Unconditionally Convergent Series of Operators on Banach Spaces
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We show that any series \( \sum_{n=1}^{\infty} K_n \) of operators in \( L(X,Y) \) that is unconditionally convergent in the weak operator topology and satisfies the condition that \( \sum_{n \in F} K_n \) is a compact operator for every index set \( F \subseteq \mathbb{N} \) is unconditionally convergent in the uniform operator topology if and only if \( X^* \), the dual space of the Banach space \( X \), contains no copy of \( c_0 \).

1. INTRODUCTION

The classical Riemann theorem states that if a series of real (or complex) numbers converges unconditionally, then it converges absolutely [10, Theorem 3.56]. By use of the equivalence of coordinatewise convergence with norm convergence in any finite-dimensional Banach space, it is easy to prove that unconditionally convergent series are absolutely convergent.

What about the Riemann theorem in infinite-dimensional Banach spaces? A conjecture that in any infinite-dimensional Banach space, there exists an unconditionally convergent series which is not absolutely convergent had been open for about twenty years until A. Dvoretzky and C. A. Rogers [3] in 1950 proved that absolute convergence is equivalent to unconditional convergence of series in a Banach space if and only if the space is finite-dimensional.

W. Orlicz [9] in 1930 proved that if \( \sum_{n=1}^{\infty} f_n \) is an unconditionally convergent series in \( L_p[0,1] \) (\( 1 \leq p < \infty \)), then \( \sum_{n=1}^{\infty} \| f_n \|_p^2 < \infty \) for \( 1 \leq p \leq 2 \) and \( \sum_{n=1}^{\infty} \| f_n \|_p^p < \infty \) for \( p > 2 \). M. Kadec [4] in 1956 proved that if \( \sum_{n} x_n \) is an unconditionally convergent series in a uniformly convex Banach space,
then $\sum_{n=1}^{\infty} \delta(\|x_n\|) < \infty$, where $\delta$ is the modulus of convexity. V. Kaftal and G. Weiss [5] in 1986 introduced a Riemann type theorem for unconditionally convergent series of operators on Hilbert spaces, i.e., if a series $\sum_n K_n$ of bounded operators on a Hilbert space $H$ is such that $\sum_n K_n$ converges unconditionally in the strong operator topology and $\sum_{n \in F} K_n$ is a compact operator for every index set $F \subseteq \mathbb{N}$, then $\sum_n K_n$ converges in the uniform operator topology.

In this paper, we introduce a Riemann type theorem for unconditionally convergent series of operators in $L(X, Y)$, the bounded linear operator space from a Banach space $X$ to a Banach space $Y$. To begin with, we introduce a Banach space valued sequence space $BMC(X)$, the space of bounded multiplier convergent series on $X$ (see Proposition 1 below) and characterize the compact sets in $BMC(X)$ (see Theorems 5, 7 below). Then by use of them, we obtain a characterization that the Riemann type theorem for unconditionally convergent series of operators in $L(X, Y)$ holds (see Theorem 9 below).

2. THE SPACE OF BOUNDED MULTIPLIER CONVERGENT SERIES

In this section, let $X$ be a Banach space, $X^*$ its dual space, and $B_X$ denote the closed unit ball of $X$.

Recall that a series $\sum_n x_n$ on $X$ is called unconditionally convergent if $\sum_n x_{\pi(n)}$ converges for each permutation $\pi$ of $\mathbb{N}$, the set of natural numbers; it is called subseries convergent if $\sum_n x_{k_n}$ converges for each increasing sequence $(k_n)$ of $\mathbb{N}$; and it is called bounded multiplier convergent if $\sum_n t_n x_n$ converges for each $(t_n) \in l_\infty$. It is known that unconditional convergence, subseries convergence, and bounded multiplier convergence of series in a Banach space are equivalent [8, p. 15]. Now we introduce a vector-valued sequence space $BMC(X)$ consisting of all bounded multiplier convergent series on $X$, i.e.,

\[
BMC(X) = \left\{ \bar{x} = (x_i) \in X^\mathbb{N} : \text{series } \sum_i t_i x_i \text{ converges for each } (t_i) \in l_\infty \right\}.
\]

For each $\bar{x} = (x_i) \in BMC(X)$, let

\[
\|\bar{x}\|_{BMC} = \sup \left\{ \left\| \sum_{i=1}^{\infty} t_i x_i \right\| : (t_i) \in B_{l_\infty} \right\}.
\]
By [7, Corollary 3], the set \( \{ \sum_{i=1}^{\infty} t_i x_i : (t_i) \in B_{l_1} \} \) is a bounded set of \( X \). So \( \| x \|_{\text{bmc}} \) is a finite number and it is easy to prove that \( \| \cdot \|_{\text{bmc}} \) is a norm on \( \text{BMC}(X) \).

**Proposition 1.** \( \text{BMC}(X) \) with the norm \( \| \cdot \|_{\text{bmc}} \) is a Banach space.

**Proof.** Let \( \{ \tilde{x}^{(n)} \} \) be a Cauchy sequence in \( \text{BMC}(X) \). Then

\[
\lim \sup_{n,m} \left\{ \sum_{i=1}^{\infty} t_i \left( x_i^{(n)} - x_i^{(m)} \right) : (t_i) \in B_{l_1} \right\} = 0. \tag{*}
\]

It follows that \( \{ x_i^{(n)} \}_{n=1}^{\infty} \) are Cauchy sequences in \( X \) for each \( i \in \mathbb{N} \) and hence there are \( x_i \in X \) such that \( \lim_n x_i^{(n)} = x_i \) for each \( i \in \mathbb{N} \). By \( (*) \), \( \lim_n \sum_{i=1}^{\infty} t_i x_i^{(n)} \) exists for each \( (t_i) \in B_{l_1} \). So [11, Theorem 3] guarantees that \( \tilde{x} = (x_i) \in \text{BMC}(X) \) and \( \lim_n \tilde{x}^{(n)} = \tilde{x} \). Thus we have proved that the norm \( \| \cdot \|_{\text{bmc}} \) is a complete norm. Q.E.D.

**Proposition 2.** For each \( \tilde{x} = (x_i) \in \text{BMC}(X) \),

\[
\| \tilde{x} \|_{\text{bmc}} = \sup \left\{ \sum_{i=1}^{\infty} |f(x_i)| : f \in B_{X^*} \right\}.
\]

**Proof.**

\[
\| \tilde{x} \|_{\text{bmc}} = \sup \left\{ \left\| \sum_{i=1}^{\infty} t_i x_i \right\| : (t_i) \in B_{l_1} \right\}
= \sup \left\{ \left\| \sum_{i=1}^{\infty} t_i f(x_i) \right\| : (t_i) \in B_{l_1}, f \in B_{X^*} \right\}
= \sup \left\{ \sum_{i=1}^{\infty} |f(x_i)| : f \in B_{X^*} \right\}. \tag{Q.E.D.}
\]

For \( \tilde{x} = (x_i) \in X^\infty \), we introduce the notation

\[
\tilde{x}(i \geq n) = (0, \ldots, 0, x_n, x_{n+1}, \ldots).
\]

**Proposition 3.** For each \( \tilde{x} \in \text{BMC}(X) \), \( \lim_n \| \tilde{x}(i \geq n) \|_{\text{bmc}} = 0 \).

**Proof.** Since \( \tilde{x} \in \text{BMC}(X) \) implies that the series \( \sum_i t_i x_i \) converges for each \( (t_i) \in l_1 \), the series \( \sum_i t_i x_i \) converges uniformly for all \( (t_i) \in B_{l_1} \) by [7, Example (1)]. So \( \lim_n \| \tilde{x}(i \geq n) \|_{\text{bmc}} = \lim_n \sup \| \sum_{i=n}^{\infty} t_i x_i \| : (t_i) \in B_{l_1} \) = 0. Q.E.D.

**Definition 4.** Let \( A \subseteq \text{BMC}(X) \). \( A \) is called a uniformly convergent set if \( \lim_n \| \tilde{x}(i \geq n) \|_{\text{bmc}} = 0 \) uniformly for all \( \tilde{x} \in A \). \( A \) is called a weakly uniformly convergent set if for each \( f \in X^* \), \( \lim_n \sum_{i=n}^{\infty} |f(x_i)| = 0 \) uniformly for all \( \tilde{x} = (x_i) \in A \).
For an index set $F \subseteq \mathbb{N}$, we define the linear map $\varphi_F : \operatorname{BM C}(X) \to X$, $\varphi_F(\bar{x}) = \sum_{i \in F} x_i$, for each $\bar{x} = (x_i) \in \operatorname{BM C}(X)$. Obviously, for each $\bar{x} \in \operatorname{BM C}(X)$, $\|\varphi_F(\bar{x})\|_X \leq \|\bar{x}\|_{\text{bmc}}$. So $\varphi_F$ is a continuous linear map. For $F = \{i\}$, denote $\varphi_i = \varphi_i$.

**Theorem 5.** Let $A \subseteq \operatorname{BM C}(X)$. Then $A$ is a relatively compact set if and only if

1. $A$ is a uniformly convergent set.
2. For each $i \in \mathbb{N}$, $\{\varphi_i(\bar{x}) : \bar{x} \in A\}$ is a relatively compact subset of $X$.

**Proof.** If $A$ is a relatively compact set, then (2) obviously holds. Next we prove that (1) holds.

Since if $A$ is relatively compact then it is totally bounded, for every $\varepsilon > 0$ there exists a finite subset $M$ of $\operatorname{BM C}(X)$ such that for each $\bar{x} \in A$ there is a $\bar{y} \in M$ satisfying $\|\bar{x} - \bar{y}\|_{\text{bmc}} \leq \varepsilon/2$. By Proposition 3, there is an $n_0 \in \mathbb{N}$ such that

$$\|\bar{y}(i \geq n)\|_{\text{bmc}} < \varepsilon/2, \quad \text{for } \bar{y} \in M \text{ and } n > n_0.$$ 

So

$$\|\bar{x}(i \geq n) - \bar{y}(i \geq n)\|_{\text{bmc}} \leq \|\bar{x} - \bar{y}\|_{\text{bmc}} \leq \varepsilon/2, \quad \text{for } n > n_0.$$ 

It follows that

$$\|\bar{x}(i \geq n)\|_{\text{bmc}} \leq \|\bar{x}(i \geq n) - \bar{y}(i \geq n)\|_{\text{bmc}} + \|\bar{y}(i \geq n)\|_{\text{bmc}}$$

$$< \varepsilon, \quad \text{for } n > n_0.$$ 

Thus we have proved that (1) holds.

Conversely, suppose the conditions (1) and (2) hold. Let $\{\bar{x}^{(n)}\} \subseteq A$. By (2), using the diagonal method, we can find a subsequence $\{n_k\}$ and $x_i^{(0)} \in X$ such that

$$\lim_{k} x_i^{n_k} = x_i^{(0)}, \quad i = 1, 2, \ldots.$$ 

For convenience, we can suppose that $n_k = k$, i.e.,

$$\lim_{n} x_i^{(n)} = x_i^{(0)}, \quad i = 1, 2, \ldots. \quad (**)$$

By (1) for each $\varepsilon > 0$, there is a $k_0 \in \mathbb{N}$ such that

$$\|\bar{x}(i \geq k_0)\|_{\text{bmc}} < \varepsilon/4, \quad \text{for } \bar{x} \in A.$$ 

And furthermore, by (**) there is an $n_0 \in \mathbb{N}$ such that for $n, m > n_0$,

$$\|x_i^{(n)} - x_i^{(m)}\| < \varepsilon/2k_0, \quad i = 1, 2, \ldots, k_0.$$
Thus for $n, m > n_0$,

$$
\|\bar{x}^{(n)} - \bar{x}^{(m)}\|_{\text{bmc}} \\
\leq \sum_{i=1}^{k_0-1} \|x_i^{(n)} - x_i^{(m)}\| + \|\bar{x}^{(n)}(i \geq k_0)\|_{\text{bmc}} + \|\bar{x}^{(m)}(i \geq k_0)\|_{\text{bmc}} < \varepsilon.
$$

So $\{\bar{x}^{(n)}\}_{n \geq 1}$ is a Cauchy sequence and hence, by Proposition 1 there exists an $\bar{x} \in \text{BMC}(X)$ such that $\lim_n \bar{x}^{(n)} = \bar{x}$. We have proved that $A$ is a relatively compact set.

**Lemma 6.** For every index set $F \subseteq \mathbb{N}$, $\varphi_F$ is c.c.t. → w.t. continuous on each weakly uniformly convergent subset of $\text{BMC}(X)$, where c.c.t. denotes the coordinatewise convergence topology on $\text{BMC}(X)$ and w.t. denotes the weak topology on $X$.

**Proof.** Let $A$ be a weakly uniformly convergent subset of $\text{BMC}(X)$ and $(\bar{x}^a)$ a net of $A$ such that $\lim_{a} x_i^a = 0$ for each $i \in \mathbb{N}$. Then for $\varepsilon > 0$ and $f \in X^*$, there is an $n_0 \in \mathbb{N}$ such that

$$
\sum_{i=n_0}^\infty |f(x_i)| < \varepsilon/2,
$$

for $\bar{x} = (x_i) \in A$.

And hence, there is an $\alpha_0$ such that for $\alpha > \alpha_0$,

$$
|f(x_i^a)| < \varepsilon/2n_0, \quad i = 1, 2, \ldots, n_0.
$$

So for $\alpha > \alpha_0$,

$$
|f(\varphi_F(\bar{x}^a))| \leq \sum_{i=1}^{n_0-1} |f(x_i^a)| + \sum_{i=n_0}^\infty |f(x_i^a)| < \varepsilon.
$$

Thus we have proved that w.t. $\lim_{a} \varphi_F(\bar{x}^a) = 0$. Q.E.D.

**Theorem 7.** Suppose $X$ contains no copy of $c_0$. Let $A \subseteq \text{BMC}(X)$. Then $A$ is a relatively compact set if and only if

1. $A$ is a weakly uniformly convergent set.
2. For every index set $F \subseteq \mathbb{N}$, $(\varphi_F(\bar{x}) : \bar{x} \in A)$ is a relatively compact subset of $X$.

**Proof.** If $A$ is a relatively compact set, then by the continuity of $\varphi_F$ and by Theorem 5, $A$ satisfies the conditions (3) and (4).

Conversely, suppose $A$ satisfies the conditions (3) and (4). Let $(\bar{x}^{(n)})_1 \subseteq A$. By the proof of Theorem 5, we can suppose that

$$
\lim_n x_i^{(n)} = x_i^{(0)} \in X, \quad i = 1, 2, \ldots. \hspace{1cm} (***)
$$

Next we prove that $\bar{x}^{(0)} = (x_i^{(0)}) \in \text{BMC}(X)$. 
For \( f \in X^* \), by (3) there exists a \( k_0 \in \mathbb{N} \) such that \( \sum_{i=1}^{\infty} |f(x_i)| \leq 1 \) for each \( x \in A \). Since (4) implies (2), \( \bigcup_{i=1}^{k_0} \{ \varphi_i(x) : x \in A \} \) is a relatively compact subset of \( X \) and hence, bounded. So there is a constant \( c > 0 \) such that

\[
|f(\varphi_i(x))| = |f(x_i)| \leq c, \quad \text{for } x \in A, i = 1, 2, \ldots, k_0.
\]

Thus

\[
\sum_{i=1}^{\infty} |f(x_i)| \leq k_0c + 1, \quad \text{for } x \in A.
\]

Now for a fixed \( m \in \mathbb{N} \), by (*** ) there is an \( n_0 \in \mathbb{N} \) such that

\[
|f(x_i^{(n_0)} - x_i^{(0)})| < 1/m, \quad i = 1, 2, \ldots, m.
\]

So

\[
\sum_{i=1}^{m} |f(x_i^{(0)})| \leq \sum_{i=1}^{m} |f(x_i^{(n_0)} - x_i^{(0)})| + \sum_{i=1}^{m} |f(x_i^{(n_0)})| \leq 1 + k_0c + 1.
\]

Since \( m \) is arbitrary, we have \( \sum_{i=1}^{\infty} |f(x_i^{(0)})| \leq k_0c + 2 < \infty \). Therefore, the series \( \sum x_i^{(0)} \) is a weakly unconditionally Cauchy series. It follows from the Bessaga–Pelczynski \( c_0 \) theorem (see [1] or see [2, p. 45, Theorem 8]) that \( \sum x_i^{(0)} \) is unconditionally convergent and hence, bounded multiplier convergent. Thus we have proved that \( \tilde{x}^{(0)} \in \text{BMC}(X) \).

Let \( D = A \cup \{ x^{(0)} \} \). For every index set \( F \subseteq \mathbb{N} \), since Lemma 6 implies that \( \varphi_F \) is c.c.t. w.t. continuous on \( D \), by (*** ) we have w.t. \( \lim_n \varphi_F(x^{(n)}) = \varphi_F(x^{(0)}) \). By use of the condition (4), we have \( \lim_n \varphi_F(x^{(n)}) = \varphi_F(x^{(0)}) \), i.e.,

\[
\lim_n \sum_{i \in F} x_i^{(n)} = \sum_{i \in F} x_i^{(0)}.
\]

It follows from [12, Proposition 4] that

\[
\lim_n \sum_{i=1}^{\infty} t_i x_i^{(n)} = \sum_{i=1}^{\infty} t_i x_i^{(0)}
\]

uniformly for all \( (t_i) \in B_{l_p} \), i.e., \( \lim_n \tilde{x}^{(n)} = \tilde{x}^{(0)} \). So we have proved that \( A \) is a relatively compact set. Q.E.D.

Remark. Condition (4) in Theorem 7 cannot be replaced by condition (2) in Theorem 5. For example, let \( X = l_p \) \( (1 < p < \infty) \), \( e_i = (0, \ldots, 0, 1^{(i)}, 0, 0, \ldots) \) and let \( A = \{ (0, \ldots, 0, e_i, 0, 0, \ldots) : n = 1, 2, \ldots \} \). Then \( A \subseteq \text{BMC}(X) \) and it is easy to see that \( A \) satisfies conditions (2) and (3) but does not satisfy (1) or (4) and it is not relatively compact.
By a proof similar to that of Theorem 7 and by [2, p. 49, Corollary 11], we have:

**Theorem 8.** Suppose $X^*$ contains no copy of $c_0$. Let $A \subseteq \text{BM}(X^*)$. Then $A$ is a relatively compact set if and only if

1. $A$ is a weak* uniformly convergent set.
2. For every index set $F \subseteq \mathbb{N}$, $(\sum_{i \in F} f_i : (f_i) \in A)$ is a relatively compact subset of $X^*$.

### 3. A Riemann Type Theorem in $L(X,Y)$

In this section, let $X, Y$ be two Banach spaces and let $L(X,Y)$ denote the space of bounded linear operators from $X$ to $Y$. Uniform operator topology, strong operator topology, and weak operator topology are simply denoted by U.O.T., S.O.T., and W.O.T., respectively.

**Theorem 9.** Every series $\Sigma_i K_i$ of operators in $L(X,Y)$ such that

1. $\Sigma_i K_i$ converges unconditionally in the W.O.T.
2. $\Sigma_i \in F K_i$ is a compact operator for every index set $F \subseteq \mathbb{N}$

is unconditionally convergent in the U.O.T. if and only if $X^*$ contains no copy of $c_0$.

**Proof.** Suppose $X^*$ contains no copy of $c_0$. Let $\Sigma_i K_i$ be a series of operators in $L(X,Y)$ which satisfies the conditions (7) and (8). For a fixed $x \in X$, by (7) the series $\Sigma_i K_i x$ of $Y$ is weakly unconditionally convergent and hence, weak subsseries convergent. So it follows from the Orlicz–Pettis Theorem [2, p. 24] that the series $\Sigma_i K_i x$ of $Y$ is subseries convergent and hence, bounded multiplier convergent. Thus $(K_i x) \in \text{BM}(Y)$. By Proposition 3, for each $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\left\| \sum_{i=n}^{\infty} t_i K_i x \right\| < \varepsilon, \quad \text{for} \ (t_i) \in B_\varepsilon.$$

For $y' \in B_{Y^*}$, let $s_i = \text{sign}(\langle K_i x, y' \rangle)$. Then $(s_i) \in B_\varepsilon$. So for $n > n_0$,

$$\left\| \sum_{i=n}^{\infty} s_i K_i x, y' \right\| \leq \left\| \sum_{i=n}^{\infty} s_i K_i x \right\| < \varepsilon.$$  

Let $K_i^*$ denote the adjoint operator of $K_i$. Then

$$\left\| \sum_{i=n}^{\infty} s_i K_i^* y' \right\| = \sum_{i=n}^{\infty} |\langle K_i x, y' \rangle| < \varepsilon, \ y' \in B_{Y^*}.$$
Now by (8), for every index set $F \subseteq \mathbb{N}$, $\sum_{i \in F} K_i^* = (\sum_{i \in F} K_i)^*$ is a compact operator. So the set

$$(11) \quad \left\{ \sum_{i \in F} K_i^* y' : y' \in B_{Y^*} \right\}$$

is a relatively compact subset of $X^*$. Thus by (10), (11), and Theorem 8 we see that the set $A = \{(K_i^* y') : y' \in B_{Y^*}\}$ is a relatively compact subset of $BM C(X^*)$. So by Theorem 5, $A$ is a uniformly convergent subset of $BM C(X^*)$ and hence, $\lim_n \|(K_i^* y')(i \geq n)\|_{l_{BM C}} = 0$ uniformly for all $y' \in B_{Y^*}$, i.e.,

$$\lim_n \sup \left\{ \left\| \sum_{i=n}^{\infty} t_i K_i^* y' \right\| : (t_i) \in B_{l_{BM C}}, y' \in B_{Y^*} \right\} = 0.$$ 

So for each $(t_i) \in B_{l_{BM C}}$,

$$\lim_n \left\| \sum_{i=n}^{\infty} t_i K_i^* \right\| = \lim_n \left\| \sum_{i=n}^{\infty} t_i K_i \right\| = \lim_n \sup \left\{ \left\| \sum_{i=n}^{\infty} t_i K_i^* y' \right\| : y' \in B_{Y^*} \right\} = 0.$$

Therefore, the series $\sum_{i=1}^{\infty} K_i$ is bounded multipliers convergent and hence, unconditionally convergent in the U.O.T.

Conversely, suppose $X^*$ contains a copy of $c_0$. Let $T : c_0 \to X^*$ be an isomorphism and $T^* : X^{**} \to l_1$ be the adjoint operator of $T$. And let $R = T^* | X$. Then for each $x \in X$, $Rx \in l_1$. Let $(Rx)_i$ denote the $i$th coordinate of $Rx$. For a fixed $y \in Y$ with $\|y\| = 1$, define

$$K_i : X \to Y, \quad K_i x = (Rx)_i y, \text{ for each } x \in X.$$ 

Then $K_i \in L(X, Y)$ for $i \in \mathbb{N}$. And for each $x \in X$, $\sum_{i=1}^{\infty} \|K_i x\| = \sum_{i=1}^{\infty} \|(Rx)_i\| = \|Rx\|_{l_1} < \infty$. So the series $\sum_{i=1}^{\infty} K_i$ converges absolutely in the S.O.T. and hence, satisfies the condition (7).

For every index set $F \subseteq \mathbb{N}$ and each $x \in X$, since $\sum_{i \in F} K_i x = \sum_{i \in F} (Rx)_i y$, $\sum_{i \in F} K_i$ is a rank one operator and hence, is a compact operator. Thus $\sum_{i=1}^{\infty} K_i$ satisfies the condition (8).

Now assume that the series $\sum_{i=1}^{\infty} K_i$ is unconditionally convergent in the U.O.T. Then $\sum_{i=1}^{\infty} K_i$ is bounded multiplier convergent in the U.O.T. Hence for each $(t_i) \in l_{BM C}$,

$$\lim_n \sup \left\{ \left\| \sum_{i=n}^{\infty} t_i (Rx) \right\| : x \in B_X \right\} = \lim_n \sup \left\{ \left\| \sum_{i=n}^{\infty} t_i K_i x \right\| : x \in B_X \right\} = \lim_n \left\| \sum_{i=n}^{\infty} t_i K_i \right\| = 0.$$
It follows from [6, p. 108, Proposition 6.11] that the set \( \{ Rx : x \in B_x \} \) is a relatively weakly compact subset of \( l_1 \) and hence, it is relatively compact. Thus \( R \) is a compact operator, which contradicts the fact that \( T \) is an isomorphism. With this contradiction we have proved that \( \Sigma, K \) is not unconditionally convergent in the U.O.T. Q.E.D.

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