < w$, then $w\sigma\sigma' < w\sigma$.*

3. The truncated cone construction

What we are going to do is covering the complex $\mathscr{C}$ by a finite set of truncated cones, $\mathscr{V} = \{V^x : x \in X_n\}$, where each $V^x \in \mathscr{V}$ is bounded by some hyperplanes in $\tilde{\mathscr{H}}$. We identify a cone with the set of alcoves contained in it. This covering will have the following three properties:

(i) $\bigcup_{x\in X_n} V^x = \mathscr{C}$,

(ii) $\bigcap_{x\in X_n} V^x = C_e$,

(iii) $C \in V \Rightarrow [C_e, C] \subset V$ for all $V \in \mathscr{V}$.

In words: (i) the truncated cones cover the complex; (ii) only the fundamental alcove C_e lies in every truncated cone; (iii) all minimal galleries between C_e and any given alcove C stay in the truncated cone in which C lies.

We will take V^x to be the smallest region bounded by hyperplanes in $\tilde{\mathscr{H}}$ and containing both C_e and V^x. This construction should be viewed in the following way (see Fig. 5). Build a *thick wall* from a pair of successive parallel hyperplanes in $\tilde{\mathscr{H}}$ enclosing the fundamental alcove C_e. In other words, each of the hyperplanes in the finite arrangement $\mathscr{H}$ is thickened to a thick wall containing C_e. Now, what we have got is a thickened version of the hyperplane arrangement for X_n, and since the thick walls overlap, they bound a set of $|X_n|$ *truncated* cones. (A truncated cone contains its thick walls.) Of course, every cone of the 'thin' arrangement $\mathscr{H}$ is contained in

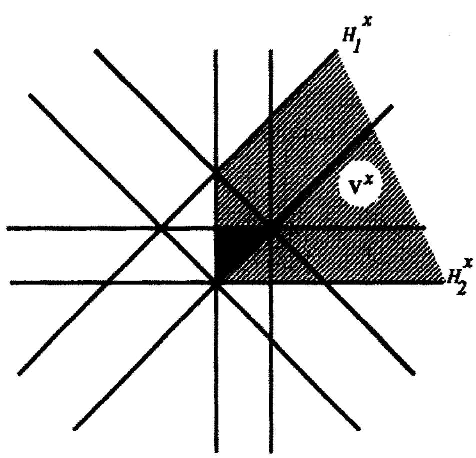


Fig. 5. The thickened arrangement of C_2, with one truncated cone V^x painted.

exactly one of the truncated cones. Label the truncated cones accordingly, so V^x is the truncated cone containing V^x.

Lemma 3. *The set* $\mathscr{V} = \{V^x : x \in X_n\}$ *of truncated cones has the properties* (i), (ii) *and* (iii) *above.*

Proof. (i) Already the $\{V^x : x \in X_n\}$ covers the complex, so a fortiori the truncated cones do. (ii) follows immediately from the construction. To prove (iii), let C be an alcove in V^x. Since a gallery from C to C_e that leaves the cone V^x must also reenter the cone, it must cross some bounding hyperplane twice. Thus it cannot be minimal, so all minimal galleries from C to C_e stay in the cone. □

Thanks to property (iii) we know that when counting minimal galleries we can restrict our attention to one truncated cone instead of the entire complex. Except for the truncation, V^x is bounded by n hyperplanes, with a natural labeling $H_1^x, H_2^x, \ldots, H_n^x$ induced by the labeling $\sigma_1, \ldots, \sigma_n$ of the corresponding walls of V^x. We shall now introduce yet another size of pieces, bigger than alcoves and smaller than cones. Make $\mathbb{R}^n$ into a lattice of *cells* by subdividing it by all hyperplanes of $\tilde{\mathscr{H}}$ that are parallel to any of the bounding hyperplanes of V^x. Fig. 3 should make the situation clear (see Fig. 6).

Note that since the subdivision is caused by a subset of $\tilde{\mathscr{H}}$, the alcoves are finer objects than the cells. Every cell will be a union of alcoves of $\mathscr{C}$. The *apex* of a cell is the vertex closest to the origin.

Lemma 4. *Every cell of a cone* V^x *is composed of alcoves in the same way, and every possible orientation of alcoves occurs exactly once in every cell. Thus, if there are k possible orientations of alcoves, then an arbitrary cell consists of k alcoves.*

Proof. The apex of any cell is of origin-type (see Section 2.1), since the n walls (of the cell) containing the apex are parallel to the n walls bounding a cone in the X_n-arrangement, so by reflections they generate an isomorphic hyperplane arrangement through the apex. Together with one of the other walls they generate the entire affine arrangement. Hence, from the viewpoint of an apex the arrangement can only look in exactly one way, so in particular every cell must look the same. Also, from the

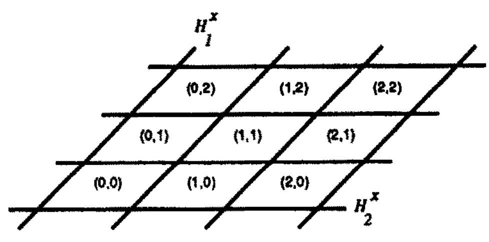


Fig. 6. The cell decomposition related to the truncated cone V^x.

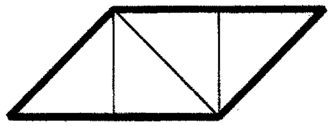


Fig. 7. A $\tilde{C}_2$-cell and its decomposition into alcoves of four types.

viewpoint of any alcove of the same orientation as the alcove at the apex of the cell, the arrangement can only look in one way. Since there are no additional hyperplanes parallel to cell walls, there can be just one alcove of this orientation in every cell, and by symmetry the same must hold for every orientation. □

We would like to treat all truncated cones in the same way. Therefore, we are not really interested in the physical orientation of an alcove, but rather the orientation relative to the truncated cone in question. Given *a specified cone* V^x, define the *type* of an alcove to be its orientation relative to the cone. If there are k different orientations of alcoves, then there are of course also k types of alcoves, and by Lemma 4 there is one alcove of each type in every cell (see Fig. 7).

We can index the cells of V^x by n integer indices such that ${}^x(0,0,\ldots,0)$ is the cell containing the fundamental alcove, and ${}^x(m_1,m_2,\ldots,m_n)$ is the cell separated from ${}^x(0,0,\ldots,0)$ by m_i hyperplanes parallel to H_i^x for all $i=1,2,\ldots,n$. See Fig. 3. (Note that, since the cones are truncated, there may be some alcoves in these cells that are not entirely contained in V^x.)

Thus, every alcove of the truncated cone V^x can be described uniquely as ${}^x_t(m_1,\ldots,m_n)$ for some indices $m_1,\ldots,m_n \geqslant 0$ and some type t.

4. Counting minimal galleries

We are interested in the number of reduced words for an element $w \in \tilde{X}_n$. Reduced words for w are equivalent to minimal galleries between the corresponding alcove C_w and the fundamental alcove C_e. C_w belongs to some truncated cone $V^x \in \mathscr{V}$ (possibly to several) and we know that the minimal galleries never leave this cone.

Let $r(C)$ denote the number of minimal galleries from C_e to the alcove C. Trivially, we have $r(C_e)=1$. For $C \neq C_e$, a fundamental observation is that, using the covering relation of the weak order,

$$r(C) = \sum r(C') \quad \text{summed over all } C' \text{ covered by } C. \tag{1}$$

By the analysis in the previous section, we know that any alcove C in a specified truncated cone V^x can be uniquely indexed ${}^x_t(m_1,\ldots,m_n)$, for some type t and indices $m_1,\ldots,m_n$. Thus, for every type t and every $x \in X_n$, we may define the function $r_t^x : \mathbb{Z}^n \longrightarrow \mathbb{N}$ by

$$r_t^x(m_1,\ldots,m_n) := \begin{cases} r({}^x_t(m_1,\ldots,m_n)) & \text{if the alcove belongs to } V^x, \\ 0 & \text{otherwise.} \end{cases}$$

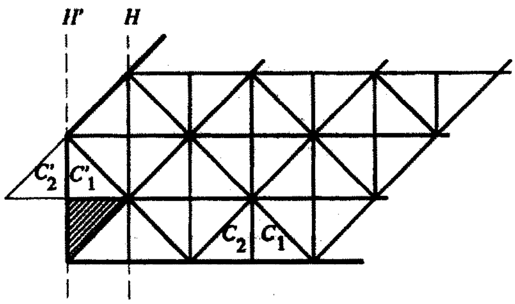


Fig. 8. Illustration of the proof.

Instead of determining $r(w)$, we shall hence determine $r_t^x(m_1,\ldots,m_n)$. We know that all covering relations for alcoves in V^x occur within V^x, so we may rewrite Eq. (1) using r_t^x instead of r:

$$r_t^x(m_1,\ldots,m_n)=\sum r_{t'}^x(\mu_1,\ldots,\mu_n),\quad {}^x_{t'}(\mu_1,\ldots,\mu_n) \text{ covered by } {}^x_t(m_1,\ldots,m_n). \quad (2)$$

Now we shall prove that, for each type t, Eq. (2) can be given a uniform look by, for the alcoves at the boundary of V^x, adding some zero terms.

Lemma 5. *Let C_1 and C_1' be two alcoves of the same type t in V^x. Let C_2 be an alcove covered by C_1, and let C_2' be the corresponding neighbor of C_1'. Then either C_2' is covered by C_1' or else C_2' does not belong to V^x.*

Proof. Let H and H' be the hyperplanes bounding the thick wall parallel with the wall between C_1 and C_2, and hence also parallel with the wall between C_1' and C_2'. By construction, one of these hyperplanes, say H, intersects the interior of V^x, while H' does not. Of course, in the half-space separated from C_e by H (including H), every wall parallel to H will have the same side facing C_e. Suppose C_1' does not cover C_2'. Then the wall in between does not have the same side facing C_e as does the wall between C_1 and C_2, hence it is not in the half-plane separated from C_e by H. But then C_2' must be on the outside of H', and thereby not inside V^x. See Fig. 8. □

Thus, if there are k types, the relation (2) can be expressed as a system of k recurrences, one for each type t. The alcove decomposition, being a reflection arrangement, is of course invariant under the orientation x of the cone. However, the fundamental alcove is located differently in different truncated cones. This means that for $x \neq y \in X_n$, the recurrences for r_t^x and r_t^y will look the same, but the boundary values will differ (or, rather, the location of the boundary will differ). It is convenient to introduce the difference operators defined by

$$\partial_i r(m_1,\ldots,m_i,\ldots,m_n)=r(m_1,\ldots,m_i-1,\ldots,m_n).$$

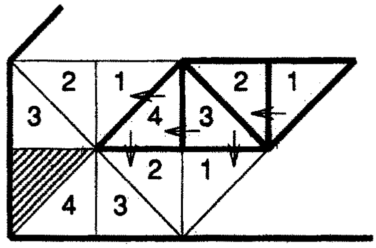


Fig. 9. Sketch of a truncated cone V^x, with the arrows signifying the covering relations between alcoves in $\tilde{C}_2$.

Example. In $\tilde{C}_2$ there are four types, which we may number 1 through 4 by decreasing distance from the apex of the cell. With V^x given by Fig. 9, the initial values are

$$r_1^x(0,0)=1,\ r_4^x(0,1)=0,\ r_t^x(-1,m_2)=r_t^x(m_1,-1)=0 \quad \text{for all } t,m_1,m_2.$$

The covering relations, as seen in Fig. 9, give the recurrences:

$$\begin{cases} r_1^x(m_1,m_2)=r_2^x(m_1,m_2) \\ r_2^x(m_1,m_2)=r_3^x(m_1,m_2) \\ r_3^x(m_1,m_2)=r_4^x(m_1,m_2)+r_1^x(m_1,m_2-1) \\ r_4^x(m_1,m_2)=r_1^x(m_1-1,m_2)+r_2^x(m_1,m_2-1) \end{cases} \Leftrightarrow \begin{cases} r_1^x=r_2^x \\ r_2^x=r_3^x \\ r_3^x=r_4^x+\partial_2 r_1^x \\ r_4^x=\partial_1 r_1^x+\partial_2 r_2^x \end{cases}$$

Elimination gives a single recurrence, which for any type t has the form

$$r_t^x(m_1,m_2)=r_t^x(m_1-1,m_2)+2r_t^x(m_1,m_2-1) \Leftrightarrow r_t^x=(\partial_1+2\partial_2)r_t^x.$$

Theorem 6. *Let T be the set of the k types in the alcove complex of $\tilde{X}_n$. The r_t^x are determined by a system of k homogeneous, first-order linear recurrences:*

$$r_t^x=\sum_{t'\in T}\Delta_{t,t'}r_{t'}^x \quad \textit{for each } t\in T,$$

where $\Delta_{t,t'}$ is a difference operator that includes only first-order terms and constants. The boundary values are given by $r(C_e)=1$ and $r_t^x(m_1,\ldots,m_n)=0$ for all alcoves ${}_t^x(m_1,\ldots,m_n)$ that do not belong to V^x, which can be described by at most $|X_n|$ conditions.

Proof. As we observed earlier, it follows from Lemma 5 that for each type t one can formulate one recurrence that holds for all alcoves of type t. The terms come from covering relations between neighbor alcoves, and two neighbors must of course either belong to the same cell or to two neighbor cells, yielding constant terms and first-order terms respectively. Finally, the boundary of a truncated cone is described by the exterior hyperplanes of the bounding thick walls, which can be at most $|X_n|$, the total number of thick walls. $\square$

Give the types any linear ordering satisfying $t' > t$ when t covers t' in the cell. Thus, in this ordering the difference operator $\Delta_{t,t'}$ has no constant term when $t' \leqslant t$.

We can now express the system of recurrences as a matrix multiplication:

$$\begin{pmatrix} (1-\Delta_{1,1}) & -\Delta_{1,2} & \cdots & -\Delta_{1,k} \\ -\Delta_{2,1} & (1-\Delta_{2,2}) & \cdots & -\Delta_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\Delta_{k,1} & -\Delta_{k,2} & \cdots & (1-\Delta_{k,k}) \end{pmatrix} \begin{pmatrix} r_1^x \\ r_2^x \\ \vdots \\ r_k^x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3}$$

All elements below the diagonal lack constant terms, while all diagonal elements have constant term 1. It is easy to verify that this property is preserved during Gaussian elimination without pivoting. Thus, by completing the Gaussian elimination, we arrive at a diagonal matrix $\mathrm{diag}(1-\Delta_t')$ or, equivalently, a system of k independent linear recurrences:

$$r_t^x = \Delta_t' r_t^x, \quad t = 1, 2, \ldots, k,$$

for some constant-free difference operators Δ_t'. Let us state this as a theorem.

Theorem 7. *For every alcove type t there is a constant-free difference operator Δ_t' of degree $\leqslant 2^{k-1}$ such that the numbers of minimal galleries $r_t^x(m_1, \ldots, m_n)$ satisfy the homogeneous n-dimensional linear recurrence*

$$r_t^x(m_1, \ldots, m_n) = \Delta_t' r_t^x(m_1, \ldots, m_n)$$

whenever all m_i are sufficiently large.

Proof. Everything is clear except for the upper bound on the degrees. During the Gaussian elimination, let $\Delta_{j,l}^{(i)}$ denote the operator at row j and column l after rows 1 through i have been eliminated. We get

$$\Delta_{j,l}^{(i)} = \Delta_{j,l}^{(i-1)} \Delta_{i,i}^{(i-1)} - \Delta_{i,l}^{(i-1)} \Delta_{j,i}^{(i-1)}.$$

Thus, the maximum degree is at most doubled for each elimination of a row. There are $k-1$ rows to be eliminated, so the final degree is bounded by 2^{k-1}. □

5. A canonical reduced word

In this section we will revisit the geometry of the alcove complex in order to find a canonical minimal gallery for each alcove. This gallery will walk in turn in each of the n directions that span the cone containing the alcove. We will then determine the reduced words that correspond to walking in each direction.

5.1. A canonical minimal gallery

Recall that in Section 2.1 we defined the unit vectors e_i^x, such that the cone V^x is their positive span. The vector e_i^x can also be decribed as pointing in the direction of the ray given by the intersection $\bigcap_{j\neq i} H_j^x$, where the $\{H_1^x,\ldots,H_n^x\}$ are the bounding hyperplanes of the cone.

In Section 3, we defined an indexing of the cells of V^x, such that the ith index changes when a wall parallel to H_i^x is crossed. Walking in the direction of e_i^x, no hyperplane parallel to H_j^x is crossed, for any $j\neq i$. Thus, walking in this direction only affects the ith index.

Our aim now is to construct, to each alcove C_w, a canonical minimal gallery from the fundamental alcove C_e. First, we will need yet another subdivision of space into cones. Let p_e be a point in the interior of C_e. Introduce a new set of hyperplanes $\mathscr{H}_{p_e}$, by translating $\mathscr{H}$ to be centered in p_e instead of in the origin. As always, $\mathscr{H}_{p_e}$ splits space in $|X_n|$ cones; let $V_{p_e}^x$ denote the cone that is the translation of V^x. Thus,

$$V_{p_e}^x = \{p_e + \lambda_1 e_1^x + \cdots + \lambda_n e_n^x : \lambda_1,\ldots,\lambda_n \geqslant 0\}.$$

Now, choose one generic point p_w in the interior of every alcove C_w. Hence, for some unique $x\in X_n$ we have

$$p_w = p_e + \lambda_1 e_1^x + \cdots + \lambda_n e_n^x \quad \text{with all } \lambda_1,\ldots,\lambda_n \geqslant 0.$$

We define the *canonical p_e, p_w-gallery* from C_e to C_w by the walk (in n straight lines) (see Fig. 10)

$$p_e \longrightarrow p_e + \lambda_1 e_1^x \longrightarrow p_e + \lambda_1 e_1^x + \lambda_2 e_2^x \longrightarrow \cdots \longrightarrow p_e + \lambda_1 e_1^x + \cdots + \lambda_n e_n^x = p_w.$$

(The points p_e, p_w should be chosen generic such that the gallery contains no degenerate crossings of walls.)

Proposition 8. *The canonical p_e, p_w-gallery is a minimal gallery from C_e to C_w.*

Proof. Were the gallery not minimal, it would have to cross some hyperplane $H\in\tilde{\mathscr{H}}$ twice. Obviously, this H cannot be parallel to any of the bounding hyperplanes of the

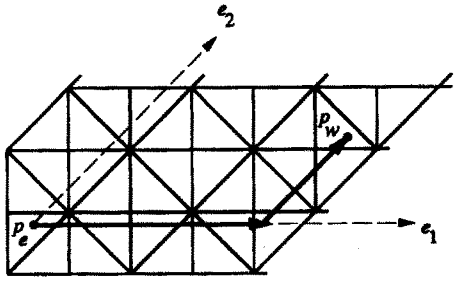


Fig. 10. A canonical p_e, p_w-gallery in $\tilde{C}_2$.

cone $V^x_{p_e}$. But all other hyperplanes in $\tilde{\mathscr{H}}$ have, by construction of the cones from $\mathscr{H}$, finite intersection with the cone, contradicting the fact that if H can be crossed twice by going in straight lines directed as $e_1, \ldots, e_n$, then H can be crossed infinitely many times by repeating the crossing pattern. □

5.2. Canonical reduced words

We shall now find out what words correspond to walking in the n basic directions of the alcove complex discussed above. Once again, let S be the generator set $\{\sigma_1, \ldots, \sigma_{n+1}\}$. Recall that σ_{n+1} is the affine reflection, so $S - \{\sigma_{n+1}\}$ generates X_n. Let $\hat{w}_i$ denote the maximal (that is, of maximal length) element of the parabolic subgroup generated by $S - \{\sigma_i\}$. Define analogously $\hat{w}_{i,j}$ to be the maximal element of the parabolic subgroup generated by $S - \{\sigma_i, \sigma_j\}$.

We use $\langle w \rangle$ to denote any reduced word for a group element w.

To begin with, we shall restrict our attention to the cone V^e that has the fundamental alcove C_e at its apex (but later we shall see that everything works similarly for every cone). V^e is bounded by the hyperplanes $H_i = H_i^e$ for $i = 1, \ldots, n$, and H_i coincides with the wall of C_e labeled σ_i. The ith direction of the cone is defined by the line $L_i = \bigcap_{j \neq i} H_j$. We shall now walk along this line, in the direction of e_i.

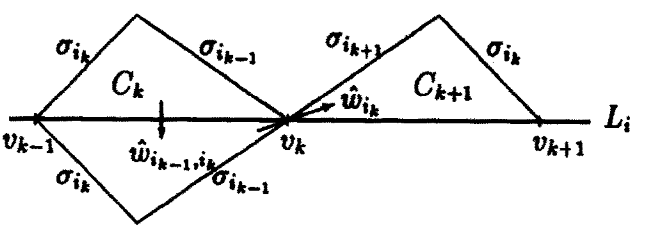
Two alcoves are *successive along the line* L_i if they are on the same side of every hyperplane through L_i, and they have a common vertex on L_i. Recall that all alcoves with one vertex in a point v have the same label on the wall opposite to v.

Lemma 9. *Let v_{k-1}, v_k, v_{k+1} be three successive vertices on the line L_i, with opposite walls labeled $\sigma_{i_{k-1}}$, σ_{i_k} and $\sigma_{i_{k+1}}$ respectively. Let C_k and C_{k+1} be the successive alcoves along L_i with common vertex v_k. Then the following holds:*

(1) $\langle \hat{w}_{i_{k-1}, i_k} \hat{w}_{i_k} \rangle$ is a minimal gallery from C_k to C_{k+1}.

(2) $\sigma_{i_{k+1}} = \hat{w}_{i_k} \sigma_{i_{k-1}} \hat{w}_{i_k}$.

Proof. The following sketch of a 2-dimensional cut through the line L_i and the alcoves C_k and C_{k+1} explains the proof:



(1) To get from C_k to C_{k+1} one must cross all hyperplanes through v_k except for those generated by $\{H_j : j \neq i\}$. It is easy to describe a nonminimal gallery: First cross all hyperplanes generated by $\{H_j : j \neq i\}$; this corresponds to the group element

$\hat{w}_{i_{k-1},i_k}$. Then cross all hyperplanes going through v_k; this corresponds to the group element $\hat{w}_{i_k}$. Thus the reduced word $\langle \hat{w}_{i_{k-1},i_k}\hat{w}_{i_k}\rangle$ gives a minimal gallery from C_k to C_{k+1}.

(2) Let H be the hyperplane coinciding with the wall of alcove C_{k+1} labeled σ_{k+1}. Crossing this wall can be done by first crossing all hyperplanes through v_k by $\hat{w}_{i_k}$, then crossing H, which here coincides with the wall $\sigma_{i_{k-1}}$, and finally crossing all hyperplanes through v_k once again by $\hat{w}_{i_k}$. Thus, $\sigma_{i_{k+1}} = \hat{w}_{i_k}\sigma_{i_{k-1}}\hat{w}_{i_k}$. □

Suppose we have a reduced word corresponding to a minimal gallery in a certain direction from a certain alcove. The following lemma tells what word might be used to travel in the same direction from other alcoves. Remember that a minimal gallery extends in the ith direction of the cone V^x if it is parallel with the line $L_i^x = \bigcap_{j\neq i} H_i^x$.

Lemma 10. *Let C be an alcove. Let $\langle w\rangle$ be a reduced word for some minimal gallery from C extending in the ith direction of V^x, let σ_j be some generator, and let C' be the alcove reached by crossing the wall σ_j from C. If the wall σ_j is parallel to some hyperplane containing L_i^x, then $\langle w\rangle$ is a minimal gallery from C' in the ith direction; otherwise, $\langle \sigma_j w\rangle$ is a minimal gallery from C' in the ith direction.*

Proof. If the wall σ_j is parallel to some hyperplane containing L_i^x, then reflection in this wall gives a gallery that still is parallel with L_i^x. Since the labels are invariant under reflections, $\langle w\rangle$ is a minimal gallery from C' in the ith direction (see Fig. 11). On the other hand, if the wall σ_j is not parallel to any hyperplane containing L_i^x, then crossing σ_j is a step (backwards or forwards) along the ith direction, and the minimal gallery $\langle \sigma_j w\rangle$ from C' extends in the ith direction. □

Proposition 11. *Every reduced word corresponding to a minimal gallery in the ith direction of V^x is a factor (that is, a contiguous subword) of the infinite periodic word*

$$\alpha_i = \dots\langle \hat{w}_{i_0,i_1}\hat{w}_{i_1}\rangle\langle \hat{w}_{i_1,i_2}\hat{w}_{i_2}\rangle\dots\langle \hat{w}_{i_{k-1},i_k}\hat{w}_{i_k}\rangle\dots,$$

where $i_0 = n+1$, $i_1 = i$, and all other indices are determined by $\sigma_{i_{k+1}} = \hat{w}_{i_k}\sigma_{i_{k-1}}\hat{w}_{i_k}$ for all k.

Proof. First, observe that the infinite word must indeed be periodic, since there are only a finite set of possible pairs i_{k-1}, i_k. Now, study the cone V^e, with apex in the fundamental alcove C_e. Going in the ith direction from C_e, the statement follows from

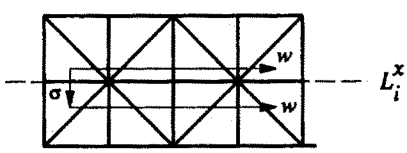


Fig. 11. Reflection in a wall parallel with L_i^x.

Lemma 9. It is easy to see from the picture of the proof, with $C_1 = C_e$, that $i_0 = n+1$ and $i_1 = i$. Thanks to Lemma 10, we know that the infinite word describing the ith direction does not change when we leave C_e for any other alcove.

Finally, the ith direction for any other cone V^x is obtained from the ith direction of V^e by the reflection sequence that takes V^e to V^x, and labels are invariant under reflections, so the statement holds for all cones. □

The canonical minimal galleries of Section 5.1 can now be expressed as reduced words:

Theorem 12. *The canonical gallery of Proposition* 8 *corresponds to a reduced word of the form* $\langle v_1\rangle\langle v_2\rangle\ldots\langle v_n\rangle$ *where* $\langle v_i\rangle$ *is a factor of the periodic word* α_i *for all* $i = 1, 2, \ldots, n$.

Proof. Immediate from Proposition 11 and the definition of canonical gallery.

Example. Let us look at $\tilde{C}_2$ again. Here we have three generators $\{\sigma_1, \sigma_2, \sigma_3\}$. The maximal elements of parabolic subgroups are

$$\hat{w}_1 = \sigma_2\sigma_3, \qquad \hat{w}_2 = \sigma_1\sigma_3\sigma_1\sigma_3, \qquad \hat{w}_3 = \sigma_2\sigma_1\sigma_2\sigma_1,$$

$$\hat{w}_{1,2} = \sigma_3, \quad \hat{w}_{1,3} = \sigma_2, \qquad \hat{w}_{2,3} = \sigma_1.$$

Furthermore, in $\tilde{C}_2$ we have $\hat{w}_j\sigma_i\hat{w}_j = \sigma_i$for all $i \neq j$. Thus, walking in direction 1 will give, repeatedly, the word

$$\langle \hat{w}_{1,3}\hat{w}_1\rangle\langle \hat{w}_{3,1}\hat{w}_3\rangle = \langle \sigma_2\sigma_2\sigma_3\rangle\langle \sigma_2\sigma_2\sigma_1\sigma_2\sigma_1\rangle = \sigma_3\sigma_1\sigma_2\sigma_1.$$

Similarly, we can compute the repeated word for direction 2 to be

$$\langle \hat{w}_{2,3}\hat{w}_2\rangle\langle \hat{w}_{3,2}\hat{w}_3\rangle = \sigma_3\sigma_1\sigma_3\sigma_2\sigma_1\sigma_2.$$

This is verified in Fig. 12.

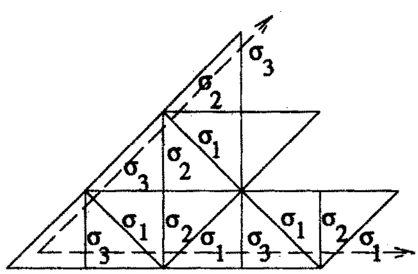


Fig. 12. The words in the two basic directions of $\tilde{C}_2$.

6. The recurrence for $\tilde{A}_n$

For the affine group $\tilde{A}_n$, there is a more combinatorial way of stating the recurrences. $\tilde{A}_n, n \geqslant 2$, is the group whose Coxeter graph is a circuit of $n+1$ nodes. Thus, the $n+1$ maximal parabolic subgroups of $\tilde{A}_n$ are all isomorphic to A_n, the group whose Coxeter graph is a path of n nodes. One consequence of this is that every vertex of the alcove complex is of origin-type. Another one is that Lemma 1 gives for $\tilde{A}_n$ that the number of types is $k=(n+1)!/(n+1)=n!$, which is equal to the number of permutations in the symmetric group S_n. S_n is isomorphic to A_{n-1}, where the generators $\{\sigma_1,\dots,\sigma_{n-1}\}$ are interpreted as the adjacent transpositions $\{(1,2),\dots,(n-1,n)\}$. The interplay between these concepts — types, permutations, and elements of A_{n-1} — will lead us to the following result.

Theorem 13. *Let $D(\pi)$ be the descent set of π, let s_i denote the transposition $(i,i+1)$ for $i=1,2,\dots,n-1$ and let τ be the rotation operator defined by $\tau(\pi_1\pi_2\dots\pi_n)=\pi_n\pi_1\dots\pi_{n-1}$, so $\tau=s_{n-1}s_{n-1}\dots s_1$. The elements in a given cone of $\tilde{A}_n$ may be indexed by a permutation π and $m_1,\dots,m_n$ such that r_π satisfies the recurrence*

$$r_\pi(m_1,\dots,m_n)= r_{\tau(\pi)}(m_1,\dots,m_{n-\pi_n}+1,m_{n-\pi_n+1}-1,\dots,m_n) + \sum_{i\in D(\pi)} r_{s_i(\pi)}(m_1,\dots,m_n).$$

(When $\pi_n=n$, the first term of the right part should be interpreted as $r_{\tau(\pi)}(m_1-1,m_2,\dots,m_n)$.)

The exact meaning of $r_\pi(m_1,\dots,m_n)$ will become clear later; we will have to change the shape of the cells, so that they coincide with cosets isomorphic to the parabolic subgroup A_{n-1}.

To begin with, recall the cells of Section 3. Every cell is bounded by n pairs of hyperplanes, each pair parallel to one of the bounding hyperplanes $H_1^x,\dots,H_n^x$ of the cone. From now on we omit the superindex x, since the combinatorics is invariant over different cones. We may number the hyperplanes bounding the cells, such that cell $(m_1,\dots,m_n)$ is bounded by the pairs $(H_{1,m_1},H_{1,m_1+1}),\dots,(H_{n,m_n},H_{n,m_n+1})$. Each cell contains exactly one alcove of each type, in total $n!$ alcoves. A cell $(m_1,\dots,m_n)$ with all $m_i\geqslant 1$ is an *inner cell*; all inner cells are properly contained in the truncated cone. We shall distinguish a special alcove in every cell: the *apex alcove* with n walls coinciding with $H_{1,m_1},\dots,H_{n,m_n}$. We will say that a wall of an alcove C leads *down* if it separates C from C_e; otherwise it leads *up*.

Lemma 14. *The apex alcove of every inner cell has n walls that leads down (viz. the walls coinciding with $H_{1,m_1},\dots,H_{n,m_n}$).*

Proof. Obvious. □

In the following, C will denote the apex alcove of the generic inner cell $(m_1,\dots,m_n)$. Relabel the walls of C such that s_i is the label of the wall parallel with H_i for $i=1,\dots,n$, and label the last wall by s_0. (Observe that this will be related to the original labeling such that for some fixed k, $s_i = \sigma_{i+k}$ for all i modulo $n+1$.) We can now state the word for a minimal gallery from C to the apex alcove of $(m_1,\dots,m_i-1,\dots,m_n)$ for any i, that is, the word for walking one step in the negative ith direction.

Lemma 15. *From an apex alcove C, a minimal gallery to the first apex alcove C' reached by walking in the negative ith direction is*

$$\langle \hat{w}_{i,0}\hat{w}_0\rangle = \langle(\hat{w}_{1,0}\hat{w}_0)^i\rangle = \langle(s_1s_2\dots s_n)^i\rangle.$$

Proof. Since all vertices are of origin-type, successive alcoves along a line will be of the same type. Hence, C' and C are successive alcoves along the line L_i. By Lemma 9, the gallery between them, in the negative direction, is $\langle\hat{w}_{i,0}\hat{w}_0\rangle$. This is an element of the parabolic subgroup generated by s_1 through s_n, which is isomorphic to S_{n+1}. Write permutations on one-line form $\pi_1\dots\pi_{n+1}$. Then $\hat{w}_0$ is the reverse permutation $(n+1)n(n-1)\dots1$, while $\hat{w}_{i,0}$ is the permutation $i(i-1)\dots1(n+1)n\dots(i+1)$. Composing these, we get that $\hat{w}_{i,0}\hat{w}_0$ is the permutation $(i+1)\dots n(n+1)1\dots(i-1)i$, which is equivalent to rotation i steps. Rotation one step can be expressed as $s_1s_2\dots s_n$. The statement follows. □

Lemma 16. *The unique periodic word for walking in the 1st negative direction from an apex alcove C is*

$$s_1s_2\dots s_ns_0s_1s_2\dots s_ns_0\dots$$

In particular, $s_is_{i+1}\dots s_ns_0s_1\dots s_{i-2}$ is always a type-preserving gallery.

Proof. As noted in the lemma above, the first type-preserving step in the negative 1st direction is $\hat{w}_{1,0}\hat{w}_0 = s_1s_2\dots s_n$, and this is the unique reduced word. As a permutation $\hat{w}_0s_1\hat{w}_0$ works by first reversing order, then switching the first two letters, and finally reversing order again. This results in a transposition of the two last letters, that is, $\hat{w}_0s_1\hat{w}_0=s_n$. Thus, Lemma 9 gives that the next type-preserving step in the 1st direction is $\hat{w}_{0,n}\hat{w}_n=s_0s_1\dots s_{n-1}$, and by symmetry the next type-preserving step is $s_ns_0s_1\dots s_{n-2}$, etc. □

Define the A_{n-1}-*block* of C to be the set of alcoves containing C and corresponding to the parabolic subgroup generated by $\{s_1,\dots,s_{n-1}\}$. Our wish is to use A_{n-1}-blocks instead of cells when indexing alcoves. We must show the following:

Lemma 17. *Every alcove lies in the A_{n-1}-block of exactly one apex alcove. Every A_{n-1}-block contains exactly one alcove of each type.*

Proof. We know that there are as many alcoves, $n!$, in an A_{n-1}-block as there are types. We shall show that every neighbor alcove of a block has the same type as one alcove within the block. This suffices, because the complex looks the same from the viewpoint of any alcove of the same type, so the neighbor alcove must also lie in an A_{n-1}-block.

The wall between the block and the neighbor must be labeled s_0 or s_n. Suppose the label is s_0. Then the gallery $s_0 s_1 \ldots s_{n-1}$ from the neighbor first leads into the block and then remains inside it (since all walls labeled s_1 through s_{n-1} are in the interior of the block), and by the previous lemma, this gallery is type-preserving. For the case when the wall is labeled s_n, use the type-preserving gallery $s_n s_{n-1} \ldots s_1$. □

Observe that in its A_{n-1}-block, the apex alcove C will be the alcove farthest from the fundamental alcove. We will regard the elements of an A_{n-1}-block as permutations in S_n, with the apex alcove being the reverse permutation $n(n-1)\ldots 1$. Thus we may index any alcove by ${}_{\pi}(m_1,\ldots,m_n)$, signifying the alcove that corresponds to permutation π in the A_{n-1}-block of the apex alcove of cell $(m_1,\ldots,m_n)$. The isomorphism between A_{n-1} and S_n gives that a wall $s_i, i=1,2,\ldots,n-1$, leads down from the alcove corresponding to permutation $\pi = \pi_1\pi_2\ldots\pi_n$ if and only if i is in the descent set of π, that is, $\pi_i > \pi_{i+1}$. But what about walls s_0 and s_n?

Lemma 18. *Let C be the apex alcove of an inner cell. For all alcoves in its A_{n-1}-block, the wall s_n leads down and the wall s_0 leads up.*

Proof. From C we know that walls s_1 through s_n lead down. It follows from Lemma 2 that from all alcoves that can be reached from C by walking down via galleries that do not contain s_n, the wall s_n will lead down. All alcoves in the block can be reached by such galleries (since, in the block, C is the alcove farthest from the fundamental alcove), hence s_n leads down in the entire block. By symmetry, s_0 must lead up for every alcove in the block. □

As noted in Lemma 16, for every $i=0,1,\ldots,n$ the n-letter word $s_i s_{i+1}\ldots s_n s_0 \ldots s_{i-2}$ is type-preserving. Suppose C_1 is an alcove, C_2 its neighbor via wall s_j, and C_1' is the alcove of the same type as C_1 reached by $s_i s_{i+1}\ldots s_n s_0 \ldots s_{i-2}$, with corresponding neighbor C_2'. What is the type-preserving gallery between C_2 and C_2'? By combining Lemmas 10 and 16, we get that $s_i s_{i+1}\ldots s_n s_0 \ldots s_{i-2}$ is transformed to

$$\begin{cases} s_{i+1}s_{i+2}\ldots s_n s_0 \ldots s_{i-1} & \text{if } j=i, \\ s_{i-1}s_i \ldots s_n s_0 \ldots s_{i-3} & \text{if } j=i-1, \\ s_i s_{i+1}\ldots s_n s_0 \ldots s_{i-2} & \text{otherwise.} \end{cases}$$

By the notation $(m_1,\ldots,m_{i-1}+1,m_i-1,\ldots,m_n)$ we mean an n-tuple where the jth entry is m_j for all $j \neq i-1, i$.

Lemma 19. *$s_i s_{i+1} \dots s_n s_0 \dots s_{i-2}$ is a minimal gallery from the apex alcove $(m_1, \dots, m_n)$ to the apex alcove $(m_1, \dots, m_{i-1} + 1, m_i - 1, \dots, m_n)$, for $i = 1, \dots, n$.*

Proof. First observe that walking $s_1 s_2 \dots s_n$ transforms

$$s_i s_{i+1} \dots s_n s_0 \dots s_{i-2} \text{ to } s_{i-1} s_i \dots s_n s_0 \dots s_{i-3};$$

the part $s_1 \dots s_{i-2}$ has no effect, the s_{i-1} does the transformation, and $s_i \dots s_n$ has no effect. Hence, walking $(s_1 s_2 \dots s_n)^{i-1}$ lowers the indices $i - 1$ times, so the word is transformed to $s_1 s_2 \dots s_n$. We know from Lemma 15 that $(s_1 s_2 \dots s_n)^i$ walks one block in the negative ith direction, and $(s_1 s_2 \dots s_n)^{i-1}$ walks one block in the negative $(i-1)$th direction. Since $(s_1 s_2 \dots s_n)^{i-1}(s_1 s_2 \dots s_n) = (s_1 s_2 \dots s_n)^i$, we have shown that the transformed gallery increases the $(i-1)$th index by one and decreases the ith index by one. □

For convenience, let us restate Theorem 13: With $D(\pi)$ the descent set of $\pi \in S_n$, and τ the rotation operator defined by $\tau(\pi_1 \pi_2 \dots \pi_n) = \pi_n \pi_1 \dots \pi_{n-1}$, that is, $\tau = s_{n-1} s_{n-1} \dots s_1$, we have the recurrence

$$r_\pi = \partial_{n-\pi_n}^{-1} \partial_{n+1-\pi_n} r_{\tau(\pi)} + \sum_{i \in D(\pi)} r_{s_i(\pi)}.$$

Proof. Let C_1 be the alcove of type $\pi = \pi_1 \pi_2 \dots \pi_n$ in block $(m_1, \dots, m_n)$. The number of minimal galleries from C_1 to C_e is the sum of such numbers for the neighbors of C_1 through walls leading down. These are the walls labeled s_i for every descent $i \in D(\pi)$, leading to the alcove of type $s_i(\pi)$ in the same block; and furthermore we have the wall labeled s_n leading to, say, C_2' outside the block. Let C_2 be the alcove reached from C_2' by the type-preserving gallery $s_n s_{n-1} \dots s_1$. Thus, C_2 is reached from C_1 by $s_{n-1} \dots s_1 = \tau$, so C_2, and hence C_2', has type $\tau(\pi) = \pi_n \pi_1 \dots \pi_{n-1}$.

Finally, we must show that C_2' is in the block $(m_1, \dots, m_{n-\pi_n} - 1, m_{n-\pi_n+1} + 1, \dots, m_n)$. Let C be the apex alcove of block $(m_1, \dots, m_n)$. Then C, being of type $n(n-1) \dots 1$, can be reached from C_2 by a gallery $\langle v \rangle s_1 s_2 \dots s_{n-\pi_n}$, where $\langle v \rangle$ sorts $\pi_1 \dots \pi_{n-1}$ in decreasing order (never using s_1), and then $s_1 s_2 \dots s_{n-\pi_n}$ puts π_n at its place, the $(n + 1 - \pi_n)$th position.

How is the word $s_1 \dots s_n$ (that takes us from C_2 to C_2') affected by the walk up to C? Well, $\langle v \rangle$, not using s_1 (nor, of course, s_0), does not affect the word $s_1 \dots s_n$, but $s_1 s_2 \dots s_{n-\pi_n}$ transforms the word to $s_{n+1-\pi_n} \dots s_n s_0 \dots s_{n-1-\pi_n}$. By Lemma 19, this is the word that takes C to the block $(m_1, \dots, m_{n-\pi_n} - 1, m_{n-\pi_n+1} + 1, \dots, m_n)$, hence C_2' is in this block. □

6.1. *Examples: $\tilde{A}_2$*

In $\tilde{A}_2$, the set of types can be identified with $S_2 = \{12, 21\}$. Theorem 13 gives the system of recurrences

$$\begin{aligned} r_{12}(m_1, m_2) &= r_{21}(m_1 - 1, m_2), \\ r_{21}(m_1, m_2) &= r_{12}(m_1 + 1, m_2 - 1) + r_{12}(m_1, m_2). \end{aligned} \tag{4}$$

Elimination down to one single recurrence yields, for both types, $r(m_1,m_2) = r(m_1,m_2-1)+r(m_1-1,m_2)$. Together with the boundary values, we obtain the binomial coefficients: $r(m_1,m_2) = \binom{m_1+m_2}{m_1}$.

6.2. Examples: $\tilde{A}_3$

In $\tilde{A}_3$, the set of types can be identified with $S_s = \{123, 132, 213, 231, 312, 321\}$. Theorem 13 gives the system of recurrences

$$\begin{aligned}
r_{123} &= \partial_1 r_{312},\\
r_{132} &= \partial_1^{-1}\partial_2 r_{213} + r_{123},\\
r_{213} &= \partial_1 r_{321} + r_{123},\\
r_{231} &= \partial_2^{-1}\partial_3 r_{123} + r_{213},\\
r_{312} &= \partial_1^{-1}\partial_2 r_{231} + r_{132},\\
r_{321} &= \partial_2^{-1}\partial_3 r_{132} + r_{231} + r_{312}.
\end{aligned} \tag{5}$$

Elimination down to one single recurrence yields, for all types,

$$r = (2\partial_1 + 4\partial_2 + 2\partial_3 - \partial_1^2 - \partial_3^2 + \partial_1\partial_3)r.$$

Remark 2. The original motivation for the author to study this problem was the following. A combinatorial one-player game, called the *numbers game*, is known to model Coxeter groups in a certain sense, see for example Eriksson's thesis [2]. In particular, to every position in the game corresponds a Coxeter group element, and the question of how many different ways there are of playing the game from a given position is equivalent to the question of how many reduced words there are for the corresponding Coxeter group element.

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