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Linear Systems of Plane Curves through Fixed "Fat" Points of $\mathbb{P}^2$

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Given any s points $P_1, ..., P_s$ in $\mathbb{P}^2$ and s positive integers $m_1, ..., m_s$, let S_n be the linear system of plane curves of degree n through P_i with multiplicity at least m_i $(1 \leqslant i \leqslant s)$. We give numerical bounds for the regularity of S_n in the following cases (a) the points P_i are non-singular points of an integral curve of degree d; (b) the P_i's are in general position; (c) the P_i's are in uniform position; (d) the P_i's are generic points of $\mathbb{P}^2$. We also study the sharpness of such bounds.

1. INTRODUCTION

In the present paper we are concerned with linear systems of plane curves through fixed base points with given multiplicities, focusing on their regularity.

The subject has been dealt with by several authors for over 100 years. Among the classical geometers we wish to mention are G. Castelnuovo [C] in 1891, F. Severi [Se] in 1926, and in later years B. Segre and M. Nagata.

B. Segre [S1] in 1961 provided sufficient conditions for the regularity of a linear system, when the base points are generic distinct points. He made extensive use of the "characteristic series" and "specialization." We remark that some results of his link the regularity of a linear system to its irreducibility, hence to the existence of an integral curve with preassigned singularities. M. Nagata [N1, N2] in 1960 made use of Cremona transformations to study linear systems of plane curves and rational surfaces.

Because of meaningful connections to other subjects, the study of linear systems, far from being exhausted, still engages mathematicians today.

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Among them, we mention E. Davis, A. Geramita, S. Greco, B. Harbourne, and A. Gimigliano; our aim is to generalize or to improve some of the results of these authors.

To be more explicit, let $\mathscr{P} = (P_1, ..., P_s)$ be an s-tuple of distinct points of $\mathbb{P}^2$, $\mathscr{M} = (m_1, ..., m_s)$ an s-tuple of non-negative integers, and let $S_n(\mathscr{P}, \mathscr{M})$ denote the linear system of plane curves of degree n passing through P_i with multiplicity at least m_i $(1 \leqslant i \leqslant s)$.

Set $\tau = \min\{n \in \mathbb{N}: S_n(\mathscr{P}, \mathscr{M}) \text{ is regular}\}$.

In Section 3, we construct the main tools, namely Theorem 3.1 and Corollary 3.2, which are repeatedly used in the sequel. In Theorem 3.1, we get inequalities involving the superabundance of $S_n(\mathscr{P}, \mathscr{M})$. The results of Corollary 3.2 provide an inductive process to lower the multiplicities.

In Section 4, under the assumption that $P_1, ..., P_s$ be nonsingular points of an integral curve of degree d, we find (Theorem 4.1) a numerical bound for τ. This generalizes a result in [H1], where the case $d = 3$ is analysed, and a result in [Gi], where it is assumed that the points impose independent conditions on curves of degree d.

In Section 5, the sharpness of this bound is investigated and related to an open problem concerning complete intersections.

In Section 6, $P_1, ..., P_s$ are assumed to be in general position (Definition 6.1). There we get a bound for τ (Theorem 6.2), possibly known to B. Segre, which improves that of [G], generalizes Propositions 5.3 and 5.4 of [DG], and extends to arbitrary characteristic an analogous result proved in [Ca] in characteristic zero by different means. The bound is given by $\tau \leqslant t$, where $t = \max(m_1 + m_2 - 1;\ [\sum_1^s m_i/2])$ $(m_1 \geqslant \cdots \geqslant m_s)$. Studying the sharpness of the bound leads to the following result: if the P_i's lie on a non-singular conic, then $S_{t-1}(\mathscr{P}, \mathscr{M})$ is not regular. We then examine under which assumptions the converse is true (Theorems 6.3 and 6.4).

In Section 7, we get bounds (Theorems 7.3 and 7.7) for τ when $P_1, ..., P_s$ are either points in uniform position (Definition 7.1) or generic points of $\mathbb{P}^2$ (Definition 7.6).

In this note we use techniques and language from both classical geometry and cohomology. In contrast to the methods of [H1, Gi] we work entirely in $\mathbb{P}^2$ rather than on various blow-ups of $\mathbb{P}^2$.

In the next section we will fix some notation. For terminology and background not mentioned see [H].

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2. PRELIMINARIES AND NOTATION

Let $\mathbb{P}^2$ be a projective plane over an algebraically closed field k. We fix some notation. Let:

$\mathcal{M} = (m_1, ..., m_s)$ be an s-tuple of non-negative integers;

$\mathcal{M} - k$ $(1 \leqslant k \leqslant s)$ be the s-tuple $(m'_1, ..., m'_s)$, where $m'_i = m_i - 1$ for $1 \leqslant i \leqslant k$ and $m_i \neq 0$; $m'_i = m_i$ otherwise;

$\mathcal{P} = (P_1, ..., P_s)$ be an s-tuple of distinct points of $\mathbb{P}^2$;

C denote an integral curve of $\mathbb{P}^2$, of degree d and arithmetic genus p_a. When $\mathcal{M}$, $\mathcal{P}$, and C are given, let $\mathcal{M} - C$ be the s-tuple $(m'_1, ..., m'_s)$, where $m'_i = m_i - 1$ for $P_i \in C$ and $m_i \neq 0$; $m'_i = m_i$ otherwise.

Let $\mathfrak{p}_i$ be the prime ideal corresponding to P_i $(1 \leqslant i \leqslant s)$ in $R = k[X_0, X_1, X_2]$. $\mathfrak{a}$ will be the homogeneous ideal $\bigcap \mathfrak{p}_i^{m_i}$, Z the scheme defined by $\mathfrak{a}$, and $\mathcal{A}$ the sheaf associated to $\mathfrak{a}$.

For any integer $t > 0$, let tH be the linear system of all the curves of $\mathbb{P}^2$ of degree t; let $S_t(\mathcal{P}, \mathcal{M})$, S_t for short, be the linear system of those curves in $\mathbb{P}^2$ of degree t which have multiplicity at least m_i at P_i $(1 \leqslant i \leqslant s)$. ($S_t$ is the projective space associated to the vector space $H^0(\mathbb{P}^2, \mathcal{A}(t))$.) The *dimension*, the *virtual dimension*, and the *superabundance* h_t of S_t are defined by

$$\dim S_t := h^0(\mathbb{P}^2, \mathcal{A}(t)) - 1;$$

$$\text{vir.}\dim S_t := h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)) - h^0(\mathbb{P}^2, \mathcal{O}_z) - 1$$

$$= \frac{t(t+3)}{2} - \sum_1^s \frac{m_i(m_i+1)}{2};$$

$$h_t := \dim S_t - \text{vir.}\dim S_t. \tag{1}$$

We have by definition that S_t is *regular* iff $h_t = 0$. Let

$$\tau := \min\{t \in \mathbb{N} : S_t(\mathcal{P}, \mathcal{M}) \text{ is regular}\}.$$

From the exact sequence

$$0 \to \mathcal{A} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_z \to 0 \tag{2}$$

twisting by an integer n and taking cohomology, and using the well-known values of cohomology for $\mathbb{P}^2$, as well as $h^0(\mathbb{P}^2, \mathcal{O}_z(n)) = \sum_1^s m_i(m_i + 1)/2$, we get for $n \geqslant 0$,

$$h^1(\mathbb{P}^2, \mathcal{A}(n)) = h_n \qquad \text{and} \qquad h^2(\mathbb{P}^2, \mathcal{A}(n)) = 0, \tag{3}$$

(note that S_n is regular iff $h^1(\mathbb{P}^2, \mathcal{A}(n)) = 0$); for $n < 0$,

$$h^1(\mathbb{P}^2, \mathcal{A}(n)) = \sum_1^s \frac{m_i(m_i+1)}{2} \qquad \text{and} \qquad h^2(\mathbb{P}^2, \mathcal{A}(n)) = \binom{-n-1}{2}. \tag{4}$$

If S is a linear system of plane curves, let $S \cdot C$ denote the linear series of Cartier divisors cut out on C by the curves of S, and let $S^{\sim}$ denote the linear system of the curves of S containing C. We recall that

$$\dim S \cdot C = \dim S - \dim S^{\sim} - 1. \tag{5}$$

If σ is a linear series on C, and E is a Cartier divisor on C, let $\sigma - E$ denote the residual series. If D is a Cartier divisor on C and K is the canonical divisor on C, we say that $i(D) = h^0(C, \mathcal{O}_c(K - D))$ is the *index of speciality* of D. By the Serre duality Theorem, we have

$$i(D) = h^1(C, \mathcal{O}_c(D)). \tag{6}$$

If σ is a linear series on C and $D \in \sigma$, we write $i(\sigma) = i(D)$ for the index of speciality of σ.

3. Linear Systems Defined by Fixed "Fat" Points

The following theorems relate the superabundance of S_t to the degree d of a curve C containing the points $P_1, \ldots, P_k$ $(1 \leqslant k \leqslant s)$.

Theorem 3.1. *With notation as in Section 2, let $k \leqslant s$ be positive integers. Let $P_1, \ldots, P_k$ be distinct, non-singular points of an irreducible reduced curve C of degree d, and let P_i, $k < i \leqslant s$, be distinct points not on C. Set $\mathcal{P} = (P_1, \ldots, P_s)$. Let D be a Cartier divisor of $tH \cdot C$, $\mathcal{M} = (m_1, \ldots, m_s)$, and let E denote the Cartier divisor on C defined by $E = \sum_1^k m_i P_i$. Set $\sigma = tH \cdot C - E$, $i = i(D - E)$, the index of speciality of $D - E$ $(= i(\sigma)$ when $\sigma \neq \emptyset)$. Denote the superabundance of $S_t(\mathcal{P}, \mathcal{M})$ by h_t.*

(a) *If $t \geqslant d$ and h_{t-d} is the superabundance of $S_{t-d}(\mathcal{P}, \mathcal{M} - k)$, then*

$$i \leqslant h_t \leqslant i + h_{t-d}.$$

(b) *If $t < d$, then*

$$i \leqslant h_t + \frac{(d-t-1)(d-t-2)}{2} \leqslant i + \sum_{i \leqslant k} \frac{m_i(m_i - 1)}{2} + \sum_{k < i \leqslant s} \frac{m_i(m_i + 1)}{2}.$$

Proof. We may assume $m_i > 0$ for every i. Let $\mathfrak{p}_i$ be the prime ideal corresponding to P_i $(1 \leqslant i \leqslant s)$ in $R = \mathscr{k}[X_0, X_1, X_2]$, let $\mathfrak{a} = \bigcap \mathfrak{p}_i^{m_i}$, $\mathfrak{b} = (\bigcap_{i \leqslant k} \mathfrak{p}_i^{m_i - 1}) \cap (\bigcap_{k < i \leqslant s} \mathfrak{p}_i^{m_i})$; let f be a polynomial defining C. $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous ideals of R: let $\mathcal{A}$ and $\mathcal{B}$ be the sheaves on $\operatorname{Proj}(R)$

associated to $\mathfrak{a}$ and $\mathfrak{b}$. Since R is an integral domain, multiplication by f gives a short exact sequence

$$0 \to \mathcal{B}(-d) \to \mathcal{A} \to \mathcal{F} \to 0. \tag{7}$$

Let $\mathfrak{q}_i = \mathfrak{p}_i \mathcal{O}_{\mathbb{P}^2,P_i}$ and let $f^\sim$ be a local equation of C at P_i. For $1 \leqslant i \leqslant k$, $\mathcal{O}_{\mathbb{P}^2,P_i}$ is a regular local ring, $\mathfrak{q}_i$ is its maximal ideal, and $f^\sim \in \mathcal{O}_{\mathbb{P}^2,P_i}$ is a regular parameter. It follows that $f^\sim \mathfrak{q}_i^{m_i-1} = (f^\sim) \cap \mathfrak{q}_i^{m_i}$, hence $\mathcal{F}_{P_i} = \mathfrak{q}_i^{m_i}/f^\sim \mathfrak{q}_i^{m_i-1} = \mathfrak{q}_i^{m_i}/(f^\sim) \cap \mathfrak{q}_i^{m_i} = (\mathfrak{q}_i^{m_i} + (f^\sim))/(f^\sim)$. It follows easily that $\mathcal{F}$ is canonically isomorphic to $\mathcal{O}_c(-E)$, i.e., to the sheaf of ideals of the subscheme $E = \sum_1^k m_i P_i$ of C.

Twisting by t the sequence (7) and taking cohomology, we obtain a long exact sequence

$$\cdots \to H^1(\mathbb{P}^2, \mathcal{B}(t-d)) \to H^1(\mathbb{P}^2, \mathcal{A}(t)) \to H^1(\mathbb{P}^2, \mathcal{F}(t))$$
$$\to H^2(\mathbb{P}^2, \mathcal{B}(t-d)) \to H^2(\mathbb{P}^2, \mathcal{A}(t)) \to \cdots.$$

Now $H^i(\mathbb{P}^2, \mathcal{F}(t)) = H^i(C, \mathcal{F}(t))$ and $\mathcal{F}(t) \simeq \mathcal{O}_c(D-E)$. Thus from (6) of Section 2, we have $h^1(\mathbb{P}^2, \mathcal{F}(t)) = i(D-E) = i$. From (3) of Section 2, we get $h^1(\mathbb{P}^2, \mathcal{A}(t)) = h_t$.

Moreover, for $t \geqslant d$, using (3) of Section 2, we have $h^1(\mathbb{P}^2, \mathcal{B}(t-d)) = h_{t-d}$ and $h^2(\mathbb{P}^2, \mathcal{B}(t-d)) = 0$.

For $t < d$, using (4) of Section 2, we have $h^1(\mathbb{P}^2, \mathcal{B}(t-d)) = \sum_{i \leqslant k} m_i(m_i-1)/2 + \sum_{k < i \leqslant s} m_i(m_i+1)/2$ and $h^2(\mathbb{P}^2, \mathcal{B}(t-d)) = (d-t-1)(d-t-2)/2$.

Thus, from the long exact sequence of cohomology, we get the conclusion. ▌

COROLLARY 3.2. *Notation as in Section 2, let* $\mathcal{P} = (P_1, ..., P_s)$, $\mathcal{M} = (m_1, ..., m_s)$. *Let C be an integral curve of degree d such that, if P_i lies on C, then P_i is a simple point of C. If moreover,*

(i) $t \geqslant d$;

(ii) $td - \sum_{P_i \in C} m_i \geqslant 2p_a - 1$;

(iii) $S_{t-d}(\mathcal{P}, \mathcal{M} - C)$ *is regular*;

then $S_t(\mathcal{P}, \mathcal{M})$ is regular.

Proof. Let k be the number of P_i's lying on C. The result being trivial for $k=0$, assume $1 \leqslant k \leqslant s$. We may suppose $P_1, ..., P_k \in C$. The conclusion follows from Theorem 3.1, since $i=0$ by (ii) and $h_{t-d}=0$ by (iii). ▌

Remark 3.3. Under the hypotheses of Corollary 3.2, in 3.1(a) we have $i = h_t = i + h_{t-d}$. Such equalities do not always hold: if $t=4$, $d=4$,

$s=k=5$, $m_1=\cdots=m_s=2$, and moreover $P_1, \ldots, P_5$ lie on a non-singular conic, we have $i=0$, $h_t=1$, $h_{t-d}=4$.

4. Points on a Curve of Degree d

In this section we give bounds for the integer $\tau=\min\{n\in\mathbb{N} : S_n$ is regular$\}$, when the points $P_1, \ldots, P_s$ are non-singular points of an irreducible, reduced curve C of degree d.

Theorem 4.1. *Notation as in Section 2, let $P_1, \ldots, P_s$ be distinct non-singular points of an integral curve C of degree d, let $m_1\geqslant m_2\geqslant\cdots\geqslant m_s>0$. If $s\geqslant d$, set $v=\max\{n\in\mathbb{N} : m_d=\cdots=m_n\}$;*

$$x_1=\sum_1^d m_i-1;$$

$$x_2=\left(\sum_1^s m_i+1\right)\Big/d+d-3;$$

$$x_3=\left(\sum_1^v m_i+(m_d-1)(d^2-v)+1\right)\Big/d+d-3;$$

$$x=\max(x_1, x_2, x_3); \qquad t_i=\min\{n\in\mathbb{N} : n\geqslant x_i\} \quad (1\leqslant i\leqslant 3).$$

Define t as follows:

(a) $t=\sum_1^s m_i-1$ *for* $s\leqslant d$;

(b) $t=\min\{n\in\mathbb{N} : n\geqslant x\}=\max(t_1, t_2, t_3)$ *for* $s\geqslant d$.

Then $S_n(\mathcal{P}, \mathcal{M})$ is regular for every $n\geqslant t$, i.e., $\tau\leqslant t$.

Remark. For $s=d$, it is easily verified that $x=x_1$, so t is well defined.

Proof. In case (a) the bound is classically known (see, for instance, [S1, p. 20]).

Let $s\geqslant d$. Now it suffices to show that $S_t(\mathcal{P}, \mathcal{M})$ is regular. We are going to use Theorem 3.1 with $k=s$, Corollary 3.2, and induction on m_1. Observe that $\mathcal{M}-C=\mathcal{M}-s$.

For $m_1=1$, notation being as in Theorem 3.1, we have $\deg(D-E)=td-s\geqslant x_2d-s=2p_a-1$, whence $i=0$. Since $t\geqslant x_1$, we have $t\geqslant d-1$. For $t\geqslant d$, since obviously $h_{t-d}=0$, $h_t=0$ follows from Theorem 3.1(a); if $t=d-1$, apply 3.1(b).

Assume $m_1>1$. The conclusion follows from Corollary 3.2: hypotheses (i) and (ii) are satisfied, for we have $t\geqslant x_1\geqslant d$ and $td-\sum_1^s m_i\geqslant x_2d-\sum_1^s m_i=2p_a-1$. We check (iii): for $m_d=1$, since $t-d\geqslant$

$x_1-d=\sum_1^d(m_i-1)-1$, then $S_{t-d}(\mathscr{P}, \mathscr{M}-s)$ is regular by (a). If $m_d>1$, let

$$x_1'=\sum_1^d(m_i-1)-1=x_1-d;$$

$$x_2'=\left(\sum_1^s(m_i-1)+1\right)\Big/d+d-3=x_2-s/d;$$

$$x_3'=\left(\sum_1^v(m_i-1)+(m_d-2)(d^2-v)+1\right)\Big/d+d-3=x_3-d.$$

If we prove that $t-d\geqslant\max(x_1', x_2', x_3')$, then $S_{t-d}(\mathscr{P}, \mathscr{M}-s)$ is regular by induction. Obviously $t-d\geqslant\max(x_1', x_3')$.

If $\sum_1^s(m_i-1)-\sum_1^v(m_i-1)\leqslant(m_d-2)(d^2-v)$, then $x_2'\leqslant x_3'$.

If $\sum_1^s(m_i-1)-\sum_1^v(m_i-1)>(m_d-2)(d^2-v)$, then $(s-v)(m_d-2)\geqslant(s-v)(m_d-1)-(s-v)\geqslant\sum_1^s m_i-\sum_1^v m_i-s+v=\sum_1^s(m_i-1)-\sum_1^v(m_i-1)>(m_d-2)(d^2-v)$. Since $m_d\geqslant 2$, we have $s>d^2$ and $x_2'<x_2-d$. In any case $t-d\geqslant\max(x_1', x_2', x_3')$. ∎

Corollary 4.2. *Notation as in Section* 2, *let* $P_1, ..., P_s$ *be distinct simple points of an integral curve* C *of degree* d. *Define* t *as follows*:

(a) *if* $d=1$,

$$t=\sum_1^s m_i-1;$$

(b) *if* $d=2$, $s\geqslant 2$, *and* $m_1\geqslant m_2\geqslant\cdots\geqslant m_s$,

$$t=\max\left(m_1+m_2-1; \left[\sum_1^s m_i/2\right]\right);$$

(c) *if* $m_i=m>0$ *for every* i *and* $s\geqslant d$,

$$t=\min\left\{n\in\mathbb{N}: n\geqslant\max\left(\frac{s+1}{d}+md-3; \frac{ms+1}{d}+d-3\right)\right\}.$$

Then in each of cases (a), (b), (c), $S_n(\mathscr{P}, \mathscr{M})$ *is regular for every* $n\geqslant t$, *i.e.*, $\tau\leqslant t$.

Proof. The assertions follow from Theorem 4.1 by straightforward calculations. ∎

Theorem 4.3. *Let the notation and hypotheses be as in Theorem* 4.1. *Assume further* $d\geqslant 3$, $d<s\leqslant d(d+3)/2$, *and the* s *points impose independent*

conditions to the curves of degree $(d+w)$, *w a non-negative integer. Define* t *as follows*:

(a) $t=\sum_1^d m_i+1$ *if* $w=0$, $m_1=\cdots=m_s\geqslant 2$, $s=d(d+3)/2$;

(b) $t=\sum_1^d m_i+w$ *otherwise.*

Then S_n *is regular for every* $n\geqslant t$, *i.e.*, $\tau\leqslant t$.

Proof. By induction on m_1, if $m_1=1$, then $t=d+w$ and the conclusion is obvious.

Assume $m_1>1$. We distinguish the following cases:

(1) $w=0$, $s=d(d+3)/2$, $m_1=\cdots=m_s$;

(2) $w>0$, $s=d(d+3)/2$, $m_1=\cdots=m_s$;

(3) $s<d(d+3)/2$, $m_1=\cdots=m_s$;

(4) $s\leqslant d(d+3)/2$, $m_1>m_s$, $m_1=\cdots=m_d$;

(5) $s\leqslant d(d+3)/2$, $m_1>m_d>1$;

(6) $s\leqslant d(d+3)/2$, $m_1>m_d=1$.

Proofs for cases (1) through (5) are similar and follow from Corollary 3.2. As an example, we prove case (4). We have $t=dm_1+w$, thus (i) of 3.2 is satisfied. For (ii) we observe that

$$td-\sum_1^s m_i-2p_a+1\geqslant d^2m_1-sm_1+1-(d-1)(d-2)+1$$

$$\geqslant d^2m_1-\frac{d(d+3)}{2}m_1-d^2+3d=(d^2-3d)\left(\frac{1}{2}m_1-1\right)\geqslant 0.$$

For (iii), $t-d=d(m_1-1)+w$. Observe that if $m_i=m_1$ for every $m_i\neq 1$, the number of points with positive multiplicity is less than $d(d+3)/2$. Hence in any case $S_{t-d}(\mathscr{P},\mathscr{M}-s)$ is regular by induction.

Case (6). The proof follows immediately from the next lemma.

LEMMA 4.4. *Let* $\mathscr{M}=(m_1,\ldots,m_s)$ *be an s-tuple of positive integers and* $\mathscr{P}=(P_1,\ldots,P_s)$ *an s-tuple of distinct points, let* $S_t(\mathscr{P},\mathscr{M})$ *be as usual and* $\mathscr{M}+1=(m_1+1,m_2,\ldots,m_s)$. *If* $S_t(\mathscr{P},\mathscr{M})$ *is regular, then* $S_{t+1}(\mathscr{P},\mathscr{M}+1)$ *is regular.*

Proof. Let C be a line through P_1, such that $P_i\notin C$, for $i\neq 1$. Apply Corollary 3.2 to $S_{t+1}(\mathscr{P},\mathscr{M}+1)$ and C. ▌

Remark 4.5. A. Gimigliano in [Gi] proves, by different means, Theorem 4.3, in the case $w=0$.

5. SHARPNESS OF THE BOUNDS

Next we shall be concerned with the sharpness of the bounds given by Theorem 4.1. Consider the following problems.

PROBLEM (A). *Let* d, s, $m_1, ..., m_s$ *be positive integers. Do there exist, for some integral curve* C *of degree* d, *simple distinct points* $P_1, ..., P_s$ *of* C *so that* $S_{t-1}(\mathscr{P}, \mathscr{M})$ *is not regular, i.e.,* $\tau = t$ (t *as in* 4.1)?

PROBLEM (B). *Let* d, s, $m_1, ..., m_s$ *be positive integers, let* e, r *be integers such that* $\sum_1^s m_i = ed + r$ $(0 \leqslant r \leqslant d-1)$. *Do there exist, for some integral curve* C *of degree* d, *a curve* C_e *of degree* e, *simple distinct points* $P_1, ..., P_s$ *of* C, *and integers* $m'_1, ..., m'_s$ *such that* $0 \leqslant m'_i \leqslant m_i$ *for every* i, $\sum_1^s m'_i = ed$, *and* $C_d \cdot C_e = \sum_1^s m'_i P_i$?

We call Problem (A′) and Problem (B′) the stronger variants obtained from Problems (A) and (B) by replacing "for some integral curve C" with "for every integral curve C."

An affirmative answer to Problem (A) means that the bound of Theorem 4.1 is sharp with respect to the given choice of d, s, $m_1, ..., m_s$. We are able to give an affirmative answer to Problem (A′) in several cases. In the other cases, Problem (A) is strongly related to Problem (B).

We will prove (Theorem 5.4) that if $(*)$ denotes the following case (notation as in Theorem 4.1),

$$s > d \geqslant 3 \qquad m_d > 1 \qquad t = t_2 \qquad t_2 > t_1 \qquad t_2 > t_3 \qquad (*)$$

then: (a) if either we are not in case $(*)$, or we are and Problem (B) has an affirmative answer, then the bound given by Theorem 4.1 is sharp; (b) if we are in case $(*)$ and d divides $\sum_1^s m_i$, then the sharpness of the bound for the given choice of d, s, $m_1, ..., m_s$ is equivalent to an affirmative answer to Problem (B).

We now state these results more precisely.

PROPOSITION 5.1. *Notation and* C, d, $\mathscr{M}$, τ, t *as in Theorem* 4.1.

(a) *If* $s \leqslant d$ *and* $P_1, ..., P_s$ *lie on a line* L, *then* $\tau = t$, *i.e.,* $S_{t-1}(\mathscr{P}, \mathscr{M})$ *is not regular.*

(b) *If* $s > d$ *and either* $d = 1$ *or* $d = 2$, *then* $\tau = t$.

(c) *Assume* $s > d \geqslant 3$:

(i) *if* $t = t_1$ *and* $P_1, ..., P_d$ *are on a line* L, *then* $\tau = t$;

(ii) *if* $m_d = 1$ *and* $t = t_2 (= t_3)$, *then there exist* $P_1, \ldots, P_s \in C$ *such that* $\tau = t$;

(iii) *if* $m_d > 1$, *and* $t = t_3$,

then there exist $P_1, \ldots, P_s \in C$ *such that* $\tau = t$.

Proof. (a), (b) If $P_1, \ldots, P_s$ lie on a line, then $t = \sum_1^s m_i - 1$ and it is classically known that $\tau = t$ (see, e.g., [DG, p. 7H]).

If $s > d$ and $d = 2$, then $t = \max(m_1 + m_2 - 1; [\sum_1^s m_i/2])$ (see 4.2). If $t = m_1 + m_2 - 1$, then the line P_1P_2 is a fixed component of all the curves of $S_{t-1}(\mathscr{P}, \mathscr{M})$; if $t = [\sum_1^s m_i/2]$, then the conic C is a fixed component of all the curves of $S_{t-1}(\mathscr{P}, \mathscr{M})$. In either case the conclusion follows by a direct calculation.

(c)(i) We have $t = \sum_1^d m_i - 1$ thus the line L is a fixed component of all the curves of $S_{t-1}(\mathscr{P}, \mathscr{M})$. The conclusion follows by a direct calculation.

(c)(ii) Let e, r be integers such that $\sum_1^s m_i = ed + r$ $(0 \leqslant r \leqslant d-1)$, set $A = \sum_1^d m_i - d$. We have $t - 1 = e + d - 3$. We may assume $r = 0$. Since $t_2 \geqslant t_1$, we have $e \geqslant A + 1$ and $t - 1 \geqslant d - 2$.

Let C_e be a curve of degree e, such that $C \cdot C_e = \sum_1^s m_i P_i = E$ and the P_i's are distinct non-singular points of C. We notice that it is always possible to construct C_e as a union of lines since $e \geqslant A + 1$. Put $\mathscr{P} = (P_1, \ldots, P_s)$.

Apply Theorem 3.1 to S_{t-1}, C, E, $\sigma = (t-1)H \cdot C - E$. Since C_e and a curve of degree $(d-3)$ give rise to a curve of degree $(t-1)$, which cuts out on C the divisor $E + K$ (K the canonical divisor), it follows that σ is special.

(c)(iii) By (c)(i), we may assume $t_3 > t_1$. Let q, r' be integers such that $v = qd + r'$ $(0 \leqslant r' \leqslant d-1)$, set $m = m_d$ and $A = \sum_1^d m_i - md$. Since $t_3 > t_1$ and $t_3 \geqslant t_2$, an easy calculation gives $v \geqslant (d-1)A + 2d$, $d^2 + d > v$, $A \leqslant d - 1$, $A + r' \leqslant 2d - 2$, $q \geqslant A + 1$, $q \geqslant 2$, $x_3 = (A + r' + 1)/d + q + md - 3$. We distinguish two cases.

Case 1. $0 \leqslant A + r' \leqslant d - 1$, thus $t - 1 = q + md - 3$.

Case 2. $d \leqslant A + r' \leqslant 2d - 2$, thus $t - 1 = q + 1 + md - 3$.

Case 1. We may assume $A + r' = 0$, $s = v$. Let C_q be a non-singular curve of degree q, which cuts C in $P_1, \ldots, P_{qd}$ distinct non-singular points of C. Take $\mathscr{P} = (P_1, \ldots, P_{qd})$. If $q = 2$, the conclusion follows from (b).

If $q > 2$, apply Theorem 3.1(a) to $S_{t-1}(\mathscr{P}, \mathscr{M})$, C_q, $E = \sum_1^{qd} mP_i$, $\sigma = (t-1)H \cdot C_q - E$. Since m times C_d and a curve of degree $(q-3)$ give rise to a curve of degree $(t-1)$, which cuts out on C_q the divisor $E + K$ (K the canonical divisor), σ is special.

Case 2. We may assume $A + r' = d$, $s = v$. Let $m_i^* = m_i - (m-1)$. It is easy to construct a curve C_{q+1} of degree $(q+1)$ so that

$C_{q+1}\cdot C_d=\sum_1^s m_i^* P_i$, the P_i's are distinct non-singular points of C_d and they are singular points of C_{q+1}, with multiplicity m_i^*. Actually, since $q+1\geqslant A+2$, it is enough to choose $P_1, ..., P_d$ on a line and the other points on A suitable lines, (m_i^*-1) of them through P_i $(1\leqslant i\leqslant d)$.

We shall prove that $S_{t-1}(\mathscr{P},\mathscr{M})$ is not regular, for every $m\geqslant 1$, by induction on m. For $m=1$, this follows from Theorem 3.1, since $i>0$. For $m>1$, let $\mathscr{M}'=(m-1, ..., m-1)$,

$$\vartheta=S_{t-1}(\mathscr{P},\mathscr{M})\cdot C_d-\sum_1^s m_iP_i,$$

$$\vartheta'=S_{(t-1)-(q+1)}(\mathscr{P},\mathscr{M}')\cdot C_d-\sum_1^s (m-1)P_i;$$

let h_{t-1} and h_{t-1-d} be the superabundances of $S_{t-1}(\mathscr{P},\mathscr{M})$ and $S_{t-1-d}(\mathscr{P},\mathscr{M}-v)$, respectively. Since $\deg\vartheta=\deg\vartheta'$ and $\vartheta\supseteq\vartheta'$, we have $\dim\vartheta\geqslant\dim\vartheta'$. From this inequality and from (1) of Section 2, since $(t-1)-(q+1)=md-3$ and both $S_{md-3}(\mathscr{P},\mathscr{M}')$ and $S_{md-3-d}(\mathscr{P},\mathscr{M}'-v)$ are regular by 4.1, we get $h_{t-1}\geqslant h_{t-1-d}$. By inductive hypothesis $h_{t-1-d}>0$, so we are done. ▌

PROPOSITION 5.2. *Let notation and* d, $\mathscr{M}$, t *be as in Theorem* 4.1. *Let* $s>d\geqslant 3$, $m_d>1$, $t=t_2$, $t_2>t_1$, $t_2>t_3$.

If Problem (B) *has an affirmative answer for* d, s, $m_1, ..., m_s$, *then the same is true for Problem* (A).

Proof. Let C, C_e, $P_1, ..., P_s$ and $m'_1, ..., m'_s$ be as given by a solution of Problem (B). It is easy to verify that $t-1=e+d-3$, $s>d^2$, $e\geqslant d$, and $t-1\geqslant d$.

Let $\mathscr{M}'=(m'_1, ..., m'_s)$ and $\mathscr{P}=(P_1, ..., P_s)$; it suffices to prove that $S_{t-1}(\mathscr{P},\mathscr{M}')$ is not regular. This conclusion follows from Theorem 3.1(a), for $k=s$. Since C_e and a curve of degree $(d-3)$ give rise to a curve of degree $(t-1)$, σ is special. ▌

In case $\sum_1^s m_i=ed$, the implication of Theorem 5.2 can be reversed. This follows from

PROPOSITION 5.3. *Hypotheses and notation as in Theorem* 4.1, *let* $s>d\geqslant 3$, $m_d>1$, $t=t_2$, $t_2>t_1$, $t_2>t_3$, *and* $\sum_1^s m_i=ed$. *If* $\tau=t$, *i.e.,* $S_{t-1}(\mathscr{P},\mathscr{M})$ *is non-regular, then Problem* (B) *has an affirmative answer.*

Proof. Claim. $S_{t-1-d}(\mathscr{P},\mathscr{M}-s)$ is regular.

By Theorem 4.1, it suffices to show that $t-1-d\geqslant\max(x'_1, x'_2, x'_3)$ (notation as in the proof of 4.1).

Now $x'_1=x_1-d$, $x'_3=x_3-d$. Thus from $t_2>t_1$ and $t_2>t_3$ it follows that

$t-1-d \geqslant \max(x'_1, x'_3)$. We also have that $x'_2 = x_2 - s/d = e+d-3+(1-s)/d$, $t-1-d=e-3$, and, from $t_2 > t_3$, $s \geqslant d^2+1$. Hence $t-1-d \geqslant x'_2$, and the claim is proved.

By Theorem 3.1(a) and the claim, we get $i(D-E)>0$, where $D \in (t-1)H \cdot C$, and E denotes the Cartier divisor on C defined by $E = \sum_1^s m_i P_i$. Let Γ_{t-1} be the linear system of all curves of degree $t-1$ passing through E. Set $\sigma = (t-1)H \cdot C - E$. By (5) of Section 2 and the Riemann–Roch Theorem, we get $\dim \sigma = \dim \Gamma_{t-1} \cdot C = (t-1)(t+2)/2 - \sum_1^s m_i + h - (t-1-d)(t+2-d)/2 - 1 = \deg(D-E) - p_a + i(D-E)$, where h is the superabundance of Γ_{t-1}. Hence it follows that $h>0$; i.e., E does not impose independent conditions on the curves of degree $t-1=d+e-3$. The conclusion follows by a well-known Theorem of B. Segre (see, e.g., [G, Corollary 3.4]). ∎

We can summarize the above results as follows.

THEOREM 5.4. *Notation as in Theorem* 4.1, *let* (∗) *denote the following case*

$$s > d \geqslant 3 \qquad m_d > 1 \qquad t = t_2 \qquad t_2 > t_1 \qquad t_2 > t_3. \tag{$*$}$$

(i) *If we are not in case* (∗), *then Problem* (A′) *has an affirmative answer*;

(ii) *if we are in case* (∗) *and Problem* (B) *has an affirmative answer, then the same is true for Problem* (A);

(iii) *if we are in case* (∗) *and d divides* $\sum_1^s m_i$, *then the sharpness of the bound for the given choice of* d, s, $m_1, ..., m_s$ *is equivalent to an affirmative answer to Problem* (B).

Proof. Obvious from 5.1, 5.2, 5.3. ∎

Remarks 5.5. (1) One can easily exhibit instances for which Problem (B′) has an affirmative answer. As an example, let s be a multiple of d, say $s=kd$; let $\mathcal{M}$ be an s-tuple such that $m_1 = \cdots = m_d \geqslant m_{d+1} = \cdots = m_{2d} \geqslant \cdots \geqslant m_{(k-1)d+1} = \cdots = m_{kd}$. Let C be any integral curve of degree d, and choose lines $L_1, ..., L_k$ that cross C at kd distinct non-singular points. Let $P_{(j-1)d+1}, ..., P_{jd}$ denote the intersection points of L_j with C. Clearly the points P_i, together with the curve arising by taking m_d times the line L_1, m_{2d} times the line L_2, and so on, provide a solution to Problem (B′).

(2) For $d=3$, in char $k=0$, Problem (B) has an affirmative answer, as we will prove next in Proposition 5.6, so it is natural to ask if this is always true.

(3) Conjecture. For $s \gg 0$ $(s \geqslant (d-1)(d-2)+1?)$ the answer to Problem (B) is in the affirmative.

PROPOSITION 5.6. *If* $d=3$ *and* $\operatorname{char} k = 0$, *then Problem* (B) *has an affirmative answer.*

Proof. We may assume $\sum_1^s m_i = 3e$. Let C be the cubic with affine equation $y = x^3$, and let $b_1, ..., b_s \in k$ be distinct elements satisfying $m_1 b_1 + \cdots + m_s b_s = 0$.

Consider the polynomial

$$(x-b_1)^{m_1} \cdots (x-b_s)^{m_s} = x^{3e} + a_2 x^{3e-2} + a_3 x^{3e-3} + \cdots + a_{3e}.$$

From this, by substituting $y = x^3$, we get the curve C_e of degree e and affine equation $y^e + a_2 y^{e-1} x + a_3 y^{e-1} + \cdots + a_{3e} = 0$. C_e cuts out on C a divisor $\sum_1^s m_i P_i$, where the P_i's are clearly non-singular on C and they are distinct, for so are $b_1, ..., b_s$. ▌

Remark 5.7. Theorem 5.1(b) shows that if the points of $\mathscr{P}$ are on a line, then S_{t-1} is superabundant $(t = \sum_1^s m_i - 1)$. It is classically known that the converse of this theorem is true; that is, if S_{t-1} is not regular, then the points of $\mathscr{P}$ are on a line. In fact, if $1 < d < s$, S_{t-1} is regular by 4.1; hence if it is not regular, $d = 1$ follows.

By Theorem 5.1(b) again, if the points are on a non-singular conic, then S_{t-1} is not regular $(t = \max(m_1 + m_2 - 1; [\sum_1^s m_i/2]))$. In the next section we will investigate when the converse of this theorem is true.

6. POINTS IN GENERAL POSITION

In Corollary 4.2 we proved that if the P_i's are on a non-singular conic, then $\tau \leqslant t = \max(m_1 + m_2 - 1; [\sum_1^s m_i/2])$. B. Segre found the same bound in the case of "generic" distinct points [S1, p. 23]. In this section we show that the same inequality holds in case of points in general position (Theorem 6.2), which implies Segre's result.

In Proposition 5.1(b), we improved the result of 4.2, proving that if the points are on a non-singular conic, then $\tau = t$. It is natural to ask when the converse is true. In Theorems 6.3 and 6.4 we answer this question.

DEFINITION 6.1. A set of s points, $P_1, ..., P_s$, is said to be in *general position* if no three of them are collinear.

THEOREM 6.2. *Let* $\mathscr{P} = (P_1, ..., P_s)$ *be an s-tuple of distinct points in general position, and let* $\mathscr{M} = (m_1, ..., m_s)$ *be an s-tuple of non-negative* *integers such that* $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$, $s \geqslant 2$. *Set* $t = \max(m_1 + m_2 - 1; [\sum_1^s m_i/2])$.

Then $S_n(\mathscr{P}, \mathscr{M})$ *is regular for every* $n \geqslant t$, *i.e.*, $\tau \leqslant t$.

Proof. Obvious for $m_2 = 0$. Let $m_2 > 0$. We may assume $m_s \neq 0$. For $s = 2$, P_1 and P_2 are on a line and the conclusion follows from Corollary 4.2. Let $s > 2$. By induction on $\sum_1^s m_i$. The proof being trivial for $\sum_1^s m_i = 3$, assume $\sum_1^s m_i > 3$. Let L be the line through P_1 and P_2. The conclusion follows from Corollary 3.2 applied to $S_t(\mathscr{P}, \mathscr{M})$, with $d = 1$ and $C = L$. In fact (i) and (ii) are obviously satisfied. For (iii) use the inductive hypothesis. ∎

We know that if the P_i's lie on a non-singular conic, then $\tau = t$. Now we prove that, under further assumptions, this implication can be reversed.

THEOREM 6.3. *Notation being as in Theorem* 6.2, *let* $m_1 \geqslant m_2 \geqslant m_i > 0$ $(3 \leqslant i \leqslant s)$, $s \geqslant 6$. *Assume*:

(i) $\sum_1^s m_i$ *is even and* $t = \sum_1^s m_i/2$;
(ii) $P_1, ..., P_s$ *are in general position*;
(iii) $\sum_1^s m_i \geqslant 2m_1 + 2m_2$;
(iv) vir. dim $S_{t-1}(\mathscr{P}, \mathscr{M}) \geqslant -1$;
(v) $S_{t-1}(\mathscr{P}, \mathscr{M})$ *is not regular.*

Then $P_1, ..., P_s$ *lie on a non-singular conic.*

THEOREM 6.4. *Notation being as in Theorem* 6.2, *let* $m_1 \geqslant m_2 \geqslant m_i > 0$ $(3 \leqslant i \leqslant s)$, $s \geqslant 6$. *Assume*:

(i) $\sum_1^s m_i$ *is odd and* $t = (\sum_1^s m_i - 1)/2$;
(ii) $P_1, ..., P_s$ *are in general position*;
(iii) $\sum_1^s m_i \geqslant 2m_1 + 2m_2 + 1$;
(iv) vir. dim $S_{t-1}(\mathscr{P}, \mathscr{M}) \geqslant -1$;
(v) $S_{t-1}(\mathscr{P}, \mathscr{M})$ *is not regular*;
(vi) $\sum_1^s m_i \geqslant 2m_1 + 7$;
(vii) $m_i \geqslant 2$ *for every* i.

Then $P_1, ..., P_s$ *lie on a non-singular conic.*

In order to prove Theorem 6.3 we need the following lemma.

LEMMA 6.5. *If* $\sum_1^s m_i \geqslant 2m_1 + 2m_2 + 2$, *then Theorem* 6.3 *holds.*

Proof. We may assume $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_5 \geqslant m_i$ $(6 \leqslant i \leqslant s)$. Let C be the non-singular conic through $P_1, ..., P_5$. Let k be the number of P_i's lying on C. We shall prove that $k<s$ implies the regularity of $S_{t-1}(\mathcal{P}, \mathcal{M})$, whence the conclusion by contradiction.

Let $k<s$. We may assume $P_1, ..., P_k \in C$, $P_{k+1}, ..., P_s \notin C$. We check that the hypotheses of Corollary 3.2 are satisfied for $S_{t-1}(\mathcal{P}, \mathcal{M})$ and C. Part (i) is obviously satisfied. For (ii) observe that $(t-1)2-\sum_1^k m_i = \sum_1^s m_i - \sum_1^k m_i - 2 \geqslant -1$. For (iii), let $t'=t-3$, let $\mathcal{M}'$ be the s-tuple $\mathcal{M}-k=\mathcal{M}-C$ rearranged by decreasing order, and let $\mathcal{P}'$ be the corresponding permutation of $\mathcal{P}$, so that $S_{t-3}(\mathcal{P}, \mathcal{M}-k)=S_{t'}(\mathcal{P}', \mathcal{M}')$. We have $t' \geqslant (\sum_1^s m_i - 6)/2 \geqslant [(\sum_1^s m_i - k)/2] = [\sum_1^s m_i'/2]$.

If $m_1+m_2 \geqslant m_1'+m_2'+1$, then $t' \geqslant (m_1+m_2+1)-3 \geqslant m_1'+m_2'-1$.

If $m_1+m_2=m_1'+m_2'$, then observe that $\sum_1^s m_i \geqslant 7m_1$. Hence $t' \geqslant (\sum_1^s m_i/2)-3 \geqslant m_1'+m_2'-1$. Then $S_{t'}(\mathcal{P}', \mathcal{M}')$ is regular by 6.2. ▌

Proof of Theorem 6.3. By induction on $\sum_1^s m_i$, for $\sum_1^s m_i=6$, apply 6.5. Let $\sum_1^s m_i \geqslant 8$. By 6.5, we may assume $\sum_1^s m_i = 2m_1+2m_2$. It is easy to verify that $m_2=1$ implies $m_1=s-3$, $t-1=s-3$, and eventually vir.dim $S_{t-1}(\mathcal{P}, \mathcal{M}) = -2$. Hence $m_2>1$.

Let $t'=t-1$, let $\mathcal{M}'$ be the s-tuple $\mathcal{M}-2$ rearranged by decreasing order and $\mathcal{P}'$ the corresponding permutation of $\mathcal{P}$. Notice that $m_i' \geqslant 1$ for every i. We will show that $S_{t'-1}(\mathcal{P}', \mathcal{M}')$ satisfies all the hypotheses of 6.3.

$S_{t'-1}(\mathcal{P}', \mathcal{M}')=S_{t-3}(\mathcal{P}, \mathcal{M}-2)$ obviously satisfies (i) and (ii) of Theorem 6.3. For (iii), notice that in case $m_1+m_2=m_1'+m_2'$, we get $\sum_1^s m_i \geqslant 4m_1+2$, whence $\sum_1^s m_i' \geqslant 2m_1'+2m_2'$. In the case $m_1+m_2>m_1'+m_2'$, (iii) is clearly satisfied. For (iv) and (v), it follows from a straightforward calculation that vir.dim $S_{t-1}(\mathcal{P}, \mathcal{M})=$ vir.dim $S_{t'-1}(\mathcal{P}', \mathcal{M}')$. Now, since $t-1=m_1+m_2-1$, the line P_1P_2 is a fixed component for all the curves of $S_{t-1}(\mathcal{P}, \mathcal{M})$, hence dim $S_{t-1}(\mathcal{P}, \mathcal{M})=$ dim $S_{t'-1}(\mathcal{P}', \mathcal{M}')$. Then (iv) and (v) hold for $S_{t'-1}(\mathcal{P}', \mathcal{M}')$, as they do for $S_{t-1}(\mathcal{P}, \mathcal{M})$.

By the inductive hypothesis, Theorem 6.3 holds for $S_{t'-1}(\mathcal{P}', \mathcal{M}')$, whence the conclusion. ▌

In order to prove Theorem 6.4, we need the following lemma.

LEMMA 6.6. *If* $\sum_1^s m_i \geqslant 2m_1+2m_2+3$, *then Theorem* 6.4 *holds.*

Proof. We may assume $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_5 \geqslant m_i$ $(6 \leqslant i \leqslant s)$. Let C be the non-singular conic through $P_1, ..., P_5$. Let k be the number of P_i's lying on C. For $5<k<s$, by the same technique used in the proof of 6.5, we obtain the regularity of $S_{t-1}(\mathcal{P}, \mathcal{M})$. Hence the conclusion by contradiction.

For $k=5$, we check that the hypotheses (i) through (iv) of Theorem 6.3 are satisfied for $S_{t'-1}(\mathcal{P}', \mathcal{M}')$, where $t'-1=t-3$, $\mathcal{M}'$ is the decreasing rearrangement of $\mathcal{M}-5$, and $\mathcal{P}'$ is the corresponding permutation of $\mathcal{P}$. Parts (i) and (ii) are evidently true. Part (iii) follows by arithmetical calculations similar to those of 6.3. For (iv) set $A=(\sum_1^s m_i')^2-2\sum_1^s m_i'$

$-4\sum_1^s m_i'^2$. By a direct calculation, wet get vir.dim $S_{t'-1}(\mathscr{P}', \mathscr{M}') \geqslant -1$ iff $A \geqslant 0$.

For $\sum_1^s m_i' \geqslant 2m_1' + 2m_2' + 2$, by (tedious) calculations, we obtain $A \geqslant 0$. It is easy to verify that the equality $\sum_1^s m_i' = 2m_1' + 2m_2'$ occurs only if $m_1 > m_2 = \cdots = m_6$ and $\sum_1^s m_i = 2m_1 + 2m_2 + 3$. In this case, we have vir.dim $S_{t-1}(\mathscr{P}, \mathscr{M}) - m_1 + 2m_2 = (2m_1 m_2 + m_1 + m_2 - 3 - 4m_2^2 - \sum_7^s m_1^2)/2 - m_1 + 2m_2 \geqslant ((2m_2 - 1)(3m_2 - 3 + \sum_7^s m_i) + 5m_2 - 4m_2^2 - \sum_7^s m_i^2 - 3)/2 \geqslant m_2^2 - 2m_2 \geqslant 0$.

Thus vir.dim $S_{t'-1}(\mathscr{P}', \mathscr{M}') =$ vir.dim $S_{t-1}(\mathscr{P}, \mathscr{M}) - 2t + \sum_1^5 m_i + 1 =$ vir.dim $S_{t-1}(\mathscr{P}, \mathscr{M}) - m_1 + 2m_2 - 1 \geqslant -1$, and (iv) is verified.

We conclude that if the P_i's are not on a non-singular conic, then $S_{t'-1}(\mathscr{P}', \mathscr{M}') = S_{t-3}(\mathscr{P}, \mathscr{M} - 5)$ is regular by Theorem 6.3.

Now apply Corollary 3.2 to $S_{t-1}(\mathscr{P}, \mathscr{M})$ and the conic C through $P_1, ..., P_5$. We have just checked that the hypothesis (iii) is satisfied. Parts (i) and (ii) of 3.2 are obviously satisfied, so that $S_{t-1}(\mathscr{P}, \mathscr{M})$ is regular, a contradiction. ▌

Proof of Theorem 6.4. By induction on $\sum_1^s m_i$, for $\sum_1^s m_i = 13$, apply Lemma 6.6. Let $\sum_1^s m_i \geqslant 15$. By 6.6, we may assume $\sum_1^s m_i = 2m_1 + 2m_2 + 1$. By (vi), $m_2 \geqslant 3$. Let $S_{t-3}(\mathscr{P}, \mathscr{M} - 2) = S_{t'-1}(\mathscr{P}', \mathscr{M}')$ as in the proof of Theorem 6.3. The proof that $S_{t'-1}(\mathscr{P}', \mathscr{M}')$ satisfies the conditions (i) through (v) of Theorem 6.4 is identical to the one given for Theorem 6.3, and will be omitted. The proof of (vi) is straightforward, that of (vii) is trivial. Thus, by inductive hypothesis, Theorem 6.4 holds for $S_{t'-1}(\mathscr{P}', \mathscr{M}')$, and we are done. ▌

Remark 6.7. The hypotheses of Theorems 6.3 and 6.4 cannot be weakened. For each hypotesis (ii) through (v), (respectively (ii) through (vii)) it is possible to construct a set of points not on a non-singular conic for which all the hypotheses, but that one, hold. In order to find such examples, the following remarks can be useful.

If we drop assumption (iii) and $\sum_1^s m_i \leqslant 2m_1 + 2m_2 - 1$, then $S_{t-1}(\mathscr{P}, \mathscr{M})$ is not regular no matter what the choice of $\mathscr{P}$ and $\mathscr{M}$ satisfying the remaining hypotheses.

Now assume that $P_1, ..., P_{s-1}$ lie on a non-singular conic C, $P_s \notin C$, and $m_1 \geqslant \cdots \geqslant m_s$.

If (vi) does not hold, then from $\sum_1^s m_i < 2m_1 + 7$ and from (iii), (iv), (vii), we get $\sum_1^s m_i = 2m_1 + 5$, $m_2 = \cdots = m_s = 2$, $m_1 = 2s - 7$, $t - 1 = m_1 + 1 = 2s - 6$, vir.dim $S_{t-1}(\mathscr{P}, \mathscr{M}) = s - 9$, $s \geqslant 8$. Now C and the lines $P_1 P_i$ $(2 \leqslant i \leqslant s)$ are fixed components for all the curves of $S_{t-1}(\mathscr{P}, \mathscr{M})$. Hence, by an easy calculation, we get dim $S_{t-1}(\mathscr{P}, \mathscr{M}) =$ dim $S_{s-7}(\mathscr{P}', \mathscr{M}') = s - 8 >$ vir.dim $S_{t-1}(\mathscr{P}, \mathscr{M})$, where $\mathscr{M}' = (s-7, 1)$, $\mathscr{P}' = (P_1, P_s)$. It follows that $S_{t-1}(\mathscr{P}, \mathscr{M})$ is not regular. Note that the points of $\mathscr{P}$ do not lie on a conic.

If (vii) does not hold, then C is a fixed component for all the curves of $S_{t-1}(\mathscr{P}, \mathscr{M})$. Thus it follows that $\dim S_{t-1}(\mathscr{P}, \mathscr{M}) = \dim S_{t-3}(\mathscr{P}, \mathscr{M} - (s-1)) > \text{vir.dim}\, S_{t-1}(\mathscr{P}, \mathscr{M})$. So we get again that $S_{t-1}(\mathscr{P}, \mathscr{M})$ is not regular, even if the points of $\mathscr{P}$ are not on a conic.

Finally, since it is easy to find examples for each hypothesis, we exhibit an example only in one case. That is the case in which (iii) of 6.4 does not hold, but all the other hypotheses of 6.4 do hold. Let $P_1, ..., P_7$ be seven points in general position and let $\mathscr{M} = (7, 4, 2, 2, 2, 2, 2)$. Obviously (iii) of 6.4 does not hold, but it is easy to prove that the other hypotheses of 6.4 hold.

7. POINTS IN UNIFORM POSITION

In this section we give a bound for τ, when $\mathscr{P}$ is an s-tuple of distinct points in uniform position. The bound we find is not sharp, but it is the best known to the author for points with that kind of genericity.

DEFINITION 7.1. A set $\mathscr{S}$ of s points of $\mathbb{P}^2$ is said to be in *uniform position*, if for any $s' \leqslant s$, for any s'-tuple $\mathscr{P}$ of points of $\mathscr{S}$, and any n, we have $\dim S_n(\mathscr{P}, \mathscr{M}) = \max(-1, \text{vir.dim}\, S_n(\mathscr{P}, \mathscr{M}))$, where $\mathscr{M}$ is the s'-tuple $(1, ..., 1)$.

Remarks 7.2. (1) The previous definition has been given by A. Geramita and F. Orecchia in [GO].

(2) Observe, on setting $s' = \binom{f+1}{2} + r$ $(0 \leqslant r \leqslant f)$, that the s' points impose independent conditions to the curves of degree f and, for $r = 0$, even to the curves of degree $f-1$.

THEOREM 7.3. *Let $\mathscr{P} = (P_1, ..., P_s)$ be an s-tuple of distinct points in uniform position, let $m_1 \geqslant \cdots \geqslant m_s \geqslant 0$, $s \geqslant 5$. Let $s_i = \max\{n \in \mathbb{N} : m_n \geqslant i\}$, $1 \leqslant i \leqslant m_1$, let r_i, f_i be the integers defined by $s_i = \binom{f_i+1}{2} + r_i$, $0 \leqslant r_i \leqslant f_i$. Set*

$$\begin{array}{lll} d_1 = f_1 - 1 & \textit{for} & r_1 = 0; \\ d_1 = f_1 & \textit{for} & r_1 > 0; \\ d_i = f_i & \textit{for} & i > 1 \ \textit{and} \ 0 \leqslant r_i \leqslant 2; \\ d_i = f_i + 1 & \textit{for} & i > 1 \ \textit{and} \ r_i > 2. \end{array}$$

Define t as

$$t = \max\left(m_1 + m_2 - 1; \left[\sum_1^5 m_i/2\right]; \sum_1^{m_1} d_i\right).$$

Then $S_n(\mathscr{P}, \mathscr{M})$ is regular for every $n \geqslant t$, i.e., $\tau \leqslant t$.

Proof. We remark that for $m_6 = 0$, the conclusion follows by Theorem 6.2, for $m_1 = 1$, by definition 7.1.

We prove the theorem by induction on $\sum_1^s m_i$. For $\sum_1^s m_i \leqslant 6$, the conclusion is obvious from the preceding remark. Let $\sum_1^s m_i \geqslant 7$. We may assume $m_1 \geqslant 2$ and $m_6 \geqslant 1$. We distinguish the following cases: (a) $1 \leqslant s_{m_1} \leqslant 4$; (b) $s_{m_1} = 5$; (c) $s_{m_1} \geqslant 6$.

Both in cases (a) and (b), the conclusion follows by Corollary 3.2, applied to $S_t(\mathscr{P}, \mathscr{M})$, when C is respectively the line P_1P_2 or the conic through $P_1, ..., P_5$. To check 3.2(iii), use the inductive hypothesis.

Case (c). To simplify notation, let f_{m_1}, r_{m_1} be denoted respectively by f, r; we have $s_{m_1} = \binom{f+1}{2} + r$, $0 \leqslant r \leqslant f$; $d_{m_1} = f$ for $0 \leqslant r \leqslant 2$; $d_{m_1} = f + 1$ for $r > 2$. By [MR1], through the points $P_1, ..., P_{s_{m_1}}$ there exists a non-singular curve C of degree d, with $d = f$ if $0 \leqslant r \leqslant 2$, while $f \leqslant d \leqslant f + 1$ if $r > 2$. If k is the number of P_i's on C, observe that $s_{m_1} \leqslant k \leqslant d(d+3)/2$. Now the conclusion follows by Corollary 3.2 applied to $S_t(\mathscr{P}, \mathscr{M})$ and C. Checking that hypotheses of 3.2 are satisfied requires lengthy, but straightforward calculations, which will be omitted. We only notice that (iii) follows by the inductive hypothesis, and that verifications are more easily made by distinguishing five cases: (1) $r = 0$, $s_{m_1} = s_1$; (2) $r = 0$, $s_{m_1} < s_1$; (3) $0 < r \leqslant 2$; (4) $r > 2$, $d = f$; (5) $r > 2$, $d = f + 1$. ▮

Remark 7.4. The bound given by Theorem 7.3 is not sharp for all possible sequences $m_1, ..., m_s$. For instance, let $\mathscr{P} = (P_1, ..., P_9)$, $\mathscr{M} = (3, 2, ..., 2)$. By 7.3, we have $t = 8$, but $S_7(\mathscr{P}, \mathscr{M})$ is regular by Corollary 3.2. In fact, let C be the line P_1P_2. The hypotheses (i) and (ii) of 3.2 are obviously satisfied. For (iii) observe that $S_6(\mathscr{P}, (2, 1, 2, ..., 2))$ is regular by 7.3.

On the other hand, we can produce cases for which $S_{t-1}(\mathscr{P}, \mathscr{M})$ is not regular. As an example, let C_4 be an integral quartic with three nodes P_1, P_2, P_3 and let $P_4, ..., P_{14}$ lie on C_4, so that $P_1, ..., P_{14}$ are distinct points in uniform position. Let $\mathscr{M} = (2, ..., 2)$. By Theorem 7.3, we get $t = 9$, and $S_8(\mathscr{P}, \mathscr{M})$ is not regular. In fact, by Bézout's Theorem, C_4 is a fixed component for all the curves of $S_8(\mathscr{P}, \mathscr{M})$, so $\dim S_8(\mathscr{P}, \mathscr{M}) = \dim S_4(\mathscr{P}', \mathscr{M}') = 3$ (where $\mathscr{P}' = (P_4, ..., P_{14})$, $\mathscr{M}' = (1, ..., 1)$), while $\text{vir.dim}\, S_8(\mathscr{P}, \mathscr{M}) = 2$.

Remark 7.5. Let U be the subset of $(\mathbb{P}^2)^s$ consisting of the s-tuples $(P_1, ..., P_s)$ of distinct points of $\mathbb{P}^2$ in uniform position and let $s = \binom{f+1}{2} + r$, $0 \leqslant r \leqslant f$. By [GO], U is a nonempty open subset of $(\mathbb{P}^2)^s$. For any s-tuple $(P_1, ..., P_s) \in U$, we know that there exists a non-singular curve C of degree d, $f \leqslant d \leqslant f + 1$, through $P_1, ..., P_s$; moreover, by [MR1, p. 189], there is a nonempty open subset $U' \subseteq U$, for which C can be found of degree $d = f$.

By this and the techniques of Theorem 7.3, one can prove the next result. We first recall the definition of generic points.

Definition 7.6. Let $(*)$ be any assertion. If there exists a nonempty open subset V of $(\mathbb{P}^2)^s$ such that $(*)$ holds for any s-tuple of V, we say that $(*)$ holds for a *generic* s-tuple of points of $\mathbb{P}^2$.

Theorem 7.7. *Let $(P_1, ..., P_s)$ be a generic s-tuple of points of $\mathbb{P}^2$, let $\mathcal{M}$, s_i, f_i, r_i be as in Theorem 7.3, and $s \geqslant 5$. Set*

$$\begin{aligned} d_1 &= f_1 - 1 && \text{for} \quad r_1 = 0; \\ d_1 &= f_1 && \text{for} \quad r_1 > 0; \\ d_i &= f_i && \text{for} \quad i > 1. \end{aligned}$$

Define t as

$$\begin{aligned} t &= m_1 d_1 + 1 && \text{either for } m_1 = m_s > 1, r_1 = f_1, s \geqslant 9 \\ & && \text{or for } m_1 = m_{s-1} > 1, m_s = 1, r_1 = 0, s \geqslant 10; \\ t &= \max\left(m_1 + m_2 - 1; \left[\sum_1^5 m_i/2\right]; \sum_1^{m_1} d_i\right) && \text{otherwise} \end{aligned}$$

Then $S_n(\mathcal{P}, \mathcal{M})$ is regular for every $n \geqslant t$, i.e., $\tau \leqslant t$.

Proof. It is sufficient to prove that the theorem holds for a nonempty open subset $V \subseteq (\mathbb{P}^2)^s$. Let $T = \{(P_1, ..., P_s) \in (\mathbb{P}^2)^s \colon P_i \neq P_j$ for $i \neq j$, and for every $s' \leqslant s$ if $s' = \binom{d'+1}{2} + r'$, $0 \leqslant r' \leqslant d'$, then there exists a non-singular curve of degree d' through any s' points taken from $\{P_1, ..., P_s\}\}$.

By induction on s, one can prove the existence of a nonempty open set $V \subseteq T$. This being trivial for $s = 2$, assume $s > 2$.

Let $V' \subseteq (\mathbb{P}^2)^{s-1}$ be given by the inductive hypothesis.

Let $V^i = \{(P_1, ..., P_s) \in (\mathbb{P}^2)^s : (P_1, ..., \hat{P}_i, ..., P_s) \in V'\}$.

Let U' be as in Remark 7.4. Then take $V = U' \cap (\bigcap_i V^i)$.

Now we proceed by induction on $\sum_1^s m_i$. After a suitable choice (varying from case to case) of s', $2 \leqslant s' \leqslant s$, consider a non-singular curve of degree d' through $P_1, ..., P_{s'}$ (which does exist by the choice of V) and apply Corollary 3.2. ∎

Remarks 7.8. (1) A. Gimigliano in [Gi] proves that the bound he gives for τ in case of points on a curve of degree d (see Remark 4.5) holds also for generic points of $\mathbb{P}^2$. The bound of Theorem 7.7 is an improvement.

(2) For generic points of $\mathbb{P}^2$, in char $k = 0$, Gimigliano in [Gi] proves that, if $m_1 = \cdots = m_s \geqslant 2$, $d \geqslant 3$, $s = d(d+3)/2$, then $\tau \leqslant m_1 d$. By his result, and by 7.7 and 3.2, it is easy to deduce that (hypotheses and notation as in 7.7), for char $k = 0$, we have

$$\tau \leqslant \max\left(m_1 + m_2 - 1; \left[\sum_1^5 m_i/2\right] : \sum_1^{m_1} d_i\right).$$

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