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c-map, very special quaternionic geometry and dual Kähler spaces

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Abstract

We show that for all very special quaternionic manifolds a different $N = 1$ reduction exists, defining a Kähler geometry which is "dual" to the original very special Kähler geometry with metric $G_{a\bar{b}} = -\partial_a \partial_b \ln V$ ($V = (1/6)d_{abc}\lambda^a\lambda^b\lambda^c$). The dual metric $g^{ab} = V^{-2}(G^{-1})^{ab}$ is Kähler and it also defines a flat potential as the original metric. Such geometries and some of their extensions find applications in type IIB compactifications on Calabi–Yau orientifolds.

1. Isometries of dual quaternionic manifolds

One of the basic constructions in dealing with the low energy effective Lagrangians of type IIA and type IIB superstrings is the so-called c-map [1], which associates to any special Kähler manifold of complex dimension n a "dual" quaternionic manifold of quaternionic dimension $n_H = n + 1$.

In particular it was shown [2] that "dual" quaternionic manifolds always have at least $2n + 4$ isometries: one scale isometry ϵ_0 and $2n + 3$ shift isometries β_I, α^I, ϵ_+ $(I = 0, \dots, n)$, whose generators close a Heisenberg algebra [3]:

$$[\beta^I, \epsilon^+] = [\alpha_I, \epsilon^+] = 0, \qquad [\beta^I, \alpha_J] = \delta^I_J \epsilon^+,$$
$$[\epsilon^0, \alpha_I] = \frac{1}{2}\alpha_I, \qquad [\epsilon^0, \beta^I] = \frac{1}{2}\beta^I,$$
$$[\epsilon^0, \epsilon^+] = \epsilon^+. \tag{1.1}$$

The corresponding generators can be written according to their ϵ^0 weight as [4–7]:

$$\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_{1/2} + \mathcal{V}_1. \tag{1.2}$$

However, it was shown in [6,7] that when the special Kähler manifold has some isometries, then some "hidden symmetries" are generated in the c-map spaces which are classified by $\mathcal{V}_{-1}$, $\mathcal{V}_{-1/2}$, with

$$\dim(\mathcal{V}_{-1}) \leqslant 1, \qquad \dim(\mathcal{V}_{-1/2}) \leqslant 2n + 2. \tag{1.3}$$

In particular, for a generic very special geometry, with a cubic polynomial prepotential

$$F(z) = \frac{1}{48} d_{abc} z^a z^b z^c \tag{1.4}$$

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with generic d_{abc}, with no additional isometries, it was shown that:

$$\dim(\mathcal{V}_{-1}) = 0, \qquad \dim(\mathcal{V}_{-1/2}) = 1,$$
$$\dim(\mathcal{V}_0) = n + 2. \tag{1.5}$$

Since the isometries of a generic very special geometry of dimension n are $n+1$, the dual manifold has then $3n+6$ isometries, where the $n+2$ additional isometries lie, $n+1$ in $\mathcal{V}_0$, denoted by ω_I ($I = 0, \dots, n$), and one $\hat{\beta}_0$ in $\mathcal{V}_{-1/2}$. For symmetric spaces the upper bound in Eq. (1.3) is saturated so that $\dim G_{\mathrm{Q}} = \dim G_{\mathrm{SK}} + 4n + 7$ where G_{SK} and G_{Q} are the isometry groups of the special Kähler and quaternionic spaces, respectively.

2. The very special σ-model Lagrangian and its $N = 1$ reduction

The quaternionic "dual" σ-model for a generic special geometry was derived in [2] by dimensional reduction of a $N = 2$ special geometry to three dimensions. By adapting the conventions of [2] to those of [6] and [8] we call the special coordinates z^a as $z^a = x^a + iy^a$ and define:

$$V = \frac{1}{6}(\kappa yyy) \equiv \frac{1}{6}\kappa, \qquad (\kappa yyy) = d_{abc} y^a y^b y^c,$$
$$\kappa_a = d_{abc} y^b y^c, \qquad \kappa_{ab} = d_{abc} y^c. \tag{2.1}$$

The $2n+4$ additional coordinates are denoted by $\zeta^I \equiv (\zeta^0, \zeta^a)$, $\tilde{\zeta}_I \equiv (\tilde{\zeta}_0, \tilde{\zeta}_a)$, D, $\tilde{\Phi}$.

The α^I, β_I isometries act as shifts on the $2n+2$ coordinates ζ^I, $\tilde{\zeta}_I$:

$$\delta\zeta^I = \alpha^I, \qquad \delta\tilde{\zeta}_I = \beta_I \tag{2.2}$$

while the ω^a shift isometries of the special geometry, $\delta x^a = \omega^a$, act as duality rotations on the ζ^I, $\tilde{\zeta}_I$ symplectic vector:

$$\delta \begin{pmatrix} \zeta \\ \tilde{\zeta} \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & -A^T \end{pmatrix} \begin{pmatrix} \zeta \\ \tilde{\zeta} \end{pmatrix} \tag{2.3}$$

with

$$A = \begin{pmatrix} 0 & 0 \\ \omega^a & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 0 & 3d_{abc}\omega^c \end{pmatrix}. \tag{2.4}$$

On the other hand the $\hat{\beta}_0$ isometry rotates ζ^a into x^a and the x^a, $\tilde{\zeta}_a$ variables are related by quaternionic isometries. It is immediate to see that the full σ-model Lagrangian [2,6,8] is invariant under the following parity operation Ω:

$$y^a \to y^a, \qquad \tilde{\zeta}_a \to \tilde{\zeta}_a, \qquad \zeta^0 \to \zeta^0,$$
$$D \to D, \tag{2.5}$$

$$x^a \to -x^a, \qquad \zeta^a \to -\zeta^a, \qquad \tilde{\zeta}_0 \to -\tilde{\zeta}_0,$$
$$\tilde{\Phi} \to -\tilde{\Phi} \tag{2.6}$$

so that, restricting to the plus-parity sector is a consistent truncation, giving rise to the following Lagrangian for $2n+2$ (real) variables:

$$\begin{aligned}(\sqrt{-g}\,)^{-1}\mathcal{L} = {} & -(\partial_\mu D)^2 - \frac{1}{4} G_{ab} \partial_\mu y^a \partial^\mu y^b \\ & - \frac{1}{8} e^{2D} V \big(\partial_\mu \zeta^0\big)^2 \\ & - 2e^{2D} V^{-1} \big(G^{-1}\big)^{ab} \partial_\mu \tilde{\zeta}_a \partial^\mu \tilde{\zeta}_b, \end{aligned} \tag{2.7}$$

where $G_{ab} = -\partial_a \partial_b \log V$. By a change of variables we can decouple the (D, ζ^0) fields from the rest as follows: define two new variables (Φ, λ^a):

$$V(y)e^{2D} = e^{2\Phi}, \qquad y^a = \lambda^a e^{\Phi/2}. \tag{2.8}$$

Thus it follows that $V(\lambda)e^{2D} = e^{\Phi/2}$ and the Lagrangian becomes:

$$\begin{aligned}(\sqrt{-g}\,)^{-1}\mathcal{L} = {} & -\frac{1}{4}(\partial_\mu \Phi)^2 - \frac{1}{8} e^{2\Phi} \big(\partial_\mu \zeta^0\big)^2 \\ & - \frac{1}{4} G_{ab} \partial_\mu \lambda^a \partial^\mu \lambda^b \\ & - \frac{1}{4} \big(\partial_\mu \log V(\lambda)\big)^2 \\ & - 2V(\lambda)^{-2} \big(G^{-1}\big)^{ab} \partial_\mu \tilde{\zeta}_a \partial^\mu \tilde{\zeta}_b \end{aligned} \tag{2.9}$$

the (Φ, ζ^0) part defines a $\mathrm{SU}(1,1)/\mathrm{U}(1)$ σ-model.

The coefficient of the two terms in the $\partial_\mu \lambda^a \partial^\mu \lambda^b$ part combine into $-(3/2)(\kappa_{ab}/\kappa - 3\kappa_a\kappa_b/\kappa^2)$. We now define a new variable $t_a = (1/2)\kappa_{ab}\lambda^b$ such that $d\lambda^b = (\kappa^{-1})^{ba} t_a$ we obtain that

$$\begin{aligned} g^{ab} &= -6\left(\frac{\kappa_{cd}}{\kappa} - 3\frac{\kappa_c\kappa_d}{\kappa^2}\right)\big(\kappa^{-1}\big)^{ac}\big(\kappa^{-1}\big)^{bd} \\ &= -\frac{6}{\kappa^2}\big[\big(\kappa^{-1}\big)^{ab}\kappa - 3\lambda^a\lambda^b\big] = \frac{36}{\kappa^2}\big(G^{-1}\big)^{ab}. \end{aligned} \tag{2.10}$$

Therefore in the $(t_a, \tilde{\zeta}_a)$ variables we finally get

$$(\sqrt{-g})^{-1}\mathcal{L} = -\frac{1}{4}(\partial_\mu \Phi)^2 - \frac{1}{8}e^{2\Phi}\big(\partial_\mu \zeta^0\big)^2 - \frac{1}{4}g^{ab}\partial_\mu t_a \partial^\mu t_b - 2g^{ab}\partial_\mu \tilde{\zeta}_a \partial^\mu \tilde{\zeta}_b. \tag{2.11}$$

Therefore by defining the complex variables

$$\eta_a = t_a + 2\sqrt{2} i \tilde{\zeta}_a \tag{2.12}$$

we get for the $2n$-dimensional σ-model:

$$-\frac{1}{4}g(\Re\eta)^{ab}\big(\partial_\mu \Re\eta_a \partial^\mu \Re\eta_b + \partial_\mu \Im\eta_a \partial^\mu \Im\eta_b\big) = -\frac{1}{4}g^{ab}\partial_\mu \eta_a \partial^\mu \bar{\eta}_b. \tag{2.13}$$

The previous Lagrangian is Kähler provided

$$g(t)^{ab} = \frac{\partial^2 \hat{K}}{\partial t_a \partial t_b}. \tag{2.14}$$

This condition is achieved by setting $\hat{K} = -2\log V(\lambda)$. Indeed

$$\frac{\partial}{\partial t_a}\log V = \big(\kappa^{-1}\big)^{ac}\frac{\partial}{\partial \lambda^c}\log V = 3\frac{\lambda^a}{\kappa},$$
$$\frac{\partial^2}{\partial t_a \partial t_b}\log V = 3\left[\frac{(\kappa^{-1})^{ab}}{\kappa} - 3\frac{\lambda^a\lambda^b}{\kappa^2}\right] = -\frac{1}{2}\times\frac{36}{\kappa^2}\big(G^{-1}\big)^{ab}. \tag{2.15}$$

3. Isometries of the $N = 1$ reduction

The σ-model isometries of the c-map, using the notations of [7] are parametrized by

$$\epsilon^+,\ \epsilon^0,\ \alpha^I,\ \beta_I,\ \omega^a,\ \omega^0,\ \hat{\beta}_0. \tag{3.1}$$

The $N = 1$ reduction projects out ϵ^+, α^a, β_0, ω^a, so the remaining isometries are $n + 4$, namely:

$$\beta_a,\ \omega^0,\ \epsilon^0,\ \alpha^0,\ \hat{\beta}_0. \tag{3.2}$$

Three of the latter generate a $\mathrm{SL}(2,\mathbb{R})$ symmetry (otherwise absent in generic dual quaternionic manifolds), the others generate a shift symmetry in $\Im\eta_a$ and a scale symmetry in the η_a variables. The dual manifold has the same isometries of the original special Kähler. Even though the $\tilde{\zeta}_a$ variables are related to the x^a variables by quaternionic isometries, the two manifolds are in general distinct. However, in the particular case of homogeneous-symmetric spaces [9], it turns out that the dual manifold coincide with the original one. The proof of this statement will be given elsewhere.

4. Connection with Calabi–Yau orientifolds

The c–map was originally studied in relation to the type II A $\to$ type II B mirror map in Calabi–Yau compactifications. In Calabi–Yau orientifolds of type II B strings with D-branes present, the bulk Lagrangian is obtained combining a world-sheet parity with a manifold parity which, for generic spaces [10], is precisely doing the truncation we have encountered in this note.

For certain Calabi–Yau manifolds more generic orientifoldings are possible where the set of special coordinates z^A is separated in two parts with opposite parity, $z^A_\pm$ $(n_+ + n_- = n)$ such that [11]

$$y_\pm \to \pm y_\pm,\qquad x_\pm \to \mp x_\pm \tag{4.1}$$

and then consequently

$$\zeta_\pm \to \mp\zeta_\pm,\quad \zeta^0 \to \zeta_0,\quad \tilde{\zeta}_\pm \to \pm\tilde{\zeta}_\pm,\quad \tilde{\zeta}_0 \to -\tilde{\zeta}_0. \tag{4.2}$$

However, in this case one must demand

$$d_{++-} = d_{---} = 0 \tag{4.3}$$

in order for the $N = 1$ reduction to be consistent [12].

In this case the σ-model Lagrangian acquires more terms and can be symbolically written as:

$$(\sqrt{-g})^{-1}\mathcal{L} = -(\partial D)^2 - \frac{1}{4}G_{++}(\partial y_+) - \frac{1}{4}G_{--}(\partial x_-)^2 - \frac{1}{8}e^{2D}V\big(\partial\zeta^0\big)^2 - \frac{1}{8}e^{2D}VG_{--}\big(x_-\partial\zeta^0 - \partial\zeta_-\big)^2 - 2e^{2D}V^{-1}\big(G^{-1}\big)^{++} \times\left(\partial\tilde{\zeta}_+ + \frac{1}{8}d_{+--}x_-x_-\partial\zeta^0 - \frac{1}{4}d_{+--}x_-\partial\zeta_-\right)^2, \tag{4.4}$$

where for the sake of simplicity space–time indices have been suppressed from partial derivatives and contraction over them is understood. In (4.4) G_{++} is as before since $d_{+++} \neq 0$, $G_{+-} = 0$ and $G_{--} = -6(d_{--+}y_+)/(d_{+++}y_+y_+y_+)$.

The total set of coordinates are: y_+, x_-, ζ_-, $\tilde{\zeta}_+$ and (Φ, ζ^0). Since in this case some of the y coordinates, namely y_-, have been replaced by x_-, the new variables define a Kähler manifold of complex dimension $n+1$ certainly distinct from the original one.

There is an $N=4$ analogue of this dual $N=1$ geometries if we consider different embeddings of $N=4$ supergravity into $N=8$. This corresponds to type II B on $T^6/\mathbb{Z}_2$ orientifold with D3- or D9-branes (type I string) or Heterotic string on T^6. In all these cases the bulk sector corresponds to $[\mathrm{SO}(6,6)/\mathrm{SO}(6)\times\mathrm{SO}(6)]\times[\mathrm{SU}(1,1)/\mathrm{U}(1)]$ σ-model but the 15 axions in $\mathrm{SO}(6,6)/\mathrm{SO}(6)\times\mathrm{SO}(6)$ are coming from C_4, C_2, B_2 [13–19].

Also cases in which a further splitting appears are realized if the orientifold projection acts [18] differently on $T^{p-3}\times T^{9-p}$ $(p=3,5,7,9)$. This is the analogue of the $y_\pm$, $x_\pm$ splitting [11]. In all these cases the dual manifolds coincide, as predicted by $N=4$ supergravity.

5. Properties of the dual special Kähler spaces and no-scale structure

The dual Kähler space, obtained by a $N=1$ truncation of the (c-map) very special quaternionic space has a metric that satisfies a "duality" relation with the original very special Kähler space:

$$g_D^{ab} = \frac{1}{V^2}\left(G^{-1}\right)^{ab}. \tag{5.1}$$

Moreover it can be shown that its affine connection is simply related to the affine connection of original Kähler space:

$$\Gamma_d^{Dbc} = \frac{1}{V}\left(G^{-1}\right)^{ca}\Gamma_{ad}^b. \tag{5.2}$$

Actually in the one-dimensional case the two connections coincide.

These dual spaces are also no-scale [20–22]. Indeed it is sufficient to prove that

$$\frac{\partial\hat{K}}{\partial\Re\eta_a}\left(g^{-1}\right)_{ab}\frac{\partial\hat{K}}{\partial\Re\eta_b} = 3. \tag{5.3}$$

But this is indeed the case since

$$\lambda^a G_{ab}\lambda^b = 3. \tag{5.4}$$

From a type II B perspective, this was anticipated in [23].

6. Concluding remarks

In this note we have shown that for an arbitrary very special geometry, through the c-map, it is possible to construct a "dual" Kähler geometry which has a dual metric, it is Kähler and it provides a dual no-scale potential. Recently such constructions have found applications in Calabi–Yau orientifolds [11,24] but the procedure considered here is intrinsic to the four dimensional context.

We have not shown that the final Lagrangian is supersymmetric but, using the reduction techniques of [12], it can be shown that this is indeed the case. It is reassuring that the $\mathrm{SL}(2,\mathbb{R})$ symmetry, related to the type II B interpretation, comes out in a pure four-dimensional context, thanks to the results of [6,7].

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