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The Yang-Mills Equations on the Universal Cosmos

Y. CHOQUET-BRUHAT

Universite de Paris VI, 75230 Paris Cedex 05, France

S. M. PANEITZ

Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

AND

I. E. SEGAL*

Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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Global existence and regularity of solutions for the Yang–Mills equations on the universal cosmos $\tilde{\mathbf{M}}$, which has the form $R^1 \times S^3$ for each of an 8-parameter continuum of factorizations of $\tilde{\mathbf{M}}$ as time $\times$ space, are treated by general methods. The Cauchy problem in the temporal gauge is globally soluble in its abstract evolutionary form with arbitrary data for the field $\oplus$ potential in $L_{2,r}(S^3) \oplus L_{2,r+1}(S^3)$, where r is an integer >1 and $L_{2,r}$ denotes the class of sections whose first r derivatives are square-integrable; if $r = 1$, the problem is soluble locally in time. When r is 3 or more the solution is identifiable with a classical one; if infinite, the solution is in $C^\infty(\tilde{\mathbf{M}})$. These results extend earlier work and approaches [1–5]. Solutions of the equations on Minkowski space-time $\mathbf{M}_0$ extend canonically (modulo gauge transformations) to solutions on $\tilde{\mathbf{M}}$ provided their Cauchy data are moderately smooth and small near spatial infinity. Precise asymptotic structures for solutions on $\mathbf{M}_0$ follow, and in turn imply various decay estimates. Thus the energy in regions uniformly bounded in direction away from the light cone is $O(|x_0|^{-5})$, where x_0 is the Minkowski time coordinate; analysis solely in $\mathbf{M}_0$ [8, 9] earlier yielded the estimate $O(|x_0|^{-2})$ applicable to the region within the light cone. Similarly it follows that the action integral for a solution of the Yang–Mills equations in $\mathbf{M}_0$ is finite, in fact absolutely convergent.

1. INTRODUCTION

The Yang–Mills equations are conformally invariant, like the Maxwell equations from which they originated, and so extend invariantly and

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maximally to the universal cover $\tilde{\mathbf{M}}$ of the conformal compactification $\bar{\mathbf{M}}$ of Minkowski space $\mathbf{M}_0$. In addition to $\mathbf{M}_0$, $\tilde{\mathbf{M}}$ contains the de Sitter spaces in a canonical way, and otherwise provides a type of maximal model for space-time known as the universal cosmos [7]. It thus appears natural to consider the Yang–Mills equations on $\tilde{\mathbf{M}}$, and extension to this larger space-time is useful even from the standpoint of Minkowski space itself since the temporal asymptotics in $\mathbf{M}_0$ are derivable from this extension.

This paper shows that the earlier results by Segal [1] and by Eardley and Moncrief [4] on the existence of a solution to the Cauchy problem for the Yang–Mills equations in $\mathbf{M}_0$ in the temporal gauge extend to $\tilde{\mathbf{M}}$. The proof uses suitable adaptations of the earlier methods for $\mathbf{M}_0$, which are simplified and generalized in certain respects, together with aspects of the geometry of the imbedding of $\mathbf{M}_0$ into $\tilde{\mathbf{M}}$. This geometry was involved in a physically related way in [6] and was used in the present connection by Christodoulou [3] and by Choquet–Bruhat and Christodoulou [2] to show global existence of solutions of conformally invariant or regular partial differential equations with small Cauchy data. Here the precise asymptotics of solutions in $\mathbf{M}_0$ are derived from the treatment of this geometry in [7] in connection with analysis of general space-time bundles. The asymptotics derivable in this fashion are in a certain sense best possible; in particular, they yield temporal decay rates significantly more rapid than those earlier obtained from the study of conserved quantities in $\mathbf{M}_0$. In this connection see Glassey and Strauss [8, 9]. For a result for fields with small data by a different approach, see [2].

A summary treatment of the global existence question on $\tilde{\mathbf{M}}$ based on the theory of Leray [10] for hyperbolic partial differential equations on general manifolds was earlier given in [5]. The present treatment takes advantage of invariance features of the Einstein Universe that facilitate subsumption under more purely functional analytic considerations. In less than four space–time dimensions, Ginibre and Velo [11, 12] have shown global existence of solutions for closely related equations. An early perturbative result is due to Kerner [13]. A survey of related matters as of 1979 was given by Choquet–Bruhat in [14]. Glassey and Strauss have studied global existence in $\mathbf{M}_0$ for a special class of solutions [15]. A recent survey in the general context of hyperbolic equations of gauge theories was given by Choquet–Bruhat in [16].

2. Nonlinear One-Parameter Groups

The present treatment appears more intelligible in the general setting of nonlinear semigroups as introduced in [17]. However no attempt at maximal generality will be made here, and only full groups will be considered.

Throughout the present section, **B** denotes a given Banach space; $W(t)$ for t in R^1 denotes a given 1-parameter continuous group of bounded linear transformations on **B**; $K_t(u)$ for t in R^1 and u in **B** denotes a given continuous map from $R^1 \times \mathbf{B}$ to **B**; ϕ a function from $[0, \infty)$ to itself that is assumed only to be bounded on bounded sets, and is otherwise generic. Thus $K_t(u)$ is said to be "boundedly lipschitzian locally in t" in case

$$\|K_t(u) - K_t(v)\| \leqslant \phi(|t| + \|u\| + \|v\|)\, \|u - v\|$$

for some ϕ. By "solution" of the equation

$$u(t) = W(t)\, u_0 + \int_0^t W(t-s)\, K_s(u(s))\, ds, \tag{1}$$

where u_0 is given in **B**, will be meant a continuous map $t \to u(t)$ from an open interval J in R_1 containing 0 that satisfies Eq. (1) in J. If J is all of R^1, the solution is called *global*, and is otherwise called a *local* (*in time*) solution.

THEOREM 1. *Let $V(\cdot)$ be a continuous 1-parameter group of bounded linear operators on* **B** *such that for all s and t in R^1 and u in* **B**

$$V(t)\, K_s(u) = K_s(V(t)\, u), \qquad V(t)\, W(s) = W(s)\, V(t).$$

Suppose also that $K_t(u)$ is of class C^n as a map from $R^1 \times \mathbf{B}$ to **B** *($n \geqslant 1$). Let P denote the infinitesimal generator of $V(\cdot)$.*

If u_0 is in the domain $\mathbf{D}(P^n)$, then $u(t)$ is in this domain for all t in the maximal interval T of existence for the datum u_0. Moreover the map $(u_0, t) \to u(t)$ is continuous from $\mathbf{D}(P^n) \times T$ to $\mathbf{D}(P^n)$ relative to the norm on $\mathbf{D}(P^n)$:

$$\|u\|_n = \|u\| + \|Pu\| + \cdots + \|P^n u\|. \tag{2}$$

Proof. Consider first the case $n = 1$. Since $K_t(u)$ is of class C^1 as a function of t and u, it is boundedly lipschitzian locally in t, so that local solutions exist for arbitrary u_0 in **B**. Now suppose that u_0 is in $\mathbf{D}(P)$, and let T denote the maximal interval of existence for this datum (open, containing 0). The linear equation

$$y(t) = W(t)\, Pu_0 + \int_0^t W(t-s)\, \partial_u K_s(u)|_{u=u(s)}\, y(s)\, ds \tag{3}$$

has a solution throughout T. Let $z_e(t) = e^{-1}(V(e) - 1)\, u(t) - y(t)$ for t in T.

Then z_e satisfies the equation

$$z_e(t) = W(t)[e^{-1}(V(e)-1)-P]\,u_0 + \int_0^t W(t-s)[e^{-1}(V(e)-1)\,K_s(u(s))$$
$$-\,\partial_u K_s(u)|_{u=u(s)}\,y(s)]\,ds.$$

Now $(V(e)-1)\,K_s(u(s)) = K_s(V(e)\,u(s)) - K_s(u(s))$, which by the mean value theorem in a Banach space is in the convex closure as h ranges over the interval $(0, e)$ of $\partial_u K_s(u)|_{u=V(h)u(s)}\,(V(e)-1)\,u(s)$. By continuity this differs from $\partial_u K_s(u)|_{u=u(s)}\,(V(e)-1)\,u(s)$ by $o(1)\|(V(e)-1)\,u(s)\|$ uniformly on any compact subinterval T_0 of T, as $e \to 0$. Noting that

$$|e|^{-1}\,\|(V(e)-1)\,u(s)\| \leqslant \|z_e(s)\| + \|y(s)\|,$$

the integrand is

$$W(t-s)[\partial_u K_s(u)|_{u=u(s)}\,z_e(s) + o(1)(\|z_e(s)\| + \|y(s)\|)].$$

It follows that uniformly on T_0

$$\|z_e(t)\| \leqslant o(1) + \int_0^t C\,\|z_e(s)\|\,ds,$$

which by Gronwall's inequality implies that $z_e(s)$ is uniformly $o(1)$ on T_0. Thus $u(s)$ is in $\mathbf{D}(P)$ for all s in T, and $Pu(s) = y(s)$. The continuity of the map $(u_0, t) \to u(t)$ in the $\|\cdot\|_1$ norm also follows.

The general case now follows by an induction argument, similar to that for the case $n = 2$, which proceeds as follows. The linear inhomogeneous equation

$$x(t) = W(t)\,Py(0) + \int_0^t W(t-s)[\partial_u K_s(u)|_{u=u(s)}\,x(s)$$
$$+\,\partial^2_{uu} K_s(u)|_{u=u(s)}\,(y(s), y(s))]\,ds$$

has a solution throughout T. Now let

$$w_e(t) = e^{-1}(V(e)-1)\,y(t) - x(t)$$

for t in T. Then w_e satisfies

$$w_e(t) = W(t)[e^{-1}(V(e)-1)-P]\,y(0)$$
$$\times \int_0^t W(t-s)[e^{-1}(V(e)-1)\,\partial_u K_s(u)|_{u=u(s)}\,y(s)$$
$$-\,\partial_u K_s(u)|_{u=u(s)}\,x(s) - \partial^2_{uu} K_s(u)|_{u=u(s)}\,(y(s), y(s))]\,ds.$$

Now for any C^2 map L from **B** to **B** with the property that $V(s)L(u) = L(V(s)u)$ for all s in R^1 and u in **B**,

$$\begin{aligned} e^{-1}&(V(e)-1)(\partial_u L(u))y \\ &= \lim_{h\to 0} (eh)^{-1}(V(e)-1)(L(u+hy)-L(u)) \\ &= \lim_{h\to 0} (eh)^{-1}[L(V(e)(u+hy))-L(V(e)u)-L(u+hy)+L(u)] \\ &= \lim_{h\to 0} (eh)^{-1}[L(u+p+q)-L(u+p)-L(u+q)+L(u) \\ &\quad + L(u+hV(e)y)-L(u+hy)], \end{aligned}$$

where $p = (V(e)-1)u$, $q = hV(e)y$. Further,

$$\lim_{h\to 0} h^{-1}[L(u+hV(e)y)-L(u+hy)] = (\partial_u L)(V(e)y-y);$$

$\lim_h (eh)^{-1}[L(u+p+q)-L(u+p)-L(u+q)+L(u)]$ is in the convex closure of $(\partial_u^2 L)|_{u_1}(p,q)$, where u_1 varies on the line segment joining u and $u+p$. It follows as earlier that w_e satisfies an inequality of the form

$$\|w_e(t)\| \leqslant o(1) + \int_0^t \|w_e(s)\|\, ds$$

on compact subintervals of T, whence $w_e(t)\to 0$ as $e\to 0$. This in turn shows that $y(t)$ is in $\mathbf{D}(P)$ and that $Py(t) = x(t)$. As before, continuity of the map $(u_0, t)\to u(t)$ in the $\|\cdot\|_2$ norm follows. Continuing this process one derivative at a time, the theorem follows.

COROLLARY 1.1. *If V is a continous unitary representation of the Lie group G on* **B** *such that for all g in G, t in R^1, and u in the Hilbert space* **B**,

$$V(g)\,W(t) = W(t)\,V(g), \qquad V(g)\,K_s(u)) = K_s(V(g)\,u)), \tag{4}$$

and if $K_t(u)$ is of class C^n as a function of (t, u), then the n-fold differentiable vectors in **B** *with respect to V are invariant under the temporal propagation defined by Eq.* (1), *within the interval of existence.*

Proof. Apply Theorem 1 to each generator and its powers, and refer to the theory of differentiable vectors in group representations (Goodman [18]).

EXAMPLE. Let G be a Lie group, F a finite-dimensional vector space, **B** an L_2-Sobolev space of functions f from G to F, and $V(g)$ of the form: $(V(g)f)(x) = f(g^{-1}x)$, x in G. If $K_t(u)$ has the form $(K_t(u))(x) = p(u(x))$, where p is a polynomial on F, then Eqs. (4) hold.

Thus the equation

$$(\partial^2/\partial t^2 - \Delta + c)\,\phi + q(\phi) = 0; \qquad \phi(0,x) = \phi_0(x), \qquad \dot\phi(0,x) = \phi_1(x),$$

with ϕ_0 in $\mathbf{H}_1$ and ϕ_1 in $\mathbf{H}_0$ and where Δ denotes the Laplace–Beltrami operator on S^3, c is positive, and q is a polynomial of degree at most 3, has local-in-time solutions to the corresponding evolutionary equation in $\mathbf{H}_1 \oplus \mathbf{H}_0$ (cf. [17]) that remain in $\mathbf{H}_{n+1} \oplus \mathbf{H}_n$ if in this space initially, n being an arbitrary non-negative integer. The same is true for polynomials q of arbitrary degree n is taken to be sufficiently large.

COROLLARY 1.2. *Suppose that $K_t(u)$ is t-independent and boundedly lipschitzian. Suppose* **D** *is a dense subset of* **B** *with the property that if u_0 is in* **D** *then the solution $u(\cdot)$ of* Eq. (1) *has $u(t)$ in* **D** *for all t in the interval $T(u_0)$ of existence. Suppose there exists a positive constant e and a continuous function ϕ from* **B** *to* $[0, \infty)$ *such that if $u(\cdot)$ and $v(\cdot)$ are local solutions with data u_0 and v_0 in* **D**, *then*

$$\|K(u(t)) - K(v(t))\| \leqslant \phi(\|u_0\| + \|v_0\| + |t|)\|u(t) - v(t)\|;$$
$$\|K(u(t)\| \leqslant \phi(\|u_0\| + |t|)\|\,u(t)\|$$

for all t that are both in $(-e, e)$ and in the common interval of existence.

Then Eq. (1) *has a global solution for arbitrary data in* **B**.

Proof. Suppose that u_0 is in **D**. Estimating from Eq. (1),

$$\|u(t)\| \leqslant C(t) + \int_0^t C(t)\,\phi(\|u_0 + |t|)\|u(s)\|\,ds$$

for t in $(-e, e)$ and $T(u_0)$. It follows from Gronwall's inequality that $\|u(t)\|$ remains bounded throughout the common part of the interval of existence and $(-e, e)$, which implies that $(-e, e)$ is contained in $T(u_0)$. Since the map $u_0 \to u(s)$ from **D** to **D** is invertible by the map $u_0 \to u(-s)$, the former map is from **D** onto **D**, for sufficiently small s, so that the mappings $L(s)$: $u_0 \to u(s)$, u_0 in **D**, form a local 1-parameter group in **D**, say for $|s| < g$. Defining $L(s)$ for arbitrary real s by

$$L(s) = L(s_1)\,L(s_2) \cdots L(s_n) \text{ if } s = s_1 + s_2 + \cdots + s_n,\ |s_j| < g,$$

it follows purely algebraically that $L(s)$ is uniquely defined, and that $L(\cdot)$ is a global 1-parameter group of transformations on **D**.

Now if u_0 is arbitrary in **B**, let $\{u_n\}$ denote a sequence of elements of **D** such that $u_n \to u_0$, and let $u_n(\cdot)$ denote the solution of Eq. (1) with Cauchy datum u_n. Subtracting Eq. (1) for $u_n(\cdot)$ from that for $u_m(\cdot)$, it follows that

$$\|u_m(t) - u_n(t)\| \leqslant C(t)\, o(1) + C(t) \int_0^t \|K(u_m(s)) - K(u_n(s)\|\, ds,$$

where $C(\cdot)$ is bounded on bounded intervals. Applying the hypothesis as to the normed term in the integrand, and Gronwall's inequality, it follows that $\|u_m(t) - u_n(t)\|$ is $o(1)$ for all t in $(-e, e)$, uniformly on compact subintervals. Hence the sequence $\{u_n(t)\}$ converges uniformly on any compact subinterval of $(-e, \epsilon)$ to a solution of Eq. (1) with Cauchy datum u_0.

This provides a local 1-parameter evolutionary group in **B** defining the propagation $u_0 \to u(t)$ corresponding to Eq. (1), and it follows as in the case of solutions with values in **D** that this extends uniquely to a global one-parameter group.

3. The Yang–Mills Equations

Let $\mathbf{\Gamma}$ be a given compact Lie group, Γ its Lie algebra, and $(\cdot,\cdot)$ a $\mathbf{\Gamma}$-invariant positive definite inner product on Γ. The Yang–Mills equations concern Γ- valued differential forms, i.e., multilinear antisymmetric forms with values in Γ, on the space of C^∞ vector fields on the manifold in question (cf. Lichnerowicz [19] and Choquet–Bruhat and DeWitt–Morette [20]). The present investigation treats the case of the universal cosmos $\tilde{\mathbf{M}}$, which becomes the Einstein Universe when a metric is imposed on $\tilde{\mathbf{M}}$ that is invariant under the maximal compact subgroup of the 15–parameter automorphism group of $\tilde{\mathbf{M}}$. Regarding $\tilde{\mathbf{M}}$, compare [7], whose notation is used here.

An admissible metric on $\tilde{\mathbf{M}}$ will be defined as one that is subordinate to the given causal (or conformal) structure on $\tilde{\mathbf{M}}$. All metrics, vector fields, etc., will be assumed C^∞ unless otherwise indicated. Given an admissible pseudo-Riemannian metric g, it will be extended by linearity from vector fields to homogeneous maps from the space of vector fields to given algebras. Thus if Y_j $(j = 0, 1, 2, 3)$ is any orthogonal basis for the vector fields near a point p of $\tilde{\mathbf{M}}$, e.g., $(Y_i, Y_j) = \pm o_{ij}$, and if P and Q are linear maps from the space of such vector fields (as a module over the algebra of C^∞ functions) to an algebra **A**, then

$$g(P, Q) = \sum_{i,j} P(Y_i)\, Q(Y_j)\, g(Y_i, Y_j).$$

When as in the present application **A** is a Lie algebra, the notation $g(P, Q) = [P; Q]$, or on occasion $[P(\cdot); Q(\cdot)]$, will be used.

The Yang–Mills equations are equations for a 1-form A and a 2-form F, both having values in Γ. They may be expressed as follows, X and Y being arbitrary vector fields on the manifold in question, which is assumed endowed with a given conformal structure:

$$F(X, Y) = dA(X, Y) - [A(X), A(Y)], \tag{5}$$

where d denotes the usual exterior derivative

$$(dA)(X, Y) = XA(Y) - YA(X) - A[X, Y];$$

and with $\delta = *d*$,

$$(\delta F)(X) = [F(\cdot, X); A(\cdot)]. \tag{6}$$

We recall the convention regarding $*$: if $s_1, \ldots, s_n$ is an ordered basis of 1-forms, the positive orientation being $s_1 s_2 \cdots s_n$, then $*(s_1 \cdots s_p) = g(s_{p+1} \cdots s_n, s_{p+1} \cdots s_n)\, s_{p+1} \cdots s_n$. Pro forma, Eq. (6) depends on the metric, but the equations are known to be conformally invariant and are unique as regards the differential equations aspects considered here but may be given a variety of equivalent formulations (loc. cit.).

The central question considered here is that of the Cauchy problem for these equations on $\tilde{\mathbf{M}}$ relative to a fixed but arbitrary factorization of $\tilde{\mathbf{M}}$ in the form $R^1 \times SU(2)$ into time and space factors. This problem may conveniently be treated in terms of scalar-valued functions defined globally on $\tilde{\mathbf{M}}$, representing the coefficients of A and F relative to the following fixed basis. Let β_j $(j = 0, 1, 2, 3)$ denote the 1-form on $\tilde{\mathbf{M}}$ such that $\beta_j(X_k) = \delta_{jk}(k = 0, 1, 2, 3)$. The β_j are invariant under left translations relative to the presentation of $\tilde{\mathbf{M}}$ as the group $\tilde{U}(2) \simeq R^1 \times SU(2) \sim R^1 \times S^3$, where here R^1 denotes the additive group of the reals.

In terms of the coordinates t, u_1, u_2, u_3, these 1-forms are given as follows: $\beta_0 = dt$;

$$u_4\beta_1 = (u_1^2 + u_4^2)\, du_1 + (u_1u_2 - u_3u_4)\, du_2 + (u_1u_3 + u_2u_4)\, du_3,$$

$$u_4\beta_2 = (u_2^2 + u_4^2)\, du_2 + (u_2u_3 - u_1u_4)\, du_3 + (u_2u_1 + u_3u_4)\, du_1,$$

$$u_4\beta_3 = (u_3^2 + u_4^2)\, du_3 + (u_3u_1 - u_2u_4)\, du_1 + (u_3u_2 + u_1u_4)\, du_2.$$

The positive orientation on $\tilde{\mathbf{M}}$ will be defined by the 4-form $\beta_0\beta_1\beta_2\beta_3$. The metric used henceforth is the invariant one on $\tilde{U}(2)$: $g(X_i, X_j) = e_i\delta_{ij}$, where $(e_0, e_1, e_2, e_3) = (1, -1, -1, -1)$. Using standard notational conventions and letting i, j, k denote the indices 1, 2, 3 in cyclic order, one finds

$$d\beta_0 = 0, \qquad d\beta_i = 2\beta_j\beta_k; \qquad d(\beta_0\beta_i) = -2\beta_0\beta_j\beta_k; \qquad d(\beta_i\beta_j) = 0;$$

$$*(\beta_0\beta_i) = \beta_j\beta_k, \qquad *(\beta_0\beta_i\beta_j) = -\beta_k, \qquad *(\beta_i\beta_j\beta_k) = -\beta_0.$$

For arbitrary 1- and 2-forms A and F, not necessarily satisfying (5) or (6), we set $A_i = A(X_i)$, $F_{ij} = F(X_i, X_j)$; thus $A = \sum_i A_i \beta_i$ and $F = \sum_{i<j} F_{ij} \beta_i \beta_j$. Straightforward computations now show that

$$dA = \sum_{i<j} (X_i A_j - X_j A_i) \beta_i \beta_j + 2 \sum' A_i \beta_j \beta_k,$$

where $\sum$ involves summation over all indices 0, 1, 2, 3 and $\sum'$ denotes summation only over the three cyclic permutations of 1, 2, 3. Also,

$$\begin{aligned} -\delta F = {} & (X_1 F_{01} + X_2 F_{02} + X_3 F_{03}) \beta_0 + (X_0 F_{01} + X_2 F_{12} + X_3 F_{13} + 2F_{23}) \beta_1 \\ & + (X_0 F_{02} + X_1 F_{21} + X_3 F_{23} + 2F_{31}) \beta_2 \\ & + (X_0 F_{03} + X_1 F_{31} + X_2 F_{32} + 2F_{12}) \beta_3 . \end{aligned}$$

It follows that in terms of scalar components, Eq. (5) takes the form

$$\begin{aligned} F_{0j} &= X_0 A_j - X_j A_0 - [A_0, A_j] \quad (j = 1, 2, 3), \\ F_{ij} &= X_i A_j - X_j A_i + 2A_k - [A_i, A_j] \quad (i, j, k = 1, 2, 3 \text{ in cyclic order}). \end{aligned} \tag{7}$$

Noting that

$$\begin{aligned} [F(\cdot, X_j), A(\cdot)] = {} & -[F_{j0}, A_0] + [F(X_j, X_1), A_1] + [F(X_j, X_2), A_2] \\ & + [F(X_j, X_3), A_3], \end{aligned}$$

Eq. (6) takes the form (for $j = 0$):

$$-(X_1 F_{10} + X_2 F_{20} + X_3 F_{30}) = [F_{10}, A_1] + [F_{20}, A_2] + [F_{30}, A_3]; \tag{8}$$

and, introducing the notations

$$\begin{aligned} E_j &= F_{j0}, \qquad E = (E_1, E_2, E_3) = (F_{10}, F_{20}, F_{30}), \\ H_i &= F_{kl}, \qquad H = (H_1, H_2, H_3) = (F_{32}, F_{13}, F_{21}), \end{aligned}$$

Eq. (8) may also be written as follows,[1] with $\nabla = (X_1, X_2, X_3)$:

$$\nabla \cdot E = [A ; E]; \tag{9}$$

and in addition the following equations are obtained:

$$\begin{aligned} -H &= \nabla \times A + 2A - A \times A, \\ X_0 E &= -\nabla \times H - 2H - [E, A_0] + [A \mathbin{\bar{\times}} H]. \end{aligned} \tag{10}$$

[1] The following notations are used: (1) $[A \mathbin{\bar{\times}} B]_i = [A_j, B_k] - [A_k, B_j]$, where i, j, k are 1, 2, 3 in cyclic order; (2) $[A; B] = [A_1, B_1] + [A_2, B_2] + [A_3, B_3]$; (3) $(\nabla \times A)_i = X_j A_k - X_k A_j$, where i, j, k are in cyclic order; (4) $(\nabla f)_i = X_i f$, f being a function.

The temporal gauge is defined by the condition $A_0 = 0$. Assuming henceforth that this is the case, and noting that then $X_0 A = -E$, it follows on applying X_0 to the first of Eqs. (10) and simplifying the second that

$$\begin{aligned} X_0 E &= -\nabla \times H - 2H + [A \overset{\times}{,} H], \\ X_0 H &= \nabla \times E + 2E - [A \overset{\times}{,} E]. \end{aligned} \tag{11}$$

4. Solution of the Cauchy Problem

In order to subsume the Yang–Mills equations under general theory, it is convenient to use the "fixed-time constraint" (9) to improve the regularity of one of the evolutionary equations. More specifically, the first of Eqs. (11) takes the form, as an equation for A,

$$\ddot{A} = \nabla \times H + 2H - [A \overset{\times}{,} H]; \qquad H = -\nabla \times A - 2A + A \times A.$$

Eliminating H by substitution yields the equation

$$\begin{aligned} \ddot{A} + \nabla \times (\nabla \times A) = {}& -4A - 4\nabla \times A + 6A \times A + \nabla \times (A \times A) \\ & + [A \overset{\times}{,} \nabla \times A] - [A \overset{\times}{,} A \times A]. \end{aligned} \tag{12}$$

Because of the partial degeneracy of the operator $A \to \nabla \times (\nabla \times A)$, the treatment given in [17] for scalar wave equations does not extend directly to this equation. The equation can however be modified so that a similar treatment becomes applicable. To this end, note first the identity

$$\nabla \times (\nabla \times A) = -LA + \nabla(\nabla \cdot A) - 2\nabla \times A, \tag{13}$$

where L denotes the Laplacian, applicable to arbitrary smooth A (irrespective of the Yang–Mills equations). Substitution in Eq. (12) now leads to

$$\begin{aligned} \ddot{A} + (I - L)A = {}& -3A - 2\nabla \times A - \nabla(\nabla \cdot A) + 6A \times A + \nabla \times (A \times A) \\ & + [A \overset{\times}{,} \nabla \times A] - [A \overset{\times}{,} A \times A]. \end{aligned} \tag{14}$$

Second, the only term on the right-hand side of Eq. (14) involving two differentiations can be adequately desingularized by using Eq. (9). Indeed, differentiation of Eq. (14) with respect to time gives the following equation, now adopting the notation: $L' = I - L$.

$$\begin{aligned} \dddot{A} + L'\dot{A} = {}& -3\dot{A} - 2\nabla \times \dot{A} - \nabla(\nabla \cdot \dot{A}) + 6[A \overset{\times}{,} \dot{A}] + \nabla \times [A \overset{\times}{,} \dot{A}] \\ & + [\nabla \times \dot{A} \overset{\times}{,} A] + \nabla \times A \overset{\times}{,} \dot{A}] - [[A \overset{\times}{,} \dot{A}] \overset{\times}{,} A] - [A \times A \overset{\times}{,} \dot{A}]. \end{aligned} \tag{15}$$

Now using the expression for $\nabla \cdot \dot{A}$ as $[A\,;\dot{A}]$ given by Eq. (9), Eq. (15) takes the more regular form

$$\ddot{A} + L'\dot{A} = -3\dot{A} - 2\nabla \times \dot{A} - \nabla[A\,;\dot{A}] + 6[A \mathbin{\overset{\times}{,}} \dot{A}] + \nabla \times [A \mathbin{\overset{\times}{,}} \dot{A}] + [\nabla \times \dot{A} \mathbin{\overset{\times}{,}} A] + [\nabla \times A \mathbin{\overset{\times}{,}} \dot{A}] - [[A \mathbin{\overset{\times}{,}} \dot{A}] \mathbin{\overset{\times}{,}} A] - [A \times A \mathbin{\overset{\times}{,}} \dot{A}], \tag{16}$$

Thus if U denotes $A \oplus \dot{A} \oplus \ddot{A}$, then $U(t, \cdot)$ satisfies the equation

$$U'(t) = QU(t) + K(U(t)), \tag{17}$$

where Q is the linear operator whose matrix decomposition relative to the 3-fold direct sum in terms of which U is defined is

$$Q = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & -L' & 0 \end{pmatrix}; \tag{18}$$

and K takes the form $K(u \oplus v \oplus w) = 0 \oplus 0 \oplus k$, where k is deduced from the right-hand side of Eq. (16) by the formal substitution $A \to u$, $\dot{A} \to v$, $\ddot{A} \to w$. Thus

$$k(u, v, w) = -3v - 2\nabla \times v - \nabla[u\,;v] + 6[u \mathbin{\overset{\times}{,}} v] + \nabla \times [u \mathbin{\overset{\times}{,}} v] + [\nabla \times v \mathbin{\overset{\times}{,}} u] + [\nabla \times u \mathbin{\overset{\times}{,}} v] - [[u \mathbin{\overset{\times}{,}} v] \mathbin{\overset{\times}{,}} u] - [u \times u \mathbin{\overset{\times}{,}} v]. \tag{19}$$

We refer to Eq. (12) as the *evolution equation*, Eq. (16) as the *auxiliary evolution equation*, and Eq. (17) as the *abstract* such equation. Equation (17) will be considered in the spaces $[\mathbf{H}_a]$, defined as the Hilbert space direct sum

$$[\mathbf{H}_a] = L_{2,a+2}(S^3, \Gamma^3) \oplus L_{2,a+1}(S^3, \Gamma^3) \oplus L_{2,a}(S^3, \Gamma^3),$$

where $L_{2,r}(M, F)$, M being a compact manifold and F being a finite-dimensional vector space, indicates the usual Sobolev space of functions that are square-integrable together with their first r derivatives, and a is a non-negative integer. Evidently the $[\mathbf{H}_a]$ are all contained in $\mathbf{H}_0$; as subsets of $\mathbf{H}_0$ in the topology of $\mathbf{H}_0$ they will be denote simply as $\mathbf{H}_a$. Note that L' is a non-negative self-adjoint operator in $L_2(S^3, F)$, assuming F is a euclidean space and the inner product of two vectors f and g in $L_2(S^3, F)$ is defined as $\int_{S^3} (f(x), g(x))\, dx$. Its non-negative square root will be denoted as B (the context indicating the relevant finite-dimensional vector space F). The norm in $[\mathbf{H}_a]$ may be formulated as: $\|u \oplus v \oplus w\|_a = (\|B^{a+2}u\|^2 + \|B^{a+1}v\|^2 + \|B^a w\|^2)^{1/2}$. By spectral theory, the operator B_a, defined as B with the modified domain consisting of vectors in $\mathbf{H}_a$ that are in the domain of B and have transforms in $\mathbf{H}_a$ (under B), is likewise self-adjoint in $[\mathbf{H}_a]$, so that continuous functions of B_a are well-defined. The context will suffice to indicate the value of a, and B_a will be denoted simply as B.

Because of the nonlinear constraints on the Cauchy data, it is convenient to solve the Cauchy problem for the Yang–Mills equations in several stages. First the local-in-time strong existence is established for Eq. (17) in the space $\mathbf{H}_0$. This implies corresponding existence in the spaces $[\mathbf{H}_a]$, which are conserved under temporal evolution. Next, the question of the preservation of the constraint, when the Cauchy data for Eq. (17) satisfy the constraint initially, is considered, resulting in local-in-time strong solution of the Yang–Mills equations in the space $\mathbf{H}_0$. An a priori estimate for solutions of the Yang–Mills equation on the Einstein Universe is then obtained from the Cronström gauge [21] estimate of Eardley and Moncrief [4] in Minkowski space-time, using the local causal equivalence of these two space-times. Global-in-time existence for the Yang–Mills equations then follows in $\mathbf{H}_1$.

THEOREM 2. *Equation* (17) *has a unique local-in-time strong solution in* $\mathbf{H}_0$ *for arbitrary Cauchy data in this space.*

LEMMA 2.1. *Q is the generator of the following continuous one-parameter linear group on* $\mathbf{H}_0$:

$$W(t) = \begin{pmatrix} I & \dfrac{\sin tB}{B} & \dfrac{I - \cos tB}{B^2} \\ 0 & \cos tB & \dfrac{\sin tB}{B} \\ 0 & -B \sin tB & \cos tB \end{pmatrix}.$$

Moreover, this group leaves invariant $\mathbf{H}_a$, *and the group obtained by restricting to* $\mathbf{H}_a$ *acts continuously on* $[\mathbf{H}_a]$.

Proof. It is straightforward to verify that $W(t)$ is a bounded linear operator on $\mathbf{H}_a$ and that $W(t+t') = W(t)\, W(t')$ for arbitrary real t and t'. It is easily seen that $W(t)$ maps $\mathbf{H}_a$ into $\mathbf{H}_a$, and that for arbitrary U in $\mathbf{H}_a$, the map $t \to W(t)\, U$ is continuous from R^1 into $[\mathbf{H}_a]$. That the group $W(\cdot)$ is generated by Q follow on differentiating the given expression for $W(t)$, in accordance with the Hille–Yosida theorem.

LEMMA 2.2. *K is boundedly lipschitzian and C^n* $(n = 0, 1, 2...)$ *from* $[\mathbf{H}_a]$ into $[\mathbf{H}_a]$ $(a = 0, 1, 2...)$.

Proof. This means that if $u \in L_{2,2+a}$, $v \in L_{2,1+a}$, and $w \in L_{2,a}$, then $k(u, v, w) \in L_{2,a}$; and that in addition the inequalities and limits corresponding to the asserted lipschitzian and differentiability properties of K hold. The latter properties follow as usual in the present context of polynomials in the elements of Sobolev spaces, once the former is shown, by the use of the Sobolev inequalities. To establish the former property, suppose first that $a = 0$. The most singular terms in k are then $[\nabla \times u \,\dot{\times}\, v]$ and $[\nabla \times v \,\dot{\times}\, u]$; it will suffice to treat these terms, the others being simpler. Both $\nabla \times u$ and v are in $L_{2,1}$, which by a Sobolev inequality implies that they are both in L_6. It follows that $[\nabla \times u \,\dot{\times}\, v]$ is in L_3; but since S^3 is compact, L_3 is contained in L_2. If now a is positive, $\nabla \times u$ is in $L_{2,b}$ with $b \geqslant 2$, and the product or commutator of $\nabla \times u$ with an element of $L_{2,c}$ for any $c \leqslant b$ remains in $L_{2,c}$. In the case of $[\nabla \times v \,\dot{\times}\, u]$ the argument is similar. With this observation the proof proceeds as earlier.

Proof of theorem. This is immediate from the lemmas and [17].

THEOREM 3. *If the Cauchy data for Eq.* (17) *are in* $\mathbf{H}_0$, *and satisfy the constraint equation* $\nabla \cdot v = [u; v]$ *and the evolutionary equation* (14), *then both of these equations are satisfied throughout the interval of existence of the strong solution.*

Proof. Setting $g = \nabla \cdot v - [u\,; v]$, and $G = \dot{E} + \nabla \times H + 2H - [A \,\dot{\times}\, H]$ as a function of u, v, and w, via the replacement of A, $\dot{A}$, and $\ddot{A}$ by u, v, and w, respectively ($\dot{E}$ and H being regarded as dependent on A, $\dot{A}$, and $\ddot{A}$ in accordance with the earlier given equations), computations similar to those in the derivation of Eq. (16) show that

$$\dot{g} = -\nabla \cdot G + [A\,; G], \qquad \dot{G} = -\nabla g. \tag{20}$$

Now in the space $L_2(S^3, \Gamma^3) \oplus L_2(S^3, \Gamma)$, the operator whose matrix relative to this direct sum decomposition is $i\,\begin{pmatrix} 0 & \nabla \\ \nabla\cdot & 0 \end{pmatrix}$ is hermitian on the submanifold of infinitely differentiable elements, and is moreover invariant under rotations on S^3, acting on Γ^3 in accordance with the action on forms of the rotation group $O(4)$ of S^3. It follows that this operator, on the indicated domain, has a unique self-adjoint extension, say P (cf. [22]). Now letting $Z(t) = e^{-itP}(G(t) \oplus g(t))$, then $Z'(t) = e^{-itP}(-iP)(G(t) \oplus g(t)) + e^{-itP}(-\nabla g \oplus -\nabla \cdot G + [A\,; G])$, inasmuch as it follows by standard approximations that the domain of P includes all $G \oplus g$ with G in $L_{2,1}(S^3, \Gamma^3)$ and g in $L_{2,1}(S^3, \Gamma)$ and that on this domain P acts as the earlier indicated differential operator.

From Eqs. (16) and (17), $Z'(t) - e^{-itP}(0 \oplus [A(t); G(t)] = 0$. Now $G(0) = 0$ because the Cauchy data satisfy Eq. (14), and $g(0) = 0$ because the constraint equation is asumed to hold initially. It follows that

$$Z(t) = \int_0^t e^{-isP}(0 \oplus [A(s); G(s)])\, ds,$$

and bounding $\|A(s)\|_\infty$ by (const.)$\|A(s)\|_{2,2}$, it results that

$$\|Z(t)\| \leqslant (\text{const.}) \int_0^t \|Z(s)\|_{2,0}\, \|A(s)\|_{2,2}\, ds.$$

This implies, together with Gronwall's inequality, that $Z(t) = 0$ throughout the interval of existence of the solution to Eq. (17). ∎

In connection with the next theorem it is convenient to define the *abstract evolution equation* as Eq. (12) considered as a differential equation for $A(t, \cdot)$ in a Banach space of functions on space. Similarly, *the abstract constraint equation* is Eq. (9) as a similar differential equation for A, with E formulated as $-\dot{A}$.

THEOREM 4. *The abstract Yang–Mills equations (evolution and constraint) have a global solution in* $\mathbf{H}_1$ *that is strong for the evolution equation and strict for the constraint equation, for arbitrary Cauchy data in* $\mathbf{H}_1$ *satisfying the constraint initially.*

LEMMA 4.1. *There exists a positive number e such that if F is the 2-form associated with a* C^∞ *local-in-time solution to the Yang–Mills equations, and if t is both in* $(-e, e)$ *and in the interval of existence of the solution, then*

$$\|F(t, \cdot)\|_\infty \leqslant \phi(\|F(0, \cdot)\|_2 + \|A(0, \cdot)\|_2).$$

Proof. By the invariance of the equations under rotations of S^3, it suffices to establish the inequality at any one point V of $SU(2)$, where $SU(2)$ is identified with S^3 as usual (cf. [7]). Moreover, the global $L_{2,a}$ bounds on the right side of the inequality may then be replaced by corresponding bounds over neighborhoods of the domain of dependence at time 0 of the point (t, V).

Now $\mathbf{M}_0$ is causally equivalent to an open submanifold of $R^1 \times SU(2)$ via the imbedding given in [7]. This carries the space-like surface $x_0 = 0$ in $\mathbf{M}_0$ into the space-like surface $t = 0$ in $\tilde{\mathbf{M}}$. Noting the conformal invariance of the equations and the equivalence in relatively compact subsets of euclidean space of the Sobolev norms of forms relative to the bases β_j on the one hand and dx_j on the other, it follows that it suffices to establish the bound claimed in the lemma in $\mathbf{M}_0$ for $|t| < e$ for some positive number e and $V = I$. This bound is readily deduced from inspection of the argument of Eardley and Moncrief [4], applied to C^∞ data, together with use of the trace theorem, $L_{2,s} \to L_{2,s-1}(\partial B)$ (with B a ball of R^3).

Proof of theorem. In Corollary 1.1, let $V(\cdot)$ be the representation of the isometry group G of S^3 that gives its action on the vectors $A \oplus \dot{A} \oplus \ddot{A}$. The satisfaction of Eqs. (4) is a slightly stronger version of the invariance of the equations under G, and virtually immediate. As shown in Lemma 2.2, $K(\cdot)$ is of class C^n for arbitrary non-negative integral n. It follows that the n-fold differentiable vector subspace of $\mathbf{H}_0$ is invariant under the temporal propagation defined by Eq. (1). But this subspace is just $\mathbf{H}_n$ [18]. It follows that the common part $\mathbf{H}_\infty$ of the $\mathbf{H}_n$ is invariant, and evidently $\mathbf{H}_\infty$ is dense in $\mathbf{H}_0$.

Consider now Corollary 1.2, with $\mathbf{B}$ taken as $\mathbf{H}_0$ and $\mathbf{D}$ taken as $\mathbf{H}_\infty$. As just shown, if the initial datum u_0 is in $\mathbf{D}$, it remains there throughout the interval of existence of the associated solution. The remaining hypothesis follows from Lemma 4.1, and the theorem follows.

COROLLARY 4.1. *If the Cauchy data in Theorem* 4 *are in* $\mathbf{H}_a$ ($a = 1, 2, \ldots,$ *or* ∞), *they remain in this space for all times.*

Proof. Included in the preceding argument.

COROLLARY 4.2. *If the Cauchy data in Theorem* 4 *are in* $\mathbf{H}_2$, *then the solution is equivalent to a function on* $\tilde{\mathbf{M}}$ *that satisfies the equation in the elementary pointwise sense.*

Proof. A, $\dot{A}$, and $\ddot{A}$ are all in $L_{2,2}$, or more regular spaces, if the data are in $\mathbf{H}_2$, and so may be uniquely defined pointwise at each fixed time so as to be continuous on S^3. It then follows by standard methods that the derivatives involved in the equation exist in the elementary pointwise sense, and yield the same functions that are defined by the in part abstract operations on function spaces involved in the earlier treatment, modulo null functions.

5. RELATIONS BETWEEN SOLUTIONS ON $\tilde{\mathbf{M}}$ AND ON $\mathbf{M}_0$

Every solution of the Yang–Mills equations on $\tilde{\mathbf{M}}$ restricts to a solution on $\mathbf{M}_0$. On the other hand, the spatial components of $\tilde{\mathbf{M}}$ and $\mathbf{M}_0$ at time 0 are the same except for the absence of the point at infinity in the latter, corresponding to the point $-I$ of $SU(2)$. It is natural in view of this and the conformal invariance of the equations to expect that solutions on $\mathbf{M}_0$ may extend under reasonable conditions to solutions on $\tilde{\mathbf{M}}$. Indeed, it is shown in this section that every solution of the equations on $\mathbf{M}_0$ that is moderately smooth and small at space-like infinity extends to a solution on $\tilde{\mathbf{M}}$. The asymptotics of solutions on $\mathbf{M}_0$ are then deducible from the regularity of solutions of the equations on $\tilde{\mathbf{M}}$ that follows from the global solubility of the

Cauchy problem in this universal space-time. In particular the temporal asymptotics will be treated in a later section.

The temporal gauges on $\tilde{\mathbf{M}}$ and $\mathbf{M}_0$ are distinct except at time 0. The transformation from the one gauge to the other is however not at issue in the next two sections. To define on $\tilde{\mathbf{M}}$ the natural extension of the temporal gauge on $\mathbf{M}_0$, let T_j denote the unique causal vector field on $\tilde{\mathbf{M}}$ that agrees on $\mathbf{M}_0$ with $\partial/\partial x_j$. A 1-form A on $\tilde{\mathbf{M}}$ will be said to be in the flat (resp. curved) temporal gauge in case $A(T_0)$ (resp. $A(X_0)$) vanishes on $\tilde{\mathbf{M}}$. Regarding the form components, the following notation is used (cf. [23]). The primed standard components are with respect to the bases for forms derived from the dx_j, while the unprimed standard components are with respect to those derived from the β_j (i.e., the flat and curved standard components, respectively). Thus $A = \sum_{j=0}^3 A_j \beta_j$; $F = \sum_{j<k} F_{jk} \beta_j \beta_k$; $A \mid \mathbf{M}_0 = \sum_{j=0}^3 A'_j \, dx_j$, etc. The term *electric field part* of the 2-form F is defined as follows: (i) curved part, $\sum_j F_{j0} \beta_j \beta_0$, denoted ${}^cE$; (ii) flat part, $\sum_j F'_{j0} \, dx_j \, dx_0$, denoted ${}^fE$. ∇' denotes the gradient operation on R^3.

Throughout the succeeding sections, $\tilde{\mathbf{M}}$ is taken in a fixed factorization as $R^1 \times SU(2)$, and functions ϕ (forms, etc.) on $\tilde{\mathbf{M}}$ correspondingly taken in the form $\phi(t, V)$, where $t \in R^1$ and $V \in SU(2)$, unless otherwise specified. As earlier, $\mathbf{M}_0$ is taken as canonically imbedded in $\tilde{\mathbf{M}}$, the origin in $\mathbf{M}_0$ corresponding to the point $0 \times I$ in $\tilde{\mathbf{M}}$. Functions ϕ on $\mathbf{M}_0$ will be taken in the form $\phi(x_0, \mathbf{x})$. If A is a 1-form on $\tilde{\mathbf{M}}$, it has restrictions in several senses including: (i) to Einstein time 0, denoted $A(0, \cdot)$; (ii) the restriction in turn of $A(0, \cdot)$ to the tangent spaces of S^3, i.e., a 1-form on S^3. For brevity the latter 1-form will be called the *restriction of A to S^3 at (Einstein) time $t = 0$*; this restriction will be denoted as ${}_{(0)}A$. The restriction in turn of ${}_{(0)}A$ to the spatial component R^3 of $\mathbf{M}_0$ at time $x_0 = 0$ will be denoted as ${}_{(00)}A$.

If F is a 2-form on $\tilde{\mathbf{M}}$, the situation is similar. As earlier, however, it is convenient to decompose F into electric and magnetic components, and to take each of these as vectors, in a temporal gauge. This complicates the transformation properties of the trace of the electric component on S^3; it transforms as a *pseudo 1-form* (cf. [23]). Specifically, the trace on S^3 (resp. R^3) of the electric component of F on $\tilde{\mathbf{M}}$ (resp. M_0) at time 0 is defined as the pseudo-form $\sum_j F_{j0}(0, \cdot) \, \beta_j$ (resp. $\sum_j F'_{j0}(0, \cdot) \, dx_j$). The components then transform under conformal transformations on S^3 (resp. R^3) as a vector of weight 2 (rather than as the vector of weight 1 corresponding to a strict 1-form) if it is required that the restriction to S^3 (resp. R^3) is to commute with conformal transformation on $\tilde{\mathbf{M}}$ (resp. $\mathbf{M}_0$). The restriction of a 2-form $L(\cdot, \cdot)$ to S^3 at time $t = 0$ (resp. R^3 at time $x_0 = 0$), contracted with the vector field X_0 (resp. $\partial/\partial x_0$) in the second position: $L(\cdot, X_0)$ (resp. $L(\cdot, \partial/\partial_0)$) will be denoted as ${}_{(0)}L$ (resp. ${}_{(00)}L$).

THEOREM 5. *If A and F are* 1- *and* 2-*forms on* $\tilde{\mathbf{M}}$, *their restrictions to*

$\mathbf{M}_0$ *have standard components related as in Eqs.* (21), *following. In particular*, $A(X_0) = 0$ *at time* $t = 0$ *in* $\tilde{\mathbf{M}}$ *if and only if* $A(\partial/\partial x_0) = 0$ *at time* $x_0 = 0$ *in* $\mathbf{M}_0$. *Moreover, the restrictions* ${}_{(0)}A$ *and* ${}_{(0)}^{c}E$ *of* A *and* ${}^{c}E$ *to* S^3 *at time* $t = 0$, *where* ${}_{(0)}^{c}E$ *is defined as* $E_1(0, \cdot)\,\beta_1 + E_2(0, \cdot)\,\beta_2 + E_3(0, \cdot)\,\beta_3$, *satisfy the curved constraint equation* (9) *if and only if the restrictions* ${}_{(00)}A$ *of* A *and* ${}_{(00)}^{f}E$ *of* ${}^{f}E$ *to* R^3 *at time* $x_0 = 0$, *where* ${}_{(0)}^{f}E = E'_1\,dx_1 + E'_2\,dx_2 + E'_3\,dx_3$ *satisfy the "flat" constraint equation*: $\nabla' \cdot {}_{(00)}^{f}E = [{}_{(00)}A\,;\,{}_{(00)}^{f}E]$.

The constraint part of the theorem could be deduced from the conformal transformation properties indicated above without a component-wise analysis, but the relations of the components given below will be useful later.

$$A'_0 = \tfrac{1}{2}(1 + u_{-1}u_4)\,A_0 - \tfrac{1}{2}u_0u_1A_1 - \tfrac{1}{2}u_0u_2A_2 - \tfrac{1}{2}u_0u_3A_3,$$

$$A'_1 = -\tfrac{1}{2}u_0u_1A_0 + (\tfrac{1}{2}u_1^2 + u_4\,p)\,A_1 + (\tfrac{1}{2}u_1u_2 + u_3\,p)\,A_2 + (\tfrac{1}{2}u_3u_1 - u_2\,p)\,A_3,$$

$$A'_2 = -\tfrac{1}{2}u_0u_2A_0 + (\tfrac{1}{2}u_1u_2 - u_3\,p)\,A_1 + (\tfrac{1}{2}u_2^2 + u_4\,p)\,A_2 + (\tfrac{1}{2}u_2u_3 + u_1\,p)\,A_3,$$

$$A'_3 = -\tfrac{1}{2}u_0u_3A_0 + (\tfrac{1}{2}u_3u_1 + u_2\,p)\,A_1 + (\tfrac{1}{2}u_2u_3 - u_1\,p)\,A_2 + (\tfrac{1}{2}u_3^2 + u_4\,p)\,A_3.$$

$$F'_{01} = p(p - \tfrac{1}{2}{}_{-1}(u_2^2 + u_3^2))\,F_{01} + \tfrac{1}{2}p(u_3 + u_{-1}(u_1u_2 + u_3u_4))\,F_{02} + \tfrac{1}{2}p(-u_2 + u_{-1}(u_3u_1 - u_2u_4))\,F_{03} + \tfrac{1}{2}u_0\,p(u_2u_4 - u_1u_3)\,F_{12} + \tfrac{1}{2}u_0\,p(u_2^2 + u_3^2)\,F_{23} - \tfrac{1}{2}u_0\,p(u_1u_2 + u_3u_4)\,F_{31},$$

$$F'_{02} = \tfrac{1}{2}p(-u_3 + u_{-1}(u_1u_2 - u_3u_4))\,F_{01} + p(p - \tfrac{1}{2}u_{-1}(u_1^2 + u_3^2))\,F_{02} + \tfrac{1}{2}(u_1 + u_{-1}(u_2u_3 + u_1u_4))\,F_{03} - \tfrac{1}{2}u_0\,p(u_2u_3 + u_1u_4)\,F_{12} + \tfrac{1}{2}u_0\,p(u_3u_4 - u_1u_2)\,F_{23} + \tfrac{1}{2}u_0\,p(u_3^2 + u_1^2)\,F_{31},$$

$$F'_{03} = \tfrac{1}{2}p(u_2 + u_{-1}(u_3u_1 + u_2u_4))\,F_{01} + \tfrac{1}{2}p(-u_1 + u_{-1}(u_2u_3 - u_1u_4))\,F_{02} + p(p - \tfrac{1}{2}u_{-1}(u_1^2 + u_2^2))\,F_{03} + \tfrac{1}{2}u_0\,p(u_1^2 + u_2^2)\,F_{12} - \tfrac{1}{2}u_0\,p(u_3u_1 + u_2u_4)\,F_{23} + \tfrac{1}{2}u_0\,p(u_1u_4 - u_2u_3)\,F_{31},$$

$$F'_{12} = \tfrac{1}{2}u_0\,p(u_1u_3 + u_2u_4)\,F_{01} + \tfrac{1}{2}u_0\,p(u_2u_3 - u_1u_4)\,F_{02} - \tfrac{1}{2}u_0\,p(u_1^2 + u_2^2)\,F_{03} + p(p - \tfrac{1}{2}u_{-1}(u_1^2 + u_2^2))\,F_{12} + \tfrac{1}{2}(u_2 + u_{-1}(u_3u_1 + u_2u_4))\,F_{23} + \tfrac{1}{2}(-u_1 + u_{-1}(u_2u_3 - u_1u_4))\,F_{31},$$

$$F'_{23} = -\tfrac{1}{2}u_0\, p(u_2^2 + u_3^2)\, F_{01} + \tfrac{1}{2}u_0\, p(u_2 u_1 + u_3 u_4)\, F_{02}$$
$$+ \tfrac{1}{2}u_0\, p(u_3 u_1 - u_2 u_4)\, F_{03} + \tfrac{1}{2}p(-u_2 + u_{-1}(u_3 u_1 - u_2 u_4))\, F_{12}$$
$$+ p(p - \tfrac{1}{2}u_{-1}(u_2^2 + u_3^2))\, F_{23} + \tfrac{1}{2}p(u_3 + u_{-1}(u_1 u_2 + u_3 u_4))\, F_{31},$$

and

$$F'_{31} = \tfrac{1}{2}u_0\, p(u_1 u_2 - u_3 u_4)\, F_{01} - \tfrac{1}{2}u_0\, p(u_1^2 + u_3^2)\, F_{02}$$
$$+ \tfrac{1}{2}u_0\, p(u_2 u_3 + u_1 u_4)\, F_{03} + \tfrac{1}{2}p(u_1 \to u_{-1}(u_2 u_3 + u_1 u_4))\, F_{12}$$
$$+ \tfrac{1}{2}p(-u_3 + u_{-1}(u_1 u_2 - u_3 u_4))\, F_{23} + p(p - \tfrac{1}{2}u_{-1}(u_3^2 + u_1^2))\, F_{31}.$$

LEMMA 5.1. *The restrictions of the β_j to $\mathbf{M}_0$ are expressible as follows in terms of the dx_j:*

$$\beta_0 = \tfrac{1}{2}(1 + u_{-1}u_4)\, dx_0 - \tfrac{1}{2}u_0(u_1\, dx_1 + u_2\, dx_2 + u_3\, dx_3),$$
$$\beta_1 = -\tfrac{1}{2}u_0 u_1\, dx_0 + (\tfrac{1}{2}u_1^2 + u_4\, p)\, dx_1 + (\tfrac{1}{2}u_1 u_2 - u_3\, p)\, dx_2$$
$$+ (\tfrac{1}{2}u_1 u_3 + u_2\, p)\, dx_3,$$
$$\beta_2 = -\tfrac{1}{2}u_0 u_2\, dx_0 + (\tfrac{1}{2}u_2^2 + u_4\, p)\, dx_2 + (\tfrac{1}{2}u_1 u_2 + u_3\, p)\, dx_1$$
$$+ (\tfrac{1}{2}u_2 u_3 - u_1\, p)\, dx_3,$$
$$\beta_3 = -\tfrac{1}{2}u_0 u_3\, dx_0 + (\tfrac{1}{2}u_3 u_1 - u_2\, p)\, dx_1 + (\tfrac{1}{2}u_2 u_3 + u_1\, p)\, dx_2$$
$$+ (\tfrac{1}{2}u_3^2 + u_4\, p)\, dx_3.$$

Proof. Since

$$T_0 = \tfrac{1}{2}(X_0 + L_{04}) = \tfrac{1}{2}(1 + u_{-1}u_4)\, X_0 - u_0 u_1 X_1 - u_0 u_2 X_2 - u_0 u_3 X_3,$$

and, for example (cf. [7]),

$$T_1 = \tfrac{1}{2}(L_{-1,1} + L_{14})$$
$$= -\tfrac{1}{2}u_0 u_1 X_0 + \tfrac{1}{2}(u_{-1}u_4 + u_1^2 + u_4^2)\, X_1 + \tfrac{1}{2}(u_{-1}u_3 + u_1 u_2 + u_3 u_4)\, X_2$$
$$+ \tfrac{1}{2}(-u_{-1}u_2 + u_1 u_3 - u_2 u_4)\, X_3,$$

we have

$$\beta_0(T_0) = \tfrac{1}{2}(1 + u_{-1}u_4), \qquad \beta_j(T_0) = -\tfrac{1}{2}u_0 u_j,$$
$$\beta_0(T_j) = -\tfrac{1}{2}u_0 u_j, \qquad \beta_j(T_j) = \tfrac{1}{2}u_j^2 + u_4\, p \qquad (j = 1, 2, 3),$$
$$\beta_2(T_1) = \tfrac{1}{2}u_1 u_2 + u_3\, p. \qquad \beta_3(T_1) = \tfrac{1}{2}u_1 u_3 - u_2\, p,$$
$$\beta_3(T_2) = \tfrac{1}{2}u_2 u_3 + u_1\, p, \qquad \beta_1(T_2) = \tfrac{1}{2}u_2 u_1 - u_3\, p,$$
$$\beta_1(T_3) = \tfrac{1}{2}u_3 u_1 + u_2\, p, \qquad \beta_2(T_3) = \tfrac{1}{2}u_2 u_3 - u_1\, p.$$

Now recalling that $p = \tfrac{1}{2}(u_{-1} + u_4)$, the lemma follows.

LEMMA 5.2. *The restrictions to* $\mathbf{M}_0$ *of the products of two of the* β_j *are expressible as follows in terms of the products of two of the* dx_j:

$$\begin{aligned}\beta_0\beta_1 = {} & p(p - \tfrac{1}{2}u_{-1}(u_2^2 + u_3^2))\, dx_0\, dx_1 \\ & + \tfrac{1}{2}p(-u_3 + u_{-1}(u_1u_2 - u_3u_4))\, dx_0\, dx_2 \\ & + \tfrac{1}{2}p(u_2 + u_{-1}(u_1u_3 + u_2u_4))\, dx_0\, dx_3 \\ & + \tfrac{1}{2}u_0\, p(u_1u_3 + u_2u_4)\, dx_1\, dx_2 \\ & - \tfrac{1}{2}u_0\, p(u_2^2 + u_3^2)\, dx_2\, dx_3 \\ & + \tfrac{1}{2}u_0\, p(u_1u_2 - u_3u_4)\, dx_3\, dx_1,\end{aligned}$$

$$\begin{aligned}\beta_0\beta_2 = {} & \tfrac{1}{2}p(u_3 + u_{-1}(u_1u_2 + u_3u_4))\, dx_0\, dx_1 \\ & + p(p - \tfrac{1}{2}u_{-1}(u_1^2 + u_3^2))\, dx_0\, dx_2 \\ & + \tfrac{1}{2}p(-u_1 + u_{-1}(u_2u_3 - u_1u_4))\, dx_0\, dx_3 \\ & + \tfrac{1}{2}u_0\, p(u_2u_3 - u_1u_4)\, dx_1\, dx_2 \\ & + \tfrac{1}{2}u_0\, p(u_1u_2 + u_3u_4)\, dx_2\, dx_3 \\ & - \tfrac{1}{2}u_0\, p(u_1^2 + u_3^2)\, dx_3\, dx_1,\end{aligned}$$

$$\begin{aligned}\beta_0\beta_3 = {} & \tfrac{1}{2}p(-u_2 + u_{-1}(u_1u_3 - u_2u_4))\, dx_0\, dx_1 \\ & + \tfrac{1}{2}p(u_1 + u_{-1}(u_2u_3 + u_1u_4))\, dx_0\, dx_2 \\ & + p(p - \tfrac{1}{2}u_{-1}(u_1^2 + u_2^2))\, dx_0\, dx_3 \\ & - \tfrac{1}{2}u_0\, p(u_1^2 + u_2^2)\, dx_1\, dx_2 \\ & + \tfrac{1}{2}u_0\, p(u_3u_1 - u_2u_4)\, dx_2\, dx_3 \\ & + \tfrac{1}{2}u_0\, p(u_2u_3 + u_1u_4)\, dx_3\, dx_1,\end{aligned}$$

$$\begin{aligned}\beta_1\beta_2 = {} & \tfrac{1}{2}u_0\, p(u_2u_4 - u_1u_3)\, dx_0\, dx_1 \\ & - \tfrac{1}{2}u_0\, p(u_2u_3 + u_1u_4)\, dx_0\, dx_2 \\ & + \tfrac{1}{2}u_0\, p(u_1^2 + u_2^2)\, dx_0\, dx_3 \\ & + p(p - \tfrac{1}{2}u_{-1}(u_1^2 + u_2^2))\, dx_1\, dx_2 \\ & + \tfrac{1}{2}p(-u_2 + u_{-1}(u_1u_3 - u_2u_4))\, dx_2\, dx_3 \\ & + \tfrac{1}{2}p(u_1 + u_{-1}(u_2u_3 + u_1u_4))\, dx_3\, dx_1,\end{aligned}$$

$$\begin{aligned}\beta_2\beta_3 = &\tfrac{1}{2}u_0\, p(u_2^2+u_3^2)\, dx_0 dx_1\\ &+\tfrac{1}{2}u_0\, p(u_3u_4-u_1u_2)\, dx_0dx_2\\ &-\tfrac{1}{2}u_0\, p(u_3u_1+u_2u_4)\, dx_0dx_3\\ &+\tfrac{1}{2}p(u_2+u_{-1}(u_3u_1+u_2u_4))\, dx_1dx_2\\ &+p(p-\tfrac{1}{2}u_{-1}(u_2^2+u_3^2))\, dx_2dx_3\\ &+\tfrac{1}{2}p(-u_3+u_{-1}(u_2u_1-u_3u_4))\, dx_3dx_1,\end{aligned}$$

$$\begin{aligned}\beta_3\beta_1 = &-\tfrac{1}{2}u_0\, p(u_1u_2+u_3u_4)\, dx_0 dx_1\\ &+\tfrac{1}{2}u_0\, p(u_3^2+u_1^2)\, dx_0dx_2\\ &+\tfrac{1}{2}u_0\, p(u_1u_4-u_2u_3)\, dx_0dx_3\\ &+\tfrac{1}{2}p(-u_1+u_{-1}(u_2u_3-u_1u_4))\, dx_1dx_2\\ &+\tfrac{1}{2}p(u_3+u_{-1}(u_1u_2+u_3u_4))\, dx_2dx_3\\ &+p(p-\tfrac{1}{2}u_{-1}(u_3^2+u_1^2))\, dx_3dx_1.\end{aligned}$$

Proof. These expressons are obtained by multiplication of the expressions given earlier for the β_j in terms of the dx_k, followed by algebraic simplification.

LEMMA 5.3. *Let A and F be given* 1- *and* 2-*forms on* $\tilde{\mathbf{M}}$ *whose fixed-time sections are in* $L_{2,2}\oplus L_{2,1}$ *on* S^3, *and such that the mapping from the Einstein time t to the section is continuous. Then when* $t=0$,

$$\sum_{j=1}^{3}(T_jF'_{0j}-[A'_j,F'_{0j}])=p^3\sum_{j=1}^{3}(X_jF_{0j}-[A_j,F_{0j}]);\qquad p=\tfrac{1}{2}(1+u_4).$$

Proof. The expression on the left will be reduced to p^3 times the sum on the right. By approximation, it is no essential restriction to assume that A and F are C^∞. Using the formulas for A'_j and F'_{0j} in terms of the A_j and F_{0j}, it results (for example) that the coefficient of $[A_2,F_{03}]$ on the right-hand side is $\frac{1}{4}p$ times

$$\begin{aligned}&(u_3+u_1u_2+u_3u_4)(-u_2+u_3u_1-u_2u_4)\\ &\quad+(u_4+u_2^2+u_4^2)(u_1+u_2u_3+u_1u_4)\\ &\quad+(-u_1+u_2u_3-u_1u_4)(u_4+u_3^2+u_4^2);\end{aligned}$$

but this equals 0 by direct computation, as do its cyclic permutations.

The coefficient of $[A_1, F_{01}]$ however is $\frac{1}{4}p$ times

$$(u_4 + u_1^2 + u_4^2)(u_4 + u_1^2 + u_4^2) + (-u_3 + u_1 u_2 - u_3 u_4)(-u_3 + u_1 u_2 - u_3 u_4) + (u_2 + u_3 u_1 + u_2 u_4)(u_2 + u_3 u_1 + u_2 u_4),$$

which equals $(1 + u_4)^2$, hence the overall factor p^3 as stated.

By the formulas for T_j when $x_0 = 0$, $\sum_{j=1}^{3} T_j F'_{01}$ is $\frac{1}{4}$ times the sum

$$[(u_4 + u_1^2 + u_4^2) X_1 + (u_3 + u_1 u_2 + u_3 u_4) X_2 + (-u_2 + u_1 u_3 - u_2 u_4) X_3] \times \{p(u_4 + u_1^2 + u_4^2) F_{01} + p(u_3 + u_1 u_2 + u_3 u_4) F_{02} + p(-u_2 + u_3 u_1 - u_2 u_4) F_{03}\}$$

+ two other cyclically related terms.

It follows readily that the resulting terms involving differentiation of the F_{0j} lead to p^3 times the differentiation terms on the right, using the same equations as before. It remains to check the terms involving differentiation of the coefficients. The coefficient of F_{01} is for example

$$T_1(p(u_4 + u_1^2 + u_4^2)) + T_2(p(-u_3 + u_1 u_2 - u_3 u_4)) + T_3(p(u_2 + u_3 u_1 + u_2 u_4)).$$

This is shown to cancel out using the equations, at time $t = 0: u_4 = p(1 - \frac{1}{4}r^2)$, $p = (1 + \frac{1}{4}r^2)^{-1}$, and $u_j = px_j$ ($j = 1, 2, 3$).

Proof of Theorem. The theorem is now a direct consequence of the lemmas

COROLLARY 5.1. *Given Cauchy data for the Yang–Mills equations in the (flat) temporal gauge on* $\mathbf{M}_0$ *at time* $x_0 = 0$, *consisting of the* 1*-forms* $A(\mathbf{x})$ *and* $E(\mathbf{x})$ $(\mathbf{x} \in R^3)$, *there exists a solution* $A \oplus E$ *of the Yang–Mills equations on* $\tilde{\mathbf{M}}$ *in the curved temporal gauge attaining these values on restrictions to time* $t = 0$ *and from* S^3 *to* R^3, *in the sense that* $A(0, \mathbf{x}) = A(\mathbf{x})$ *and* $E(0, \mathbf{x}) = E(\mathbf{x})\, dx_0$, *in the space (at each fixed time)* $L_{2,r+1}(S_3) \oplus L_{2,r}(S^3)$, *where* $r > 1$, *if and only if*

(i) $A(\mathbf{x})$ *and* $E(\mathbf{x})$ *are in* $L^{\mathrm{loc}}_{2,r+1}(R^3)$ *and* $L^{\mathrm{loc}}_{2,r}(R^3)$, *respectively*;

(ii) *the same is true of the transforms of* $A(\mathbf{x})$ *and* $E(\mathbf{x})$ *under conformal inversion on* R^3.

Moreover there exists an integer s depending only on r such that if $A(\cdot) \oplus E(\cdot)$ *and the function* $|\mathbf{x}|^s(A(\mathbf{x}) \oplus E(\mathbf{x}))$ *are both in* $L_{2,r+1}(R^3) \oplus L_{2,r}(R^3)$, *then* (i) *and* (ii) *are satisfied.*

Proof. Conditions (i) and (ii) mean that on S^3, $A(\cdot)\oplus E(\cdot)$ is locally in $L_{2,r+1}\oplus L_{2,r}$ over arbitrary subregions of $SU(2)$ that exclude neighborhoods of $-I$ and I, respectively. It is therefore in this space globally on S^3. The fixed-time constraint remaining valid on S^3 by the theorem, the existence of a solution to the equations on $\tilde{\mathbf{M}}$ in the curved temporal gauge, having the indicated Cauchy data at time $t=0$, follows.

Noting that under conformal inversion the multipliers for the coefficients of either strict or pseudo 1-forms are dominated by powers of $|\mathbf{x}|$ (cf. e.g., [7] regarding multipliers), and noting the relation between the curved and flat components of forms on S^3 given in Eqs. (21), it follows that the norms in $L^{\rm loc}_{2,r+1}\oplus L^{\rm loc}_{2,r}$ on R^3 of the transforms of $A(\mathbf{x})$ and $E(\mathbf{x})$ under conformal inversion are dominated by the global norms over R^3 in these spaces of a sufficiently high power of $|\mathbf{x}|$ times the given functions $A(\mathbf{x})$ and $E(\mathbf{x})$.

6. Asymptotics of Fields on $\tilde{\mathbf{M}}$ on Restriction to $\mathbf{M}_0$

Given a field on $\mathbf{M}_0$ that extends in a continuous manner to the closure of $\mathbf{M}_0$ in $\tilde{\mathbf{M}}$, its asymptotics in $\mathbf{M}_0$ along paths tending to infinity in various directions take a precise form, by virtue of the compactness of the closure of $\mathbf{M}_0$. The asymptotics considered here will be along paths that are respectively null-, time-, or space-like, or more specifically, of the forms:

(N) $[x=(x_0,\mathbf{x})=sn+c : s\to\infty]$ for some $n=(1,\mathbf{n})$ where n^2 $(=1-|\mathbf{n}|^2)=0$ and $c=(c_0,\mathbf{c})\in\mathbf{M}_0$;

(T) $[x : x_0>0, x^2\geqslant ex_0^2]$, for some $e\in(0,1)$;

(S) $[x : x_0>0, x^2\leqslant -ex_0^2]$, for some $e>0$.

In cases (T) and (S) the regions in question are cones that are bounded away from the null cone whose vertex is $(0,\mathbf{0})$ in $\mathbf{M}_0$, and the asymptotics as $x_0\to\infty$ will be uniform over space. In the case of (N) the asymptotics below will be uniform over all null directions $(1,\mathbf{n})$ and over a bounded range of values of the vector c; usage of "uniformly" below is in precisely this sense.

The asymptotic expressions given below for a given field f are all of the form bg, where b is a nonvanishing purely geometrical factor dependent only on the point of space-time in question, and g is a constant dependent on the field. The notation $f\sim bg$ then means that $f/b\to g$. Thus in case $g\neq 0, f\sim bg$ is equivalent to the relation $f/bg\to 1$ as usual; if $g=0, f\sim bg$ is in general materially distinct from the relation $f\to 0$.

The form of the function ϕ_0 treated in Theorem 6 is motivated by the relation between a solution ϕ of the curved wave equation and a solution ϕ_0 of the flat wave equation: $\phi_0=p\phi$ (cf. [7]).

THEOREM 6. *Let ϕ be a continuous function on $\tilde{\mathbf{M}}$, and let $\phi_0 = p\phi$. Then along the null lines* (N),

$$\phi_0(x_0, \mathbf{x}) \sim (L(n \cdot c)/x_0)\, \phi(B(n, c))$$

uniformly as $x_0 \to \infty$, where $n \cdot c$ denotes the Lorentz-invariant inner product. $L(c \cdot n)$ is a positive constant, and $B(n, c)$ is a point on the boundary of $\mathbf{M}_0$ in $\tilde{\mathbf{M}}$.

If on the other hand $(x_0, \mathbf{x})$ tends to infinity within the sectors (T) *or* (S), *the asymptotics are as follows (uniformly; $|x_0| \to \infty$):*

$$\text{(T)}: \phi_0(x_0, \mathbf{x}) \sim (4/x^2)\, \phi(\pi \times I);$$
$$\text{(S)}: \phi_0(x_0, \mathbf{x}) \sim -(4/x^2)\, \phi(0 \times -I).$$

Remark. The functions involved in the asymptotics along null directions are given specifically as follows: $L(n \cdot c) = [1 + \frac{1}{4}(n \cdot c)^2]^{-1/2}$; $B(n, c) = (u_{-1}, u_0, u_1, \ldots, u_4)$ (where these are the coordinates in $S^1 \times S^3$ for the relevant point of $R^1 \times S^3$; cf. [7]), with $u_{-1} = -\frac{1}{2}(n \cdot c)\, L(n \cdot c) = -u_4$, $u_0 = L(n \cdot c)$, $u_j = n_j L(n \cdot c)$ $(j = 1, 2, 3)$, where $\mathbf{n} = (n_1, n_2, n_3)$.

Note that within the sectors (T) and (S), $|x^2|$ is bounded on either side by (const.) x_0^2.

Proof. Since $u_j = x_j p$ $(j = 0, 1, 2, 3)$, it suffices to determine the asymptotics of u_{-1}, u_4, and p, as in

LEMMA 6.1. *Along the null lines* (N),

$$p(x_0, \mathbf{x}) \sim L(n \cdot c)\, x_0^{-1} \qquad \textit{uniformly as } x_0 \to \infty.$$

Within the sectors (T) *or* (S),

$$p(x_0, \mathbf{x}) \sim 4\, |x^2|^{-1} \qquad \textit{uniformly as } x_0 \to \infty.$$

Along the null lines (N), *uniformly as earlier,*

$$u_{-1} \sim x_0^{-1} L(n \cdot c)\, (1 - \tfrac{1}{2} x_0 (n \cdot c) - \tfrac{1}{4} c^2);$$
$$u_4 \sim x_0^{-1} L(n \cdot c)(1 + \tfrac{1}{2} x_0 (n \cdot c) + \tfrac{1}{4} c^2).$$

Within the sector (T), $u_{-1} \to -1$, *and* $u_4 \to 1$; *and within the sector* (S), $u_{-1} \to 1$, $u_4 \to -1$, *in both cases uniformly as $x_0 \to \infty$.*

Remark. The asymptotics for u_{-1} and u_4 along the null lines cover several relativistically distinct cases. If $n \cdot c \neq 0$, then u_{-1} and u_4 approach nonzero constants. If $n \cdot c = 0$, they approach 0 as x_0^{-1}, unless additionally $c^2 = \pm 4$. If $c^2 = 4$, $u_{-1} = 0$, while if $c^2 = -4$, $u_4 = 0$ on the ray in question.

Proof of lemma. Recall that $u_{-1} = p(1 - \frac{1}{4}x^2)$, $u_4 = p(1 + \frac{1}{4}x^2)$, and $p = \frac{1}{2}(u_{-1} + u_4) = ((1 - \frac{1}{4}x^2)^2 + x_0^2)^{-1/2} = (1 + \frac{1}{2}(x_0^2 + \mathbf{x}^2) + (x^2)^2/16)^{-1/2}$. In the sectors (T) and (S) the $\frac{1}{4}x^2$ and $(x^2)^2/16$ terms dominate all the others, leading to the stated asymptotics. On the null line where $x = sn + c$ (s tending to $+\infty$), $x^2 = 2s(n \cdot c) + c^2$ and $\frac{1}{2}(x_0^2 + \mathbf{x}^2) + (x^2)^2/16$ is asymptotic to $x_0^2 + \frac{1}{4}x_0^2(n \cdot c)^2$. Referring to the definition of $L(n \cdot c)$, the asymptotics for p follows.

The theorem now follows directly.

The asymptotics in $\mathbf{M}_0$ of 1-form and 2-form fields that extend continuously to $\tilde{\mathbf{M}}$ are given next. Only the generic order of decay (power of $|x_0|^{-1}$) is given for each case; the more precise asymptotics of the scalar coefficient functions in Theorem 5 are given in Lemma 7.1.

For estimates of the energy outside the region treated in Corollary 7.1, see Theorem 8. The order of decay valid for arbitrary 2-forms, not necessarily solutions of the Yang–Mills equations, is substantially greater than that obtained for solutions of these equations by analysis entirely within $\mathbf{M}_0$ [8, 9]. Moreover, this order is best possible even for solutions of the equations because of the convergence of, for example, the sectors under (T) to the point $\pi \times I$, at which some regular solution of the equations takes on generic nonvanishing values.

THEOREM 7. *Let A be a continuous 1-form on $\tilde{\mathbf{M}}$ (or defined merely on the closure of $\mathbf{M}_0$ in $\tilde{\mathbf{M}}$). Then along null rays of the form* (N), *the flat components A'_j ($j = 0, 1, 2, 3$) defined in $\mathbf{M}_0$ are $O(1)$ as $x_0 \to \infty$ and are uniformly $O(x_0^{-2})$ as $x_0 \to \infty$ in the sectors* (T) *and* (S).

Assume further that A is in the Minkowski temporal gauge ($A'_0 = 0$). Then the asymptotics of A'_1, A'_2, and A'_3 in (T) *and* (S) *remain $O(x_0^{-2})$, but along the null rays* (N) *become $O(x_0^{-1})$.*

If F is a continuous 2-form on $\tilde{\mathbf{M}}$ (or again merely on the closure of $\mathbf{M}_0$), then along the null rays (N) *the flat components F'_{ij} are uniformly $O(x_0^{-1})$, and in the sectors* (T) *and* (S) *the F'_{ij} are uniformly $O(x_0^{-4})$.*

Proof. It suffices to determine the asymptotics of the coefficient functions appearing in Theorem 5, listed as follows. Let i, j, and k be any permutation of 1, 2, and 3.

(a)	$1 + u_{-1}u_4$,	(f)	$\dfrac{x_j}{x_0}u_{-1}p$,
(b)	$u_0 u_j$,	(g)	$pu_0(u_i^2 + u_j^2)$,
(c)	$u_i u_j$,	(h)	$pu_0(u_i u_j \pm u_k u_4)$,
(d)	$u_4 p$,	(i)	$p(1 + u_{-1}u_4)u_k \pm u_{-1}u_i u_j)$
(e)	$u_j p$,	(j)	$p(p - \frac{1}{2}u_{-1}(u_i^2 + u_j^2))$.

The case (N) of null rays must be divided into the subcases

(N1) $n \cdot c = 0$ and $c^2 = 4$,

(N2) $n \cdot c = 0$ and $c^2 = -4$,

(N3) $n \cdot c = 0$ and $|c^2| \neq 4$,

(N4) $n \cdot c \neq 0$.

The functions in case (f) appear when the 1-form equations of Theorem 5 are specialized to the case $A_0' = 0$; then

$$(1 + u_{-1} u_4) A_0 = u_0 (u_1 A_1 + u_2 A_2 + u_3 A_3)$$

and thus for example

$$A_1' = p(u_4 A_1 + u_3 A_2 - u_2 A_3) + \frac{x_1}{x_0} u_{-1} p A_0.$$

Asymptotics for the functions (a) to (j) are given in

LEMMA 7.1. *The following asymptotics in* $\mathbf{M}_0$ *are valid uniformly as* $x_0 \to +\infty$:

(a) $1 + u_{-1} u_4 \sim L(n \cdot c)^2$ for (N1–N4),

$1 + u_{-1} u_4 \sim \dfrac{8(x_0^2 + \mathbf{x}^2)}{x^4}$ for (T), (S)

(b) $u_0 u_j \sim \dfrac{L(n \cdot c)^2 x_j}{x_0}$ for (N_1–N4),

$u_0 u_j \sim \dfrac{16 x_0 x_j}{x^4}$ for (T), (S)

(c) $u_i u_j \sim \dfrac{L(n \cdot c)^2 x_i x_j}{x_0^2}$ for (N1–N4),

$u_i u_j \sim \dfrac{16 x_i x_j}{x^4}$ for (T), (S),

(d) $u_4 p = 0$ for (N2)

$u_4 p \sim \dfrac{1 + \frac{1}{4} c^2}{x_0^2}$ for (N1) and (N3),

$$u_4\, p \sim \frac{1}{2} \frac{(n \cdot c)\, L(n \cdot c)^2}{x_0} \qquad \text{for (N4)},$$

$$u_4\, p \sim \frac{4}{x^2} \qquad \text{for (T) and (S)},$$

(e) $$u_j\, p \sim \frac{L(n \cdot c)^2\, x_j}{x_0^2} \qquad \text{for (N1–N4)},$$

$$u_j\, p \sim \frac{16 x_j}{x^4} \qquad \text{for (T), (S)},$$

(f) $$\frac{x_j}{x_0} u_{-1}\, p = 0 \qquad \text{for (N1)},$$

$$\sim \frac{(1 - \frac{1}{4}c^2)\, x_j}{x_0^3} \qquad \text{for (N2) and (N3)},$$

$$\sim -\frac{1}{2} \frac{(n \cdot c)\, L(n \cdot c)^2\, x_j}{x_0^2} \qquad \text{for (N4)},$$

$$\sim -\frac{4 x_j}{x_0 x^2} \qquad \text{for (T), (S)},$$

(g) $pu_0(u_i^2 + u_j^2)$

$$\sim \frac{L(n \cdot c)^4\, (x_i^2 + x_j^2)}{x_0^3} \qquad \text{for (N1–N4)},$$

$$\sim \frac{256 x_0 (x_i^2 + x_j^2)}{(x^2)^4} \qquad \text{for (T), (S)},$$

(h) $pu_0(u_i u_j \pm u_k u_4)$

$$\sim \frac{x_i x_j}{x_0^3} \qquad \text{for (N2)},$$

$$\sim \frac{1}{x_0^3} (x_i x_j \pm x_k (1 + \tfrac{1}{4}c^2)) \qquad \text{for (N1), (N3)},$$

$$\sim \frac{L(n \cdot c)^4}{x_0^3} (x_i x_j \pm \tfrac{1}{2}(n \cdot c)\, x_0 x_k)$$

for (N4),

$$\sim \frac{256 x_0}{(x^2)^4} (x_i x_j \pm \tfrac{1}{4} x^2 x_k) \qquad \text{for (T), (S)},$$

(i) $p((1 + u_{-1}u_4)\, u_k \pm u_{-1} u_i u_j)$

$$\sim \frac{x_k}{x_0^2} \qquad \text{for (N1)},$$

$$\sim \frac{x_k}{x_0^2} \pm \frac{(1 - \frac{1}{4}c^2)\, x_i x_j}{x_0^4} \qquad \text{for (N2), (N3)},$$

$$\sim \frac{L(n \cdot c)^4\, x_k}{x_0^2} \pm \left(-\frac{1}{2}\right) \frac{L(n \cdot c)^4\, x_i x_j}{x_0^3}$$

for (N2),

$$\sim \frac{128 x_k (x_0^2 + \mathbf{x}^2)}{(x^2)^4} \pm (-1) \frac{64 x_i x_j}{(x^2)^3}$$

for (T), (S),

(j) $p(p - \frac{1}{2}u_{-1}(u_i^2 + u_j^2))$

$$\sim \frac{1}{x_0^2} \qquad \text{for (N1)},$$

$$\sim \frac{1}{x_0^2} - \frac{1}{2} \frac{(1 - \frac{1}{4}c^2)\, x_i x_j}{x_0^4} \qquad \text{for (N2), (N3)},$$

$$\sim \frac{L(n \cdot c)^2}{x_0^2} + \frac{1}{4} \frac{(n \cdot c)\, L(n \cdot c)(x_i^2 + x_j^2)}{x_0^3}$$

for (N4),

$$\sim \frac{16}{x^4} + \frac{32(x_i^2 + x_j^2)}{(x^2)^3} \qquad \text{for (T), (S)}.$$

COROLLARY 7.1. *Given a two-form F as in the theorem with (flat) components* **E** *and* **H**, *and given* $R > 0$ *and* $e \in (0, 1)$, *then*

$$\int_{|x| < R + e x_0} (|\mathbf{E}|^2 + |\mathbf{H}|^2)\, d_3 x = O(x_0^{-5}) \qquad \textit{as} \quad x_0 \to +\infty.$$

Proof. It follows that $|\mathbf{E}|^2 + |\mathbf{H}|^2 = O(x_0^{-8})$ in such sectors, whose volumes at a fixed time are proportional to x_0^3.

7. Asymptotics of Solutions to the Yang-Mills Equations in $\mathbf{M}_0$

The asymptotics of Yang–Mills fields are relatively simply deducible from their regularity on $\tilde{\mathbf{M}}$ by virtue of gauge invariance of the norms of the fields on restriction to $\mathbf{M}_0$. It suffices that the requisite gauge transformation, from the curved to the flat temporal gauge, exists in every bounded open set in $\mathbf{M}_0$, as it does with the loss of one order of differentiability. The asymptotics of the potentials requires a study of the regularity of the gauge transformation involved at the limiting points $\pm\pi \times I$, which represent the limits under temporal displacement in $\mathbf{M}_0$ of points in $\mathbf{M}_0$. This section first treats the simpler question of asymptotics of fields, and then studies the asymptotics of the curved-to-flat temporal gauge transformation, and applies this to the asymptotics of the potentials.

Theorem 8. *Let A and F denote the* 1- *and* 2*-forms of a solution to the Yang–Mills equations on $\mathbf{M}_0$ in the flat temporal gauge. Let r be a given integer greater than* 1. *Then there exists an integer s (depending only on r) such that if at time $x_0 = 0$,*

$$(1 + |\mathbf{x}|^s)(A(0, \mathbf{x}) \oplus E(0, \mathbf{x})) \in L_{2,r+1}(R^3) \oplus L_{2,r}(R^3).$$

then F decays temporally in accordance with the following estimates:

(1) *For any $R > 0$ and $e \in (0, 1)$,*

$$\sup_{\substack{\mathbf{x} \\ |\mathbf{x}| < R + ex_0}} |F'_{ij}(x_0, \mathbf{x})| = O(x_0^{-4})$$

and

$$\int_{|\mathbf{x}| < R + ex_0} \sum_{i<j} |F'_{ij}|^2 \, d_3x = O(x_0^{-5}).$$

(2) $\sup_{\mathbf{x} \in R^3} |F'_{ij}(x_0, \mathbf{x})| = O(|x_0^{-1}|)$.

(3) *For any $R > 0$ and $e \in (0, 1)$,*

$$|F'_{ij}(x_0, \mathbf{x})| = O((x_0^2 + \mathbf{x}^2)^{-2}) \qquad \textit{whenever } 0 < x_0 < R + e\,|\mathbf{x}|$$

and

$$\int_{|x_0| < R + e|\mathbf{x}|} \sum_{i<j} |F'_{ij}|^2 \, d_3x = O(x_0^{-5}).$$

The generic constants on the right-hand sides are bounded by Sobolev norms of the Cauchy data.

Proof. The solution $(\tilde{A}, \tilde{F})$ of the equations in the Einstein temporal gauge with the same Cauchy data at $t=0$ is continuous on $\tilde{\mathbf{M}}$, and restricted to $\mathbf{M}_0$ clearly differs from (A, F) at any given point in $\mathbf{M}_0$ by a local gauge transformation. The quantities estimated in the theorem being gauge-invariant, the results of the previous section for 2-forms extending continuously to the closure of $\mathbf{M}_0$ in $\tilde{\mathbf{M}}$ apply here.

(1) This is Theorem 7 and Corollary 7.1.

(2) By Theorem 5, each F'_{ij} is equal to p times a continuous function on $\tilde{\mathbf{M}}$, and

$$p = ((1 - \tfrac{1}{4}x^2)^2 + x_0^2)^{-1/2} = O(|x_0|^{-1}) \qquad (\text{as } |x_0| \to \infty)$$

in $\mathbf{M}_0$ throughout space.

(3) The fixed coefficients (g)–(j) examined in Lemma 7.1 are easily estimated to be $O((x_0^2 + \mathbf{x}^2)^{-2})$ in the stated region by use of the asymptotics provided since $x^4 \geqslant (\text{const.})(x_0^2 + \mathbf{x}^2)^2$ there. Thus the energy at time x_0 in this region is bounded by a constant times

$$\int_{|\mathbf{x}| \geqslant |x_0|} (x_0^2 + \mathbf{x}^2)^{-4}\, d_3x.$$

By the change of variables $d_3x = r^2\, dr\, d\Omega$ and $r = x_0 s$, the integral is easily estimated to be $O(x_0^{-5})$.

The detailed consideration of the gauge transformation from the curved (Einstein) temporal gauge to the flat (Minkowski) temporal gauge leads to extremely precise results regarding the asymptotics of the Minkowski temporal gauge fields, as x_0 tends to $\pm\infty$. These estimates are uniform in $\mathbf{x}$ as $\mathbf{x}$ ranges over a compact set and may be evaluated to any desired order in x_0. In order to deal with the gauge transformation issue it is convenient to work in part in the spaces C^n in addition to Sobolev spaces, as in

THEOREM 9. *Let $A(\mathbf{x})$ and $E(\mathbf{x})$ be Cauchy data for the Yang–Mills equations in the flat temporal gauge on M_0 at $x_0 = 0$. Given any integer $r \geqslant 4$, there exists an integer s such that if*

$$(1 + |\mathbf{x}|^s)(A(\mathbf{x}) \oplus E(\mathbf{x})) \in L_{2,r+1}(R^3) \oplus L_{2,r}(R^3)$$

and $\nabla \cdot E = [A\,; E]$, then the Cauchy data extend to $L_{2,r+1}(S^3) \oplus L_{2,r}(S^3)$, the solution $\tilde{A} \oplus \tilde{F}$ of the Yang–Mills equations in the curved temporal gauge given by Corollary 5.1 is in $C^{r-1} \oplus C^{r-2}$ on $\tilde{\mathbf{M}}$, and the standard components $\tilde{A}_j$ and $\tilde{F}_{ij}$, together with their derivatives by the X_j ($j = 0, 1, 2, 3$) up to orders $r-1$ and $r-2$ resp., vanish at the point $0 \times -I$ in $\tilde{\mathbf{M}}$.

Moreover there exists a C^{r-1} gauge transformation defined on the open *region* $(-\pi, \pi) \times S^3$ *in* $\tilde{\mathbf{M}}$ *and transforming* $\tilde{A} \oplus \tilde{F}$ *into a solution in the flat temporal gauge, whose restriction to* $\mathbf{M}_0$ *is identical to the solution to the Yang-Mills equations on* $\mathbf{M}_0$ *with Cauchy data* $A(\mathbf{x})$ *and* $E(\mathbf{x})$.

Proof. Extendability of the Cauchy data follows from Corollary 5.1, and $\tilde{A} \oplus \tilde{F} \in C^{r-1} \oplus C^{r-2}$ by the Sobolev inequalities. The vanishing of the spatial derivatives of $\tilde{A}$ and $\tilde{F}$ at $0 \times -I$ follows from the observation that if f is a continuous function on S^3 such that $\int_{R^3} |x|^b \; |f| \; d_3x < \infty$ for some $b \geqslant -3$, then $f(-I) = 0$ (use $d_3x = r^2 \, dr \, d\Omega$ where $r = |\mathbf{x}|$); the vanishing of the time derivatives then follows from use of the Yang–Mills equations.

Let (A, F) be the solution of the Yang–Mills equations in $\mathbf{M}_0$ in the Minkowski temporal gauge. It remains to define a C^{r-1} extension to $(-\pi, \pi) \times S^3$ of the solution of the equation defining the requisite gauge transformation $U(x_0, \mathbf{x})$ in $\mathbf{M}_0$ from the curved to the flat temporal gauge:

$$U^{-1}(\partial/\partial x_0)\, U = U^{-1}\tilde{A}_0'\, U, \qquad U(0, \mathbf{x}) = I; \tag{22}$$

and then show that for this extension U, $(U^{-1}\tilde{A}U - U^{-1}\partial U, U^{-1}\tilde{F}U)$ solves the Yang–Mills equations in the Minkowski temporal gauge and equals (A, F) in $\mathbf{M}_0$. Since $\tilde{A}_0 = 0$ (Einstein temporal gauge), $\tilde{A}_0' = -\frac{1}{2}u_0(u_1\tilde{A}_1 + u_2\tilde{A}_2 + u_3\tilde{A}_3)$ by Theorem 5, and thus $\tilde{A}_0'$ is in $C^{r-1}(\tilde{\mathbf{M}})$. Thus clearly U has continuous derivatives in $\mathbf{M}_0$ up to order $r-1$. We extend U by solving the canonical extension of (22) to $\tilde{\mathbf{M}}$ using

$$\frac{\partial}{\partial x_0} = \frac{1}{2}(1 + u_{-1}u_4)\, X_0 - \tfrac{1}{2}u_0(u_1X_1 + u_2X_2 + u_3X_3)$$

(which, for $|t| < \pi$, vanishes only at $0 \times -I$, as $u_{-1} = -u_4 = 1$ there), and show that there is a unique solution U with the stated regularity.

The question is a purely local one at $0 \times -I$, so it is convenient to treat the problem in conformally inverted coordinates x_j $(j = 0, 1, 2, 3)$, so that $0 \times -I$ corresponds to all $x_j = 0$. The vector field to integrate is then (cf. [24])

$$-\hat{T}_0 = -\tfrac{1}{4}x^2 \frac{\partial}{\partial x_0} + \frac{x_0}{2} S = \frac{1}{4}(x_0^2 + \mathbf{x}^2)\frac{\partial}{\partial x_0} + \frac{1}{2}x_0\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right),$$

whose integral curves are given locally by $x_j/(x_0^2 - \mathbf{x}^2) = \text{const.}$ for $j = 1, 2, 3$ (conformally inverted coordinates). It may be seen that each such curve near $0 \times -I$ intersects the x_0-plane at one point, and that each such curve may be parametrized locally by x_0. One computes easily that along any specific such curve $-\hat{T}_0$ is thus represented by $\frac{1}{4}(x_0^2+\mathbf{x}^2)(\partial/\partial x_0)$. Recalling now the previous expression for $\hat{A}_0'$, and noting that $u_0 u_j/(x_0^2+\mathbf{x}^2)$ is bounded in a neighborhood of $0\times -I$, it follows that the solution U is uniformly continuous in a neighborhood of $0\times -I$ and so extends to a continuous function. (Here we need only that A_1, A_2, A_3 are bounded near $0\times -I$.) In the following set $|x|^2=x_0^2+\mathbf{x}^2$.

To obtain the higher derivatives of U, it suffices to consider a locally equivalent scalar problem:

$$\left(\frac{1}{4}|x|^2\partial_0+\frac{1}{2}x_0\left(\sum_{j=1}^{3}x_j\partial_j\right)\right)\phi=\sum_{j=1}^{3}x_0x_jf_j,$$
$$\phi=0 \qquad \text{for} \quad x_0=0, \tag{23}$$

where ϕ is a local coordinate function on the gauge group Γ, and the functions f_j (like the A_j) are C^{r-1} and have derivatives that vanish at the origin to order $r-1$, whence $\partial^{(r-1)}f_j=O(1)$ and

$$\partial^{(r-2)}f_j=O(|x|),\ldots, \qquad f_j=O(|x|^{r-1}). \tag{24}$$

That is for $q\leqslant r-2$, for all x in some neighborhood of 0, and for some constant $c\leqslant c_r\|A\|_{C^{r-1}}$, $|\partial^{(q)}f_j(x)|\leqslant c\,|x|^{r-1-q}$.

Let the integral curve ending at $(x_0,\mathbf{x})$ be parametrized by (s,v_j), with s between 0 and x_0, so that

$$\frac{s^2-\mathbf{y}^2}{y_j}=\frac{x_0^2-\mathbf{x}^2}{x_j} \qquad (j=1,2,3).$$

Solving for the y_j, get

$$y_j=y_j(x_0,\mathbf{x},s)=x_j\frac{((x_0^2-\mathbf{x}^2)^2+4s^2\mathbf{x}^2)^{1/2}-(x_0^2-\mathbf{x}^2)}{2\mathbf{x}^2},$$

and then clearly $\partial^{(q)}y_j=O(|x|^{1-q})$, uniformly for $|s|\leqslant|x_0|$, for all $q\geqslant 0$ (∂ indicating formation of the complex of derivatives with respect to the x_k; $k=0,1,2,3$). Also, set $F_j=F_j(x_0,\mathbf{x})=4x_0x_j/|x|^2$; clearly $\partial^{(q)}F_j=O(|x|^{-q})$ for $q\geqslant 0$. Then the solution ϕ of (23) is given by

$$\phi(x_0,\mathbf{x})=\int_0^{x_0}\sum_{j=1}^{3}F_j(s,\mathbf{y}(x_0,\mathbf{x},s))f_j(s,\mathbf{y}(x_0,\mathbf{x},s))\,ds.$$

It follows from the bounds (24) and above estimates for the derivatives of the y_j and F_j that the integrands obtained by differentiating under the integral sign up to order $r-1$ with respect to the x_j ($j=0,1,2,3$) remain bounded as $|x|\to 0$, so that ϕ is in C^{r-1}.

Thus $(U^{-1}\tilde{A}U - U^{-1}\partial U, U^{-1}\tilde{F}U), = (A'', F'')$ say, is defined in $(-\pi, \pi) \times S^3$, is of class $C^{r-2} \times C^{r-2}$ and thus at least $C^2 \times C^1$, and is a classical strict solution of the Yang–Mills equations by gauge invariance. When $x_0 = 0$, $U^{-1}\tilde{A}'_j U - U^{-1}\partial_j U = A'_j$ and $U^{-1}\tilde{F}'_{ij} U = F'_{ij}$ for $i, j = 0, 1, 2, 3$, so (A'', F'') has the same Cauchy data as (A, F), and (A'', F'') also satisfies the Minkowski temporal gauge condition precisely by Eq. (22). Thus $(A, F) = (A'', F'')$ by uniqueness for the Cauchy problem for the Minkowski temporal gauge, completing the proof.

The system (22) seems not so easily set up on $(-\pi, \pi) \times S^3$ in a Sobolev-space context. Nevertheless, solving (22) by ordinary differential equation methods, it results that the C^{r-1} norm of U over any compact subregion of $(-\pi, \pi) \times S^3$ is bounded by a constant depending on the region times the C^{r-1} norm of A in that region. Unfortunately, control over one derivative of the potential is lost in the extension: A is C^{r-1} in $\mathbf{M}_0$ by the Sobolev inequality, but its extension $U^{-1}\tilde{A}U - U^{-1}\partial U$ (outside $\mathbf{M}_0$) is only clearly C^{r-2}. For this reason we take $r \geqslant 4$.

The line of argument earlier indicated for the determination of the asymptotics of the potentials will now be completed, with the aid of the gauge transformation established in the preceding theorem in a more restrictive context.

THEOREM 10. *With the hypotheses and notation of Corollary* 5.1 *and assuming further that r is at least* 3, *there exists a* C^2 $\mathbf{\Gamma}$*-valued function* $V(\mathbf{x})$ *on* R^3 (*where* Γ *is the given compact Lie gauge group*) *and unique elements* c_k *and* $f_{ij} = -f_{ji}$ *of the Lie algebra* Γ ($k = 1, 2, 3$ *and* $i, j = 0, 1, 2, 3$) *such that*

$$x_0^4 |A'_1(x_0, \mathbf{x}) + V(\mathbf{x})^{-1}\, \partial_1 V(\mathbf{x}) + \tfrac{1}{3} x_0^{-3} V(\mathbf{x})^{-1} f_{01} V(\mathbf{x}) + \tfrac{1}{2} x_0^{-4} V(\mathbf{x})^{-1} (x_2 f_{12} - x_3 f_{31} + c_1) V(\mathbf{x})| \to 0$$

as $x_0 \to +\infty$, *uniformly for* $\mathbf{x}$ *restricted to a compact set, and likewise for* 1, 2, 3 *cyclically permuted*; *moreover*

$$x_0^4 |F'_{ij} - x_0^{-4} V(\mathbf{x})^{-1} f_{ij} V(\mathbf{x})| = O(x_0^{-1})$$

as $x_0 \to +\infty$ *for any pair* $i, j = 0, 1, 2, 3$, *again uniformly for* $|\mathbf{x}| < R$.

Remark. The formal context may be clarified by the observation that the above series for A'_k and F'_{ij} must be and in fact are consistent with the Minkowski temporal gauge equations $(\partial/\partial x_0) A'_k = F'_{0k}$ and e.g., $F'_{12} = \partial_1 A'_2 - \partial_2 A'_1 - [A'_1, A'_2]$ to the order x_0^{-4}.

Proof. Let $(\tilde{A}, \tilde{F})$ be the solution of the Yang–Mills equations in the Einstein temporal gauge on $\tilde{\mathbf{M}}$ having the same Cauchy data at $t = 0$. The components $\tilde{A}_j$ and $\tilde{F}_{ij}$ are then resp. C^2 and C^1 on $\tilde{\mathbf{M}}$. The solution $U(x_0, \mathbf{x})$ of

$$U^{-1}\frac{\partial}{\partial x_0}U = U^{-1}\tilde{A}_0'(x_0, \mathbf{x})\,U,$$
$$U(0, \mathbf{x}) = I$$

in $\mathbf{M}_0$ is then, like A and $\tilde{A}'$ also at least C^2. By Theorem 5,

$$\tilde{A}_0' = -\tfrac{1}{2}u_0(u_1\tilde{A}_1 + u_2\tilde{A}_2 + u_3\tilde{A}_3). \tag{25}$$

By Lemma 7.1, each $u_0 u_j$ is uniformly $O(x_0^{-3})$ for $|\mathbf{x}| < R$, and it may also be checked that $\partial_k\tilde{A}_0'$ for $k = 1, 2, 3$ is $O(x_0^{-3})$ there. (The more precise asymptotic expansions of $\tilde{A}_0'$ and $\partial_k\tilde{A}_0'$ for $k > 0$ are derived in Lemma 10.1 below.) Thus there is no difficulty in defining

$$V(\mathbf{x}) = \lim_{x_0\to+\infty} U(x_0, \mathbf{x}),$$

so that

$$\partial_k V(\mathbf{x}) = \lim_{x_0\to+\infty} \partial_k U(x_0, \mathbf{x}) \qquad (k = 1, 2, 3).$$

Note also that for $k = 1, 2, 3$,

$$U(x_0, \mathbf{x})^{-1}\,\partial_k U(x_0, \mathbf{x}) = \int_0^{x_0} U(s, \mathbf{x})^{-1}\,(\partial_k\tilde{A}_0')\,U(s, \mathbf{x})\,ds,$$

and that the above two limits take place at the rate $O(x_0^{-2})$ uniformly as before.

It follows that

$$A_k' = U^{-1}\tilde{A}_k'U - U^{-1}(\partial/\partial x_k)\,U \tag{26}$$

for $k = 1, 2, 3$, and

$$F_{ij}' = U^{-1}\tilde{F}_{ij}'U$$

for all i, j by uniqueness of the solution of the Minkowski temporal gauge Cauchy problem. The approach will next be to expand $U^{-1}\tilde{A}_k'U$ and $V^{-1}\partial_k V - U^{-1}\partial_k U$ in asymptotic series out to x_0^{-4} and add them together (two terms of order x_0^{-2} then cancelling) to obtain the stated asymptotic series for $A_k' + V^{-1}\partial_k V$. The estimate for the F_{ij}' is more immediate, however, and identifies the f_{ij} in the statement of the theorem: by Theorem 5 and Lemma 7.1,

$$\tilde{F}'_{ij}(x_0, \mathbf{x}) = 16x_0^{-4}\tilde{F}_{ij}(\pi \times I) + O(x_0^{-5})$$

for $|\mathbf{x}| < R$ since the $\tilde{F}_{ij}$ are C^1 in $\tilde{\mathbf{M}}$. Thus $f_{ij} = 16\tilde{F}_{ij}(\pi \times I)$.

The Taylor series of the $\tilde{A}_j$ $(j = 1, 2, 3)$ out to second order will be needed, using $u = (u_0, u_1, u_2, u_3)$ as coordinates near $\pi \times I$ and then substituting $u_0 = x_0 p \sim 4x_0^{-1}$ and $u_j = x_j p \sim 4x_j x_0^{-2}$. The expansion for $\tilde{A}_1$ up to the term x_0^{-2} is then

$$\begin{aligned}\tilde{A}_1(u) &= \tilde{A}_1(\pi \times I) + u_0\left(\left(\frac{\partial}{\partial_0}u_0\right)\tilde{A}_1\right)(\pi \times I)\\ &\quad + \sum_{j=1}^{3} u_j\left(\frac{\partial \tilde{A}_1}{\partial u_j}\right)(\pi \times I) + \frac{1}{2}u_0^2\left(\left(\frac{\partial^2}{\partial u_0^2}\right)\tilde{A}_1\right)(\pi \times I) + \cdots.\end{aligned}$$

By use of Table I and $u_4^2 = 1 - u_1^2 - u_2^2 - u_3^2$, $\partial/\partial u_j$ may be replaced by X_j for $j > 0$, and since $X_0 = u_{-1}(\partial/\partial u_0)$, the desired series for $\tilde{A}_1$ is

$$\begin{aligned}\tilde{A}_1(u) = a_1 - x_0^{-1}4b_{01} + \sum_{j=1}^{3} x_0^{-2}4x_j b_{j1}\\ + x_0^{-2}8d_1 + x_0^{-2}R(x_0, \mathbf{x}),\end{aligned}$$

where $a_j = \tilde{A}_j(\pi \times I)$, $b_{ij} = (X_i\tilde{A}_j)(\pi \times I)$, $d_j = (X_0^2\tilde{A}_j)(\pi \times I)$, and $R(x_0, \mathbf{x}) \to 0$ as $x_0 \to +\infty$ (uniformly in $\mathbf{x}$ at least for a given region $|\mathbf{x}| < R$) since the $\tilde{A}_j$ are C^2.

The first three terms of the asymptotic series for the flat components $\tilde{A}'_0$, $\partial_k\tilde{A}'_0$, and $\tilde{A}'_k$ may consequently be written as follows.

TABLE I

$u_4\dfrac{\partial}{\partial u_1} = (u_1^2 + u_4^2)X_1 + (u_1u_2 + u_3u_4)X_2 + (u_1u_3 - u_2u_4)X_3$
$u_4\dfrac{\partial}{\partial u_2} = (u_1u_2 - u_3u_4)X_1 + (u_2^2 + u_4^2)X_2 + (u_1u_4 + u_2u_3)X_3$
$u_4\dfrac{\partial}{\partial u_3} = (u_1u_3 + u_2u_4)X_1 + (-u_1u_4 + u_2u_3)X_2 + (u_3^2 + u_4^2)X_3$

LEMMA 10.1. *Uniformly over* $\mathbf{x}$ *subject to* $|\mathbf{x}| < R$ *for some* R, *we have*

$$\tilde{A}_1' = x_0^{-2} 4a_1 - x_0^{-3} 16b_{01} + 8x_0^{-4}\Big((x_1x_2 + 2x_3)\, a_2 + (x_3x_1 - 2x_2)\, a_3$$

$$+ 4d_1 + (x_1^2 - 2 + \tfrac{1}{2}r^2)\, a_1 + 2\sum_{j=1}^{3} x_j b_{j1}\Big) + \cdots,$$

$$\tilde{A}_0' = -x_0^{-3} 8\sum_{j=1}^{3} x_j a_j + x_0^{-4} 32 \sum_{j=1}^{3} x_j b_{0j} + x_0^{-5} 64(1 - \tfrac{1}{4}r^2)\sum_{j=1}^{3} x_j a$$

$$- x_0^{-5} 64 \sum_{j=1}^{3} x_j d_j - x_0^{-5} 32 \sum_{i,j=1,2,3} x_i x_j b_{ij} + \cdots,$$

and

$$\left(\frac{\partial}{\partial x_1}\right) \tilde{A}_0' = -x_0^{-3} 8a_1 + x_0^{-4} 32b_{01} + x_0^{-5} 64\left(1 - \frac{1}{4}r^2\right) a_1$$

$$- x_0^{-5}\, 32x_1 \sum_{j=1}^{3} x_j a_j - x_0^{-5} 64d_1$$

$$- x_0^{-5} 32 \sum_{j=1}^{3} x_j (b_{1j} + b_{j1}) + \cdots,$$

where the omitted terms $+\cdots$ *times resp.* x_0^4, x_0^5, *and* x_0^5, *go to* 0 *as* $x_0 \to +\infty$, *and similarly for the other expansions obtained by cyclic permutation of* 1, 2, 3.

Proof. The third expansion follows from the second essentially by differentiation. The second uses equation (25), the above expansion for the $\tilde{A}_j$ ($j = 1, 2, 3$), and the expansion derived below for the $u_0 u_j$. One checks that uniformly as before

$$p = x_0^{-2} 4 - x_0^{-4} 16(1 - \tfrac{1}{4}r^2) + \cdots,$$

so

$$p^2 = x_0^{-4} 16 - x_0^{-6}\, 128(1 - \tfrac{1}{4}r^2) + \cdots,$$

so

$$-\tfrac{1}{2}u_0 u_1 = -\tfrac{1}{2}x_0 x_1\, p^2 = -x_0^{-3} 8x_1 + x_0^{-5} 64x_1(1 - \tfrac{1}{4}r^2) + \cdots.$$

To expand A_1', use Theorem 5 and Lemma 7.1 to obtain

$$\tilde{A}_1' = u_4\, p\tilde{A}_1 + 8x_0^{-4}(x_1^2 a_1 + (x_1x_2 + 2x_3)\, a_2$$

$$+ (x_1x_3 - 2x_2)\, a_3) + \cdots$$

and

$$u_4 p\tilde{A}_1 = (x_0^{-4}4 - x_0^{-4}16(1-\tfrac{1}{4}r^2))\Big(a_1 - x_0^{-1}4b_{01} + x_0^{-2}8d_1 + x_0^{-2}4\sum_{j=1}^{3} x_j b_{j1}\Big) + \cdots$$

$$= x_0^{-2}4a_1 - x_0^{-3}16b_{01} + 16x_0^{-4} \times \Big(2d_1 + \sum_{j=1}^{3} x_j b_{j1} + (\tfrac{1}{4}r^2 - 1)\, a_1\Big) + \cdots$$

concluding the proof of the lemma.

Completion of proof of the Theorem. The first nontrivial term in the expansion of $U(x_0, \mathbf{x})$ in terms of $V(\mathbf{x})$ will affect the x_0^{-4} term for the A'_k. Using Lemma 10.1 for $\tilde{A}'_0$ (more precisely only the first term thereof) and the equation that $U(x_0, \mathbf{x})$ satisfies, it follows easily that

$$V(\mathbf{x})\, U(x_0, \mathbf{x})^{-1} = I - x_0^{-2}4(x_1 a_1 + a_2 x_2 + x_3 a_3) + O(x_0^{-3}).$$

Thus

$$V(\mathbf{x})^{-1}\, \partial_1 V(\mathbf{x}) - U(x_0, \mathbf{x})^{-1}\, \partial_1 U(x_0, \mathbf{x})$$

$$= \int_{x_0}^{\infty} U(s, \mathbf{x})^{-1}\, (\partial_1 A'_0)\, U(s, \mathbf{x})\, ds$$

$$= V(\mathbf{x})^{-1} \left[\int_{x_0}^{\infty} (\partial_1 A'_0)\, ds\right] V(\mathbf{x}) + 32V(\mathbf{x})^{-1}\left(\int_{x_0}^{\infty} x_0^{-5}\right)[x_1 a_1 + x_2 a_2 + x_3 a_3, a_1]\, V(\mathbf{x}) + \cdots$$

$$= V(\mathbf{x})^{-1}\Big\{-4x_0^{-2}a_1 + x_0^{-3}\frac{32}{3}b_{01} + 8x_0^{-4}\Big(2(1-\tfrac{1}{4}r^2)\, a_1 - x_1\sum_{j=1}^{3} a_j x_j - 2d_1 + x_2[a_2, a_1] + x_3[a_3, a_1] - \sum_{j=1}^{3} x_j(b_{1j} + b_{j1})\Big)\Big\}\, V(\mathbf{x}) + R(x_0, \mathbf{x}),$$

such that $x_0^4 R(x_0, \mathbf{x}) \to 0$ uniformly as $x_0 \to +\infty$.

By Ec. (26), the other term needed is $U^{-1}\tilde{A}_1' U$; up to the term x_0^{-4} it is

$$V(\mathbf{x})^{-1} \Big\{ x_0^{-2} 4a_1 - x_0^{-3} 16 b_{01} + 8x_0^{-4} \Big((x_1 x_2 + 2x_3)\, a_2 + 4d_1$$
$$+ (x_3 x_1 - 2x_2)\, a_3 + (x_1^2 - 2 + \tfrac{1}{2} r^2)\, a_1 + 2 \sum_{j=1}^{3} x_j b_{j1} \Big) \Big\} V(\mathbf{x})$$
$$- x_0^{-4}\, 16 V(\mathbf{x})^{-1} \left[\sum_{j=1}^{3} x_j a_j, a_1 \right] V(\mathbf{x}).$$

Therefore (cancelling several terms),

$$\begin{aligned} A_1'(x_0, \mathbf{x}) &+ V(\mathbf{x})^{-1} \partial_1 V(\mathbf{x}) \\ &:= U^{-1}\tilde{A}_1' U + (V^{-1}\partial_1 V - U^{-1}\partial_1 U) \\ &:= V(\mathbf{x})^{-1} \{ -\tfrac{16}{3} x_0^{-3} b_{01} + 8x_0^{-4} (2x_3 a_2 - 2x_2 a_3 + 2d_1 - x_2 [a_2, a_1] \\ &\quad + x_2 (b_{21} - b_{12}) + x_3 (b_{31} - b_{13}) - x_3 [a_3, a_1]) \} V(\mathbf{x}) \\ &= V(\mathbf{x})^{-1} \{ - x_0^{-3} \tfrac{16}{3} b_{01} + 8x_0^{-4} (x_3 (b_{31} - b_{13} + 2a_2 - [a_3, a_1]) \\ &\quad - x_2 (b_{12} - b_{21} + 2a_3 - [a_1, a_2]) + 2d_1) \} V(\mathbf{x}). \end{aligned}$$

However,

$$\begin{aligned} &8(b_{12} - b_{21} + 2a_3 - [a_1, a_2]) \\ &\quad = 8(X_1 \tilde{A}_2 - X_2 \tilde{A}_1 + 2\tilde{A}_3 - [\tilde{A}_1, \tilde{A}_2])(\pi \times I), \end{aligned}$$

Likewise,

$$\begin{aligned} 16 b_{01} &= 16 (X_0 \tilde{A}_1)(\pi \times I) \\ &= 16 \tilde{F}_{01} (\pi \times I) \\ &= f_{01}, \end{aligned}$$

whence the stated asymptotics for $A_1'(x_0, \mathbf{x})$ is valid provided in addition

$$c_k = 32 d_k = 32 (X_0^2 \tilde{A}_k)\,(\pi \times I).$$

In theoretical physical applications of the Yang-Mills equations, the action integral is of considerable importance (cf. [25]). As a final application and illustration of the utility of the view of $\mathbf{M}_0$ as imbedded in $\tilde{\mathbf{M}}$ it is deduced that the action integral for these equations is absolutely convergent when evaluated for mildly regular solutions of the equations on $\mathbf{M}_0$.

THEOREM 11. *With hypotheses and notation as in Corollary* 5.1, *the integral*

$$\int_{\mathbf{M}_0} \left| \sum_{j=1}^{3} |F'_{0j}|^2 - \sum_{i>j>0} |F'_{ij}|^2 \right| d_4 x$$

is finite, and is a continuous function of the Cauchy data (A, E) *in* $L_{2,3}(S^3) \oplus L_{2,2}(S^3)$ *at* $t = 0$.

Remark. It follows that the action integral is also a continuous function of the Minkowski Cauchy data (A'_j, F'_{0j}) at $x_0 = 0$, topologized by the norm

$$\|(1 + |\mathbf{x}|^s)(A' \oplus E')\|_{(L_{2,3} \oplus L_{2,2})(R^3)}$$

for sufficiently large s, as in the second paragraph of Corollary 5.1.

Proof. Let $\tilde{A} \oplus \tilde{F}$ (resp. $A \oplus F$) be the solutions in the curved (resp. flat) temporal gauge having the given Cauchy data. By use of the gauge transformation of the previous theorem, it follows that $|F'_{ij}|^2 = |\tilde{F}'_{ij}|$ for all i, j. But by Theorem 5,

$$\sum_{j=1}^{3} |\tilde{F}'_{0j}|^2 - \sum_{i>j>0} |\tilde{F}'_{ij}|^2 = p^4 \left(\sum_{j=1}^{3} |\tilde{F}_{0j}|^2 - \sum_{i>j>0} |\tilde{F}_{ij}|^2 \right),$$

and the $\tilde{F}_{ij}$ are continuous and bounded on finite t-intervals in $\tilde{\mathbf{M}}$ by the Sobolev inequality. It remains to observe that

$$\int_{\mathbf{M}_0} p^4 d_4 x = \tfrac{1}{2} \int_{S^1 \times S^3} d_4 u = 2\pi^3$$

(cf. [7]).

Remark. It has been difficult to treat hyperbolic equations on $\mathbf{M}_0$ by variational methods because of the indefiniteness of the Lagrangian, whereby it lacks the coercive power it has in elliptic contexts. However, the foregoing treatment opens up the possibility of establishing generalized solutions of the Yang–Mills equations as extremals of the action on finite covers of $\tilde{\mathbf{M}}$, within the space of sections the (finite) time integrals of whose energy is bounded by a given limit.

Correction to [1]. The author of [1] (I.E.S.) thanks several individuals (J. Ginibre and G. Velo, and independently D. Eardley and V. Moncrief) for noting that the argument for Theorem 3 in [1] requires the assumption of one additional derivative for the Cauchy data. (Thus on p. 185, line 19, $b + 1$ should read b, and on p. 190, line 16, "two" should be inserted following "first" and "second" changed to "third" (loc. cit.), as far as the proof given in [1] is concerned.) The theorem is actually correct as stated, by the desingularization method used in the present article, or by a different argument given by Eardley and Moncrief [4, Part I], which however is lengthy and clearly applicable only to R^3. In any event, the number of derivatives required of the Cauchy data for the local existence result, apart from its finiteness, has not been an issue in theoretical physical applications.

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