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Simple Connectedness of the 3-Local Geometry of the Monster

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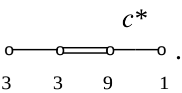
We consider the 3-local geometry $\mathcal{M}$ of the Monster group M introduced as a locally dual polar space of the group $\Omega_8^-(3)$ and independently in the context of minimal p-local parabolic geometries for sporadic simple groups. More recently the geometry appeared implicitly within the Z_3-orbifold construction of the Moonshine module $V^\natural$. In this paper we prove the simple connectedness of $\mathcal{M}$. This result makes unnecessary the refereeing to the classification of finite simple groups in the Z_3-orbifold construction of $V^\natural$ and realizes an important step in the classification of the flag-transitive c-extensions of the classical dual polar spaces. We make use of the simple connectedness results for the 2-local geometry of M and for a subgeometry in $\mathcal{M}$ which is the 3-local geometry of the Fischer group $M(24)$. © 1997 Academic Press

1. INTRODUCTION

The Monster group M acts flag-transitively on a diagram geometry $\mathcal{M}$ which is described by the diagram



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The elements of $\mathcal{M}$ corresponding to the nodes from the left to the right on the diagram are called *points*, *lines*, *planes*, and *quadrics*, respectively. The residue of a quadric is the classical polar space associated with the group $\Omega_8^-(3)$. The quadrics and planes incident to a line form the geometry of vertices and edges of a complete group on 11 vertices. The existence of $\mathcal{M}$ was independently established in [BF, RS]. We follow [BF] to review briefly the construction of $\mathcal{M}$ and to formulate its basic properties. The starting point is the description of conjugacy classes of the subgroups of order 3 in the Monster [At].

LEMMA 1.1. *In the Monster group M every element of order* 3 *is conjugate to its inverse and there are exactly three conjugacy classes of subgroups of order* 3 *with representatives σ, μ, and τ, so that*

(a) $N_M(\sigma) \sim 3^{1+12} \cdot 2 \cdot \mathrm{Suz} \cdot 2$, *where Suz is the Suzuki sporadic simple group*;

(b) $N_M(\mu) \sim 3 \cdot M(24)$, *where $M(24)$ is the largest sporadic Fischer* 3-*transposition group*;

(c) $N_M(\tau) \sim \mathrm{Sym}(3) \times F_3$, *where F_3 is the sporadic simple group discovered by Thompson.*

We define a subgroup of order 3 in M to be of *Suzuki*, *Fischer*, or *Thompson type* if it is conjugate to σ, μ, or τ from Lemma 1.1, respectively.

A crucial role in the construction of $\mathcal{M}$ is played by a subgroup $M_8 \sim 3^8 \cdot \Omega_8^-(3) \cdot 2$ in $\mathcal{M}$. If $Q_8 = O_3(M_8)$ then M_8/Q_8 is an extension of the simple orthogonal group $\Omega_8^-(3)$ by an automorphism of order 2, Q_8 is the natural orthogonal module for M_8/Q_8, and $N_M(Q_8) = M_8$.

LEMMA 1.2. *Let φ be the orthogonal form of minus type on Q_8 preserved by M_8/Q_8. Then M_8/Q_8 acting on the subgroups of order* 3 *in Q_8 has two orbits I and N such that*

(a) $|I| = 1066$, *the subgroups in I are isotropic with respect to φ and of Suzuki type in M; for $\sigma \in I$ we have $N_{M_8/Q_8}(\sigma) \sim 3^6 \cdot 2 \cdot U_4(3) \cdot 2^2$;*

(b) $|N| = 2214$, *the subgroups in N are non-isotropic with respect to φ and of Fischer type in M; for $\mu \in N$ we have $N_{M_8/Q_8}(\mu) \sim Q_7(3) \cdot 2$.*

Proof. Under the action of $O^2(M_8/Q_8) \sim \Omega_8^-(3)$ the set of order 3 subgroups in Q_8 splits into three orbits I, N_1, N_2 with lengths 1066, 1107, 1107 and stabilizers isomorphic to $3^6 \cdot 2 \cdot U_4(3) \cdot 2$, $\Omega_7(3) \cdot 2$, $\Omega_7(3) \cdot 2$, respectively (cf. [At]). As 3^{17} divides the order of each of the stabilizers there are no Thompson type subgroups in Q_8 and as the elements of I are 3-central they are of Suzuki type. By Lagrange $3^7\Omega_7(3)$ is not involved in Suz and since 3^{1+12} has no elementary abelian subgroup or order 3^8, N_1

and N_2 consist of Fischer type subgroups. Finally, in $M(24)$ all subgroups of order 3^7 whose normalizers involve $\Omega_7(3)$ are conjugated. Hence N_1 and N_2 fuse into a single M_8/Q_8-orbit. ∎

By Lemma 1.2 the polar space acted on flag-transitively by M_8/Q_8 can be identified with the Suzuki-pure subgroups in Q_8 with two subgroups being incident if one of them contains the other one. Let Q_1, Q_2, Q_3 be Suzuki-pure subgroups in Q_8 with $Q_1 < Q_2 < Q_3$, so that $|Q_i| = 3^i$ for $1 \leq i \leq 3$. Then the points, lines, planes, and quadrics in $\mathcal{M}$ are defined to be the subgroups in M conjugate to Q_1, Q_2, Q_3, and Q_8, respectively, with $\mathcal{F} = \{Q_1, Q_2, Q_3, Q_8\}$ being a maximal flag. Let $M_i = N_M(Q_i)$ be the maximal parabolic subgroup corresponding to the flag-transitive action of M on $\mathcal{M}$. Then M_8 is as above while M_1 is the normalizer of a Suzuki type subgroup Q_1 (which we will also denote by σ) and $M_1 \sim 3^{1+12} \cdot 2 \cdot \mathrm{Suz} \cdot 2$ by Lemma 1.1(a). The stabilizer of $\mathcal{F}$ in M contains a Sylow 3-subgroup of M. Hence for two elements of $\mathcal{M}$ to be incident it is necessary for their common stabilizer in M to contain a Sylow 3-subgroup. Let $P_i = O_3(M_i)$, P_i^* be the kernel of the action of M_i on the residue of Q_i in $\mathcal{M}$ and $\overline{M}_i = M_i/P_i^*$ for $i = 1$, 2, 3, and 8. It is clear that $Q_i \leq P_i$ and that $Q_8 = P_8 = P_8^*$. For $i = 1$, 2, 3, and 8 we denote by $\mathcal{M}_i$ the set of points, lines, planes, and quadrics in $\mathcal{M}$, respectively. For an element α in $\mathcal{M}$ we denote by $\mathcal{M}_i(\alpha)$ the set of elements in $\mathcal{M}_i$ incident to α.

Let Σ be the graph on the Suzuki type subgroups in Q_8 in which two subgroups are adjacent if they are orthogonal with respect to φ. Then Σ is strongly regular with parameters

$$v = 1066, \qquad k = 336, \qquad l = 729, \qquad \lambda = 92, \qquad \mu = 112$$

(that is, Σ has $v = 1066$ vertices, every vertex has $k = 336$ neighbors and $l = 729$ vertices in distance two, two adjacent vertices have $\lambda = 92$ common neighbors, and two vertices of distance two have $\mu = 112$ common neighbors).

The quotient M_8/Q_8 induces a rank 3 action on Σ, so that if $\sigma \in \Sigma$ then $N_{M_8/Q_8}(\sigma)$ acts transitively on the set $\Sigma_1(\sigma)$ of points adjacent to σ in Σ and on the set $\Sigma_2(\sigma)$ of points at distance 2 from σ.

The next statement follows from standard properties of classical groups.

LEMMA 1.3. *Let $L = N_{M_8/Q_8}(\sigma)$ and z be an involution from $O_{3,2}(L)$. Then*

(a) *$L \sim 3^6 \cdot 2 \cdot U_4(3) \cdot 2^2$ and Q_8, as a module for L, has a unique composition series*:

$$1 < \sigma < \langle \Sigma_1(\sigma) \rangle = \sigma^{\perp} < Q_8;$$

(b) *both* $\sigma^{\perp}/\sigma$ *and* $O_3(L)$ *are isomorphic to the natural orthogonal module for* $O^2(L)/O_3(L) \sim 2 \cdot U_4(3) \sim 2 \cdot P\Omega_6^-(3)$;

(c) $O_3(L)$ *acts regularly on* $\Sigma_2(\sigma)$;

(d) z *acts fixed point-freely on* $\sigma^{\perp}/\sigma$ *and on* $O_3(L)$; *it centralizes a unique subgroup* $\varepsilon \in \Sigma_2(\sigma)$ *and* $C_{Q_8}(z) = \langle \sigma, \varepsilon \rangle$ *is* 2-*dimensional containing two subgroups of Suzuki and two subgroups of Fischer type*.

Since $N_{M_8}(\sigma)$ contains a Sylow 3-subgroup of M, it contains $P_1 = O_3(N_M(\sigma))$. By Lemmas 1.1(a) and 1.3, we have $P_1 \cap Q_8 = \sigma^{\perp}$, $P_1 Q_8 = O_3(N_{M_8}(\sigma))$ and z acts fixed-point freely on P_1/Q_1. This shows that all the points collinear to σ are contained in P_1 and that $P_1^* = P_1\langle z \rangle$. Let ε be as in Lemma 1.3(d). Then $Q_8 = \langle \varepsilon, C_{P_1}(\varepsilon)\rangle$ is uniquely determined by ε and σ. So if Q_8^* is another quadric containing σ then $Q_8 \cap Q_8^*$ is a point, a line, or a plane. Furthermore, the image δ of Q_8 in $\overline{M}_1 = M_1/P_1^* \sim \mathrm{Suz} \cdot 2$ is a subgroup of order 3. Moreover, $\overline{N_{M_8}(\sigma)/P_1^*} = N_{\overline{M}_1}(\delta) \sim 3 \cdot U_4(3) \cdot 2^2$ and by [At] is a maximal subgroup in $\overline{M}_1$. Thus the quadrics from $M_8(\sigma)$ correspond to 3-central subgroups of order 3 in $\overline{M}_1$. The next lemma (cf. [BCN, Sect. 13.7]) describes the action of $\overline{M}_1$ on its 3-central subgroups of order 3.

LEMMA 1.4. *The group* $S \sim \mathrm{Suz} \cdot 2$ *acting on the set* Δ *of its subgroups of order* 3 *with normalizer* $U \sim 3 \cdot U_4(3) \cdot 2^2$ *has rank* 5 *with subdegrees* 1, 280, 486, 8505, *and* 13,608. *If* Δ *denotes also the graph of valency* 280 *invariant under this action, then*:

(a) *two distinct vertices of* Δ *commute* (*as subgroups in* S) *if and only if they are adjacent*;

(b) Δ *is distance-transitive with distribution diagram given on Fig*. 1 *and* S *is the full automorphism group of* Δ.

(c) *if* K *is a maximal clique in* Δ *then* $|K| = 11$, *the setwise stabilizer* T *of* K *is a maximal subgroup in* S, *and* $T \sim 3^5 \cdot (2 \times Mat_{11})$, *so that* $O_3(T)$ *is generated by the subgroups from* K *and* $T/O_{3,2}(T) \sim Mat_{11}$ *acts* 5-*transitively on the vertices of* K *while* $O_3(T)$ *fixes none of the vertices outside* K;

(d) *let* δ *be the vertex of* Δ *stabilized by* U, *then the geometry of cliques and edges containing* δ *with the incidence relation via inclusion is isomorphic*

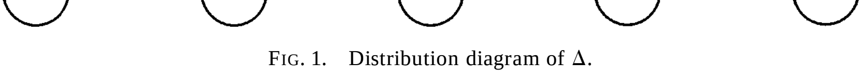

FIG. 1. Distribution diagram of Δ.

to the geometry of 1- *and* 2-*dimensional totally isotropic subspaces in* 6-*dimensional orthogonal* $GF(3)$-*space of minus type and it is acted on flag-transitively by* $O^2(U)/O_3(U) \sim U_4(3) \sim P\Omega_6^-(3)$;

(e) *if* ϱ *is a vertex at distance* 2 *from* δ *in* Δ *then the subgraph induced on the vertices adjacent to both* δ *and* ϱ *is the complete bipartite graph* $K_{4,4}$;

(f) *if* ϱ *is a vertex of distance* 2, 3, *or* 4 *from* δ *then* $\langle \delta, \varrho \rangle$ *is isomorphic to* $SL_2(3)$, $\mathrm{Alt}(5)$, *and* $\mathrm{Alt}(4)$, *respectively*.

Next we make use of the following information about the action of M_1 on the set of subgroups of order 9 in P_1 containing σ (cf. [Wi]).

LEMMA 1.5. $\overline{M}_1$ *has two orbits L and K on the set of subgroups of order* 9 *in* P_1 *containing* σ, *moreover*

(a) *if* $l \in L$ *then* $N_{\overline{M}_1}(l) \sim 3^5 \cdot (2 \times \mathrm{Mat}_{11})$ *is the stabilizer of a maximal clique in the graph* Δ *as in Lemma* 1.4(*c*) *and all subgroups of order* 3 *in* l *are of Suzuki type*;

(b) *if* $k \in K$ *then* $N_{\overline{M}_1}(k) \sim U_5(2) \cdot 2$ *and all subgroups of order* 3 *in* k *except* σ *are of Fischer type*.

Since P_1 is extraspecial, it follows from the above lemma that the subgroups of order 3 in P_1 other than σ form exactly two conjugacy classes $\tilde{L}$ and $\tilde{K}$ of M_1 with normalizers

$$(3 \times 3^{1+10}) \cdot 2 \cdot 3^5 \cdot (2 \times \mathrm{Mat}_{11}) \qquad \text{and} \qquad (3 \times 3^{1+10}) : (2 \times U_5(2) \cdot 2),$$

respectively.

It is clear that the subgroups from L in Lemma 1.5 are exactly the lines from $\mathcal{M}_2(\sigma)$. Comparing Lemmas 1.5(a) and 1.4(c) we can identify $\mathcal{M}_2(\sigma)$ with the set of cliques in the graph Δ on $\mathcal{M}_8(\sigma)$. Since a flag of $\mathcal{M}$ is stabilized by a Sylow 3-subgroup of M, it follows from Lemma 1.4(c) that a line $l \in \mathcal{M}_2(\sigma)$ and a quadric $\delta \in \mathcal{M}_8(\sigma)$ are incident if and only if l, as a clique of Δ, contains δ. By Lemma 1.4(d) two cliques l_1 and l_2 of Δ of maximal possible intersection have exactly two vertices, say δ_1 and δ_2 in common. Then the lines l_1 and l_2 are in two different quadrics and hence they generate an element of $\mathcal{M}$ which has to be the plane p which is the intersection of δ_1 and δ_2. This enables us to identify p with the edge $\{\delta_1, \delta_2\}$ of Δ.

Thus the elements from $\mathcal{M}_i(\sigma)$ for $i = 2$, 3, and 8 can be considered as cliques, edges, and vertices of the graph Δ with the natural incidence relation. In particular the planes and quadrics incident to a given line are edges and vertices of the corresponding clique of size 11 in Δ. Hence we have that the diagram of $\mathcal{M}$ is as given above and also (compare [Wi]) that

$$M_2 \sim 3^{2+5+10}:(GL_2(3) \times \mathrm{Mat}_{11}), \qquad \overline{M}_2 \sim \mathrm{Sym}(4) \times \mathrm{Mat}_{11};$$
$$M_3 \sim 3^{3+6+8}:(L_3(3) \times D_8:2), \qquad \overline{M}_3 \sim L_3(3) \times 2.$$

The main result of the paper is the following.

THEOREM 1.6. *The 3-local geometry $\mathcal{M}$ of the Monster is simply connected, equivalently, M is the universal completion of the amalgam of maximal parabolic subgroups M_1, M_2, M_3, and M_8 corresponding to the action of M on $\mathcal{M}$.*

Here and elsewhere a tuple $\{H_i \mid \in I\}$ of subgroups in a group will also be viewed as the amalgam obtained by considering the intersection of the H_i and the inclusion maps.

To prove the theorem we define G to be the universal completion of the amalgam $(M_i \mid i = 1, 2, 3, 8)$. Identify M_i with its image in G. Then there is a unique homomorphism χ of G onto M with $\chi_{M_i} = \mathrm{id}_{M_i}$ for all i. We will show eventually that χ is an isomorphism.

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2. $M(24)$-SUBGEOMETRY

In this section we discuss a subgeometry $\mathcal{M}(\mu)$ in $\mathcal{M}$ stabilized by a subgroup $F := N_M(\mu) \sim 3 \cdot M(24)$, where μ is a subgroup of order 3 of Fischer type in M. The elements of $\mathcal{M}(\mu)$ are some (not all) elements of $\mathcal{M}$ centralized by μ and the incidence relation is induced by that in $\mathcal{M}$.

As above, let $\mathcal{F} = \{Q_1, Q_2, Q_3, Q_8\}$ be a maximal flag in $\mathcal{M}$ with $Q_1 = \sigma$ and let μ be a Fischer type subgroup of Q_8 contained in $Q_3^{\perp}$. Define $\mathcal{M}(\mu)$ to be the subgeometry in $\mathcal{M}$ induced by the images under $F = N_M(\mu)$ of the elements in $\mathcal{F}$. We discuss the diagram of $\mathcal{M}(\mu)$ and the structure of the parabolic subgroups $F_i := N_F(Q_i) = N_{M_i}(\mu)$ corresponding to the action of F on $\mathcal{M}(\mu)$.

Since μ is non-isotropic with respect to φ we have

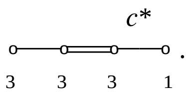
$$Q_8 = \mu \oplus \mu^{\perp},$$

where $\mu^{\perp}$ is the natural orthogonal module for $F_8/Q_8 \sim \Omega_7(3) \cdot 2$. Moreover, Q_1, Q_2, and Q_3 are contained in $\mu^{\perp}$ and form a maximal flag in the polar space defined on $\mu^{\perp}$. Thus $F_8/\mu \sim 3^7 \cdot \Omega_7(3) \cdot 2$ and the residue of Q_8 in $M(\mu)$ is the non-degenerate orthogonal polar space in dimension 7 over $GF(3)$.

By Lemma 1.5 we have $F_1/\mu \sim 3^{1+10} \cdot (2 \times U_5(2) \cdot 2)$ and one can see that the image of Q_8 in $\overline{F}_1 = F_1/(F_1 \cap P_1^*) \sim U_5(2) \cdot 2$ is a 3-central subgroup of order 3 with the normalizer isomorphic to $(3 \times U_4(2)) \cdot 2$. Let Θ be the graph on all these subgroups of order 3 in $\overline{F}_1$ in which two subgroups are adjacent if they commute. Then Θ is strongly regular with parameters

$$v = 176, \qquad k = 40, \qquad l = 135, \qquad \lambda = 12, \qquad \mu = 8$$

and clearly it is a subgraph in the graph Δ as in Lemma 1.4. In these terms the quadrics, planes, and lines in $M(\mu)$ incident to σ are the vertices, edges, and cliques (of size 5) in Θ with the natural incidence relation. This shows that the diagram of $M(\mu)$ is



It is easy to deduce the structure of two other parabolic subgroups (compare [RS]):

$$F_2/\mu \sim 3^{2+4+8} \cdot \left(GL_2(3) \times \mathrm{Sym}(5)\right), \quad F_3/\mu \sim 3^{3+7+3} \cdot 2 \cdot \left(L_3(3) \times 2\right).$$

In [IS] the geometry $M(\mu)$ was proved to be simply connected.

LEMMA 2.1. *The geometry* $M(\mu)$ *is simply connected and hence* $3 \cdot M(24)$ *is the unique faithful completion of the amalgam consisting of the subgroup* F_1, F_2, F_3, *and* F_8.

This immediately implies the following.

LEMMA 2.2. *Let X be a faithful completion of the amalgam consisting of the Monster subgroups M_i, $i = 1, 2, 3, 8$. Let μ be a non-isotropic subgroup of order* 3 *in Q_8 contained in $Q_3^{\perp}$. Then X contains a subgroup $M_\mu \sim 3 \cdot M(24)$, which normalizes μ, such that $M_\mu \cap M_i = N_{M_i}(\mu) = N_{M_\mu}(Q_i)$ for $i = 1, 2, 3$, and* 8. *If $X = M$ then $M_\mu = N_M(\mu)$.*

A subgroup μ as in the above lemma will be said to be of Fischer type. We remark that the subgroup M_μ of X does not only depend on μ but a priori also on the flag (M_1, M_2, M_3, M_8). But as the reader might check M_μ is already determined by μ together with any one of the M_i's, $i = 1, 2, 3$, or 8.

3. THE 2-LOCAL GEOMETRY OF THE MONSTER

There are exactly two classes of involutions in M, called the *Conway type* and *Baby Monster type* involutions with representatives z and t, such that

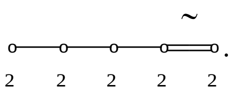
$$C_M(z) \sim 2_+^{1+24} \cdot Co_1 \qquad \text{and} \qquad C_M(t) \sim 2 \cdot F_2,$$

where Co_1 is the first Conway sporadic simple group and F_2 is the Fischer Baby Monster group [At].

Let $C = C_M(z)$. Then for $i = 4$ and 8 up to conjugation in C there is a unique Conway-pure subgroup E_i of order i in $O_2(C)$ containing z, whose normalizer in M contains a Sylow 2-subgroup of M. Moreover these two subgroups can be chosen so that $E_4 < E_8$ and we will assume that the inclusion holds. Let $N = N_M(E_4)$ and $L = N_M(E_8)$. Then

$$C \sim 2_+^{1+24} \cdot Co_1, \qquad N \sim 2^{2+11+22} \cdot (\mathrm{Sym}(3) \times \mathrm{Mat}_{24}),$$
$$L \sim 2^{3+6+12+18} \cdot (L_3(2) \times 3 \cdot \mathrm{Sym}(6)).$$

Furthermore C, N, and L are the stabilizers of a point, a line, and a plane from a maximal flag in the 2-local minimal parabolic geometry of the Monster group [RS] having the diagram



This geometry was proved to be 2-simply connected in [Iv1] and by standard principles this result is equivalent to the following.

LEMMA 3.1. *The Monster group M is the universal completion of the amalgam of its subgroups C, N, and L defined as above.*

Our strategy to prove Theorem 1.6 is to show that the universal completion G of the amalgam of the 3-local parabolics M_i is also a completion of the amalgam consisting of the subgroups C, N, and L as in Lemma 3.1.

LEMMA 3.2. *Let μ be a subgroup of Fischer type in M. Then M_μ has exactly four classes of involutions and for an involution $t \in M_\mu$ exactly one of the following holds*:

(a) *t inverts μ, t is of Baby Monster type, and $C_{M_\mu}(t) \cong M(23) \times C_2$.*

(b) *t centralizes μ, t is of Baby Monster type, and $C_{M_\mu}(t) \sim 3 \cdot 2^2 M(22) \cdot 2$.*

(c) *t inverts μ, t is of Conway type, and $C_{M_\mu}(t) \sim 2^3 \cdot U_6(2) \cdot \mathrm{Sym}(3)$.*

(d) *t centralizes μ, t is of Conway type, and* $C_{M_\mu}(t) \sim 3.2_+^{1+12} \cdot 3 \cdot U_4(3) \cdot 2^2$.

Proof. By [At], M_μ has four classes of involutions with centralizers as given. By Lagrange, Co_1 involves neither $M(23)$ nor $3 \cdot M(22) \cdot 2$ and so the first two classes are of Baby Monster type. Since Conway type involutions both invert and centralize groups of Fischer type the remaining two classes must be of Conway type. ∎

LEMMA 3.3. *Let z be an involution from $P_1^* = O_{3,2}(M_1)$. Then every involution in M_1 is conjugated to an involution $s \in C_{M_1}(z)$ and one of the following holds*:

(a) $s = z$, $C_{M_1}(s) \sim 6 \cdot \mathrm{Suz} \cdot 2$, *and s is of Conway type in M*;

(b) *s inverts σ*, $C_{M_1}(s)P_1^*/P_1^* \sim 2 \cdot \mathrm{Mat}_{12}$; *$s$ and sz are conjugated in M_1 and the centralizer of s in P_1/σ has order* 3^6;

(c) *s centralizes σ and* $C_{M_1}(s)P_1^*/P_1^* \sim 2_-^{1+6} \cdot O_6^-(2)$ (*two conjugacy classes*).

Proof. The conjugacy classes of involutions in $M_1/P_1 \sim 2 \cdot \mathrm{Suz} \cdot 2$ can be read from [At]. Since z centralizes σ and acts fixed point-freely on P_1/σ, the structure of $C_{M_1}(s)$ in (a) follows. Since the Baby Monster has no elements of order 3 with normalizer of the shape $3 \cdot \mathrm{Suz} \cdot 2$, z is of Conway type. In (b) we have $C_{P_1/\sigma}(s) = [P_1/\sigma, sz]$ and since s and sz are conjugated, both subspaces have dimension 6. ∎

4. THE 3-LOCAL GEOMETRY FOR Co_1

In [Iv2], a relationship between the 3- and 2-local geometries of the Monster via a 2^{24}-cover of the 3-local geometry of the Conway group [BF] was noticed.

Let X be an arbitrary faithful completion of the amalgam (M_1, M_2, M_3, M_8) of the 3-local parabolics in M which has M has a quotient and let X be the geometry whose elements are the cosets in X of M_i for $i = 1, 2, 3, 8$ and where two cosets are incident if their intersection is non-empty. If $X = M$ or $X = G$ where as above G is the universal completion of the amalgam, then X is M or the universal cover G of M, respectively. For an element x of X let M_x denote the stabilizer of x in X which is a conjugate of M_i for $i = 1$, 2, 3, or 8 depending on the type of x. If $x = M_i g$ put $Q_x = Q_i^g$, $P_x = P_i^g$, and $P_x^* = P_i^{*g}$. When working in the residue of an element we can and will identify x with Q_x. If μ is a subgroup of order 3 of Fischer type in Q_8^g, then M_μ denotes the subgroup as in Lemma 2.2, i.e., if $\mu \in Q_3^{g\perp}$ then $M_\mu = \langle N_{M_i^g}(\mu) \mid i = 1, 2, 3, 8 \rangle$.

Let us pick an involution z from $P_1^* = O_{3,2}(M_1)$. Then by Lemma 3.3(a), $C_{M_1}(z) \sim 6 \cdot \mathrm{Suz} \cdot 2$. Let $\Xi = \Xi^X$ be the set of points of X such that $x \in \Xi$ if and only $z \in O_{2,3}(M_x)$. Let Ξ denote also the graph on Ξ in which two points are adjacent if they are incident to a common quadric. It is clear that $C_X(z)$ preserves Ξ as a whole as well as the adjacency relation on Ξ.

LEMMA 4.1. *Locally Ξ is the commuting graph Δ of 3-central subgroups of order 3 in $\overline{M}_1 \sim \mathrm{Suz} \cdot 2$ as in Lemma 1.4. Let Ω be a maximal clique in Ξ containing σ and H be the setwise stabilizer of Ω in $C_X(z)$. Then $|\Omega| = 12$ and there is a unique point α collinear to σ such that $H = C_{M_\alpha}(z)$. Moreover, $H \sim 2.3^6 \cdot (2 \cdot \mathrm{Mat}_{12})$, $O_3(H) = P_\alpha \cap H$, H induces the natural action of Mat_{12} on the vertices of Ω, and $O_3(H)$ is an irreducible $GF(3)$-module for $H/O_3(H)\langle z\rangle \sim 2 \cdot \mathrm{Mat}_{12}$.*

Proof. Abusing the notation we denote by σ the point stabilized by M_1 so that $\sigma \in \Xi$. By Lemma 1.3(d) every quadric incident to σ contains besides σ exactly one point ε centralized by z and ε is not collinear to σ. This means that the set $\Xi(\sigma)$ of points adjacent to σ in Ξ is in a natural bijection with the set of quadrics incident to σ, i.e., with the vertices of the graph Δ as in Lemma 1.4. Moreover, if $\delta \in \Delta$ then there is a unique point centralized by z which maps onto δ under the homomorphism of M_1 onto $\overline{M}_1$. We will identify δ with this unique point. By definition if x and y are adjacent points in Ξ then $[Q_x, Q_y] = 1$. Hence if $\delta_1, \delta_2 \in \Xi(\sigma)$ are adjacent in Ξ, then the corresponding vertices of Δ are adjacent. In particular a maximal clique in Ξ contains at most 12 vertices. We are going to show that this bound is attained.

Let l be a line incident to σ and let σ, α, β, and γ be all the points incident to l. Since z acts fixed-point freely on $P_1/\sigma \sim 3^{12}$, we can choose our notation so that z inverts α and permutes β and γ. So on every line incident to σ there is exactly one point which is inverted by z. Since $C_{M_1}(z)P_1 = M_1$, $C_{M_1}(z)$ permutes transitively the lines incident to σ and hence also the points collinear to σ and inverted by z. This implies that $C_{M_\alpha}(z)$ permutes transitively the points collinear to α and centralized by z.

Let Q_8 denote a quadric incident to l and let ε be the point in Q_8 other than σ centralized by z. Then ε is collinear to exactly one point on l. We know that σ and ε are not collinear and since β and γ are permuted by z, ε is collinear to α. Thus in every quadric incident to l besides σ there is exactly one point collinear to α and centralized by z. By the diagram of X there are exactly 11 such quadrics which correspond to a clique K of Δ. Let $\Omega = \{\sigma\} \cup K$ and H be the setwise stabilizer of Ω in $C_X(z)$. Since locally Ξ is Δ, K is a maximal clique in $\Delta, C_{M_1}(z)$ acts transitively on the set of cliques in Δ and since $C_X(z)$ is vertex-transitive on Ξ, we see that H

acts transitively on Ω. Since α is the only point which is collinear to every point in Ω, it is clear that $H \leq C_{M_\alpha}(z)$. Since z acts fixed-point freely on P_1/σ, $C_{P_1^*}(z) = \sigma \times \langle z \rangle$. By Lemma 1.4(c) and the Frattini argument $(H \cap M_1)P_1^*/P_1^* \sim (M_l \cap M_1)/P_1^* \sim 3^5 \cdot (2 \times \mathrm{Mat}_{11})$. Since $H \cap M_1$ induces the natural action of Mat_{11} on the points in K, H induces on the points in Ω the natural action of Mat_{12}. Thus $O_3(H)$ is elementary abelian of order 3^6 generated by the 12 points in Ω and $H/O_3(H)\langle z \rangle \sim 2 \cdot \mathrm{Mat}_{12}$ induces a non-trivial action on $O_3(H)$. By [MoAt], Mat_{12} does not have a faithful $GF(3)$-representation of dimension less than or equal to 6 and the smallest faithful $GF(3)$-representation of $2 \cdot \mathrm{Mat}_{12}$ has dimension exactly 6. Thus we have shown that $H \sim 2.3^6 \cdot (2 \cdot \mathrm{Mat}_{12})$ and by Lemma 3.3(b), $H = C_{M_\alpha}(z)$.

In what follows we will need the detailed information on the structure of 6-dimensional $GF(3)$-modules of $2 \cdot \mathrm{Mat}_{12}$ contained in the following lemma.

LEMMA 4.2. *Let $\overline{H} \sim 2 \cdot \mathrm{Mat}_{12}$ and A be a faithful irreducible 6-dimensional $GF(3)\overline{H}$-module. Then the following assertions hold*:

(a) *$\overline{H}$ has a unique orbit A of length 12 on the 1-spaces of A.*

(b) *Any five elements from A are linearly independent.*

(c) *$\overline{H}$ has a unique orbit L of length less or equal to 12 on the hyperplanes of A. Moreover, $|\mathsf{L}| = 12$ and if $L \in \mathsf{L}$ then L contains no element from A.*

(d) *Let B be the set of 1-spaces of A of the form $\langle a_1 + a_2 \rangle$, where $\langle a_1 \rangle$ and $\langle a_2 \rangle \in \mathsf{A}$ are different elements of A. Then $|\mathsf{B}| = 132$ and H acts transitively on B.*

(e) *If $F \in \mathsf{B}$ then there exist unique elements D_1 and D_2 in A with $F \leq D_1 + D_2$. If $L \in \mathsf{L}$ and $\tilde{F}$ is the 1-space in $D_1 + D_2$ different from D_1, D_2, and F, then $F \leq L$ if and only if $\tilde{F} \not\leq L$.*

(f) *Define $L \in \mathsf{L}$ and $B \in \mathsf{B}$ to be incident if $B \leq L$. Then (L, B) is a Steiner system of type $(5, 6, 12)$.*

(g) *Let $\mathsf{T} \subset \mathsf{L}$ with $|\mathsf{T}| = 4$ and put $F = \bigcap \mathsf{T}$. Then F is a 2-subspace of A, all 1-spaces of F are in B, and $N_H(F)/C_H(F) \cong GL_2(3)$.*

Proof. Let X and Y be two non-conjugate subgroups in $\overline{H}$ isomorphic to Mat_{11}. Then every proper subgroup of index at most 12 in $\overline{H}/Z(\overline{H}) \cong \mathrm{Mat}_{12}$ is conjugate to the image of either X or Y. Moreover, $\overline{H} = \langle X, Y \rangle$ and $X \cap Y \cong L_2(11)$. Let Z be one of the subgroups X, Y and $X \cap Y$. By [MoAt] every faithful irreducible $GF(3)Z$-module is 5-dimensional. This means that Z normalizes in A at most one 1-subspace and at most one 5-subspace. Suppose that A contains a 1-subspace normalized by X and a

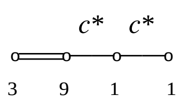
1-subspace normalized by Y. Then both these 1-spaces are normalized by $X \cap Y$ and hence this is the same 1-space, normalized by the whole $\overline{H} = \langle X, Y \rangle$, a contradiction to the irreducibility of A. Applying the same argument to the module dual to A, we obtain that the subspaces in A normalized by X and Y have different dimensions and we can choose our notation so that X normalizes a 1-space D and Y normalizes a 5-space E. In this case $A = D \oplus E$ as a module for $X \cap Y$. Moreover, $\mathcal{A} := D^{\overline{H}}$ is the only orbit of length 12 of $\overline{H}$ on 1-spaces in A and $\mathcal{L} := E^{\overline{H}}$ is the only orbit of length 12 of $\overline{H}$ on hyperplanes in A and (c) holds.

The actions induced by $\overline{H}$ on $\mathcal{A}$ and $\mathcal{L}$ are two non-equivalent 5-transitive actions of Mat_{12}. Since A is irreducible, $\mathcal{A}$ spans A and so there is a set of six linearly independent elements in $\mathcal{A}$. Since $\overline{H}$ induces on $\mathcal{A}$ a 5-transitive action, every set of five elements in $\mathcal{A}$ is linearly independent and thus (b) holds.

Let $D_1 \neq D_2 \in \mathcal{A}$ and let $D_1, D_2, F, \tilde{F}$ be the set of all 1-spaces in $D_1 + D_2$. Then $F, \tilde{F} \in \mathcal{B}$. We are going to show that $\mathcal{B}$ satisfies the properties stated in (d)–(f). If there are $D_i, D_j \in \mathcal{A}$ with $\{i, j\} \neq \{1, 2\}$ such that F is contained in $D_i + D_j$ then the set $\{D_k \mid k = 1, 2, i, j\}$ of size at most four in $\mathcal{A}$ would be linearly dependent, a contradiction to (b). Hence the pair $\{D_1, D_2\}$ is uniquely determined by F. Let $L \in \mathcal{L}$. Since L is a hyperplane in A, its intersection with $D_1 + D_2$ is at least 1-dimensional. By (c) neither D_1 nor D_2 are in L, hence (e) follows. Moreover, F or $\tilde{F}$ is contained in at least 6 elements of $\mathcal{L}$. Since the action of $\overline{H}$ on $\mathcal{L}$ is 5-transitive, we conclude that the intersection of any five elements of $\mathcal{L}$ is in $\mathcal{B}$. Let $\mathcal{D}$ be the set of elements of $\mathcal{L}$ containing F. Suppose that $|\mathcal{D}| \geq 7$. Then by 5-transitivity of $\overline{H}$ on $\mathcal{L}$ there exists $h \in \overline{H}$ with $|\mathcal{D} \cap \mathcal{D}^h| \geq 5$ and $\mathcal{D} \neq \mathcal{D}^h$. But then the intersection of the elements on $\mathcal{D}$, $\mathcal{D} \cap \mathcal{D}^h$, and $\mathcal{D}^h$, respectively, are all equal to F, a contradiction to $\mathcal{D} \neq \mathcal{D}^h$. Hence $|\mathcal{D}| \leq 6$ and both F and $\tilde{F}$ are contained in exactly six elements of $\mathcal{L}$. Thus (f) holds. As $\overline{H}$ acts transitively on the blocks of any associate Steiner systems, (d) follows.

By (f), $\mathcal{T}$ is incident to exactly four elements say B_1, B_2, B_3, B_4 of $\mathcal{B}$. By the dual of (b), F is a 2-space and so B_1, B_2, B_3, B_4 are exactly the 1-spaces of F. Since $N_{\overline{H}}(\mathcal{T})$ induces Sym(4) on $\{B_1, B_2, B_3, B_4\}$ we conclude $N_{\overline{H}}(F)/C_{\overline{H}}(F) \cong GL_2(3)$. ∎

By Lemmas 4.1 and 1.4(d) two maximal cliques in Ξ are either disjoint or have intersection of size 1, 2, or 3. Moreover, if $\mathcal{C} = \mathcal{C}^X$ is a geometry whose elements are maximal cliques, triangles, edges, and vertices of Ξ^X with respect to the incidence relation given by inclusion, then $\mathcal{C}$ corresponds to the diagram



The geometry $\mathcal{C}$ is connected precisely when Ξ is connected. Let $\sigma = \Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \Omega_{12} = \Omega$ be the maximal flag in $\mathcal{C}$. Then Ω_i is a complete subgraph of size i in Ξ. Let C_i denote the stabilizer in $C_X(z)$ of Ω_i. Then

$$C_1/\langle z \rangle \sim 3 \cdot \mathrm{Suz} \cdot 2, \qquad C_2/\langle z \rangle \sim 3^2 \cdot U_4(3) \cdot D_8,$$
$$C_3/\langle z \rangle \sim 3^{3+4} \cdot [2^3] \cdot S_4 \cdot S_3, \qquad C_{12}/\langle z \rangle \sim 3^6 \cdot 2 \cdot \mathrm{Mat}_{12}.$$

Consider the situation when $X = M$. By Lemma 3.3(a), z is of Conway type and $C_M(z) = C \sim 2_+^{1+24} \cdot CO_1$. Put $R = O_2(C)$.

LEMMA 4.3. *The graph Ξ^M is connected.*

Proof. Let A be the setwise stabilizer in $C_M(z)$ of the connected component of Ξ^M which contains σ. Then A contains $C_1 \sim 6 \cdot \mathrm{Suz} \cdot 2$. Let ε be a vertex adjacent to σ in Ξ^M. Then $[\sigma, \varepsilon] = 1$ and since σ acts fixed-point freely on $R/\langle z \rangle$, we have $\sigma R \neq \varepsilon R$. Since $C_1 R$ is maximal in C, this means that $AR = C$. Finally, $C/\langle z \rangle$ does not split over $R/\langle z \rangle$ and hence $A = C$ and Ξ^M is connected. ∎

The homomorphism $\chi: G \to M$ induces morphisms $\mathcal{G} \to \mathcal{M}$ and $\mathcal{C}^G \to \mathcal{C}^M$ of geometries which will be denoted by the same letter χ. Our goal is to show that the restriction of χ to the connected component of $\mathcal{C}^G$ containing σ is an isomorphism onto $\mathcal{C}^M$. This will immediately imply that the setwise stabilizer in $C_G(z)$ of the connected component of C^G maps isomorphically onto $C \sim 2_+^{1+24} \cdot Co_1$. An important role in the realization of this step will be played by a simply connected subgeometry in $\mathcal{G}$.

Let μ be a subgroup of Fischer type as in Section 2. Then $k := \sigma\mu$ is a subgroup of order 9 in P_1 which is not a line (so that k is as in Lemma 1.5(b)). Since z acts fixed-point freely on P_1/Q_1, as in the proof of Lemma 4.1 we have a unique subgroup of order 3 in k which is normalized and inverted by z. Hence we can and do choose μ so that z inverts μ. By Lemma 2.2 there is a subgroup $M_\mu \sim 3 \cdot M(24)$ in X which normalizes μ such that $M_\mu \cap M_i = N_{M_i}(\mu)$ for $i = 1, 2, 3$, and 8. Let $W = C_{M_\mu}(z)$ and let Ψ be the orbit of W on Ξ which contains σ.

LEMMA 4.4. (a) $|\Psi| = 2688$ *and* $W/\langle z \rangle \sim 2^2 \cdot U_6(2) \cdot \mathrm{Sym}(3)$ *acts faithfully on* Ψ;

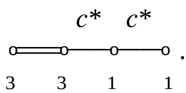
(b) *locally* Ψ *is the commuting graph* Θ *on the* 3-*central subgroups of order* 3 *in* $U_5(2) \cdot 2$.

Proof. By Lemma 1.5(b) and since $M_\mu \cap M_1 = N_{M_1}(\mu)$, $C_{M_\mu}(z) \cap M_1 \sim 2 \cdot (3 \times U_5(2)) \cdot 2$. By Lemma 3.2 and since z is of Conway type and inverts μ, $W \sim 2^3 \cdot U_6(2) \cdot \mathrm{Sym}(3)$. Thus (a) holds.

For (b) we may by (a) assume that $X = M$. The subgroups of Fischer type in P_1 normalized by z, are permuted transitively by $C_{M_1}(z)$ and hence

Ψ contains a vertex x of Ξ if and only if μ is contained in P_x, or equivalently if x is contained in $M(\mu)$ and hence (b) follows. ∎

Since Ψ is locally Θ, its maximal cliques have size 6 and two such cliques are either disjoint or have intersection of size 1, 2, or 3. Define U to be a geometry whose elements are maximal cliques, triangles, edges, and vertices of Ψ with the natural incidence relation. Since $\Psi = \Xi \cap M(\mu)$, it is easy to see that the diagram of U is



As follows from Lemma 4.4, the isomorphism type of U is independent on whether $X = M$ or $X = G$, since U is contained in $M(\mu)$ which is simply connected. It is worth mentioning that U itself is simply connected as proved in [Me] and that Ψ is distance-transitive with the distribution diagram given on Fig. 2.

5. A CHARACTERIZATION OF C^M

It is not known whether the geometry C^M is simply connected. In this section we establish a sufficient condition for a covering of C^M to be an isomorphism in terms of the subgeometry U and its images under $C_M(z)$. Let $R = O_2(C_M(z))$ which is extraspecial of order 2^{25}. We start by defining the folding $\overline{C}$ of C^M with respect to the action of R.

The kernel of the action of $C = C_M(z)$ on C^M is $\langle z \rangle$ and since $O_2(C_i/\langle z \rangle) = 1$ for $i = 1, 2, 3$, and 12, the action of $R/\langle z \rangle$ is fixed-point free. Let $\overline{C}$ be the folding of C^M with respect to the action of R. This means that $\overline{C}$ is a geometry whose elements are the orbits of R on C^M with two such orbits O_1 and O_2 incident if and only if an element from O_1 is incident in C^M to an element from O_2. Since $R/\langle z \rangle$ acts fixed-point freely on C^M, it is easy to see that if O_1 and O_2 are incident in $\overline{C}$ then each element from O_1 is incident in C^M to exactly one element from O_2. Let $\overline{\Xi}$ be the collinearity graph of $\overline{C}$ which is also the folding with respect to the action of R of the collinearity graph Ξ^M of C^M.

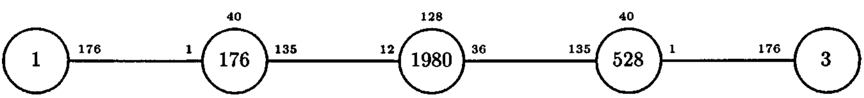


FIG. 2. Distribution diagram of Ψ.

We put $\overline{C} = C/R$ and use the bar notation for the images of $\overline{C}$ of subgroups of C. Then $\overline{\sigma}$ is a subgroup of order 3 in $\overline{C}$ and $N_{\overline{C}}(\overline{\sigma}) \sim 3 \cdot \mathrm{Suz} \cdot 2$ which is a maximal subgroup in $\overline{C}$. This enables us to identify the vertices of $\overline{\Xi}$ with the Suzuki-type subgroups of order 3 in $\overline{C} \sim Co_1$. We will use the following properties of the action of $\overline{C}$ on $\overline{\Xi}$.

LEMMA 5.1. *Let $\overline{C} \cong Co_1$, $\overline{\Xi}$ be the set of Suzuki-type subgroups of order 3 in $\overline{C}$, $\overline{\sigma} \in \overline{\Xi}$, and $\overline{C}(\overline{\sigma}) = N_{\overline{C}}(\overline{\sigma}) \sim 3 \cdot \mathrm{Suz} \cdot 2$. Then $\overline{C}$ acts primitively on $\overline{\Xi}$ while $\overline{C}(\overline{\sigma})$ has 5 orbits on $\overline{\Xi}$: $\{\overline{\sigma}\}, \overline{\Xi}_1(\overline{\sigma})$, $\overline{\Xi}_2(\overline{\sigma})$, $\overline{\Xi}_3(\overline{\sigma})$, and $\overline{\Xi}_4(\overline{\sigma})$ with lengths 1, 22,880, 405,405, 1,111,968, and 5346, respectively. Let $\overline{\Xi}$ denote also the graph on $\overline{\Xi}$ invariant under the action of $\overline{C}$, in which $\overline{\sigma}$ is adjacent to the vertices from $\overline{\Xi}_1(\overline{\sigma})$. Let $\overline{\mu}_i \in \overline{\Xi}_i(\overline{\sigma})$ and $\overline{B}_i = \overline{C}(\overline{\sigma}) \cap \overline{C}(\overline{\mu}_i)$ for $i = 1, 2, 3, 4$. Then*

(a) $\overline{\delta} \in \overline{\Xi} \setminus \{\overline{\sigma}\}$ *is adjacent to $\overline{\sigma}$ in $\overline{\Xi}$ if and only if $[\overline{\sigma}, \overline{\delta}] = 1$, so that $\overline{\Xi}$ is the folding of Ξ^M with respect to the action of R; the distribution diagram of $\overline{\Xi}$ is given on Fig.* 3;

(b) $\overline{B}_1 \sim 3^2 \cdot U_4(3) \cdot 2^2$, *locally $\overline{\Xi}$ is the commuting graph Δ of central subgroups of order 3 in $\overline{C}(\overline{\sigma})/\overline{\sigma} \sim \mathrm{Suz} \cdot 2$;*

(c) $\overline{B}_2 \sim 2^{1+6} \cdot U_4(2) \cdot 2$ *acts transitively on $\overline{\Xi}_i(\overline{\sigma}) \cap \overline{\Xi}_1(\overline{\mu}_2)$ for $i = 1$, 2, and 3, the subgraph induced on $\overline{\Xi}_1(\overline{\mu}_2) \cap \overline{\Xi}_1(\overline{\sigma})$ is the disjoint union of 40 copies of the complete 3-partite graph $K_{4,4,4}$, these copies are permuted primitively by $\overline{B}_2/O_2(\overline{B}_2) \sim U_4(2) \cdot 2$, $\langle \overline{\sigma}, \overline{\mu}_2 \rangle \cong SL_2(3)$;*

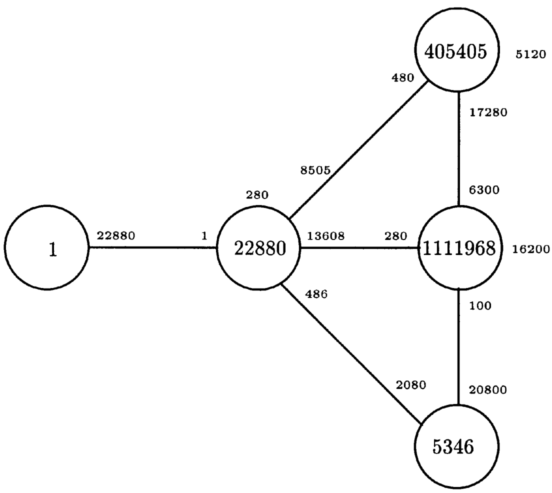


FIG. 3. Distribution diagram of $\overline{\Xi}$.

(d) $\overline{B}_3 \sim J_2\colon 2 \times 2$ *acts primitively on* $\overline{\Xi}_1(\overline{\mu}_3) \cap \overline{\Xi}_i(\overline{\sigma})$ *for* $i = 1, 4$ *and transitive for* $i = 2$, $\langle \overline{\sigma}, \overline{\mu}_3 \rangle \cong \mathrm{Alt}(5)$;

(e) $\overline{B}_4 \sim G_2(4) \cdot 2$ *acts primitively on* $\overline{\Xi}_1(\overline{\mu}_4) \cap \overline{\Xi}_i(\overline{\sigma})$ *for* $i = 1$ *and* 3, $\langle \overline{\sigma}, \overline{\mu}_4 \rangle \cong \mathrm{Alt}(4)$;

(f) *the subgraph induced on* $\overline{\Xi}_1(\overline{\mu}_i) \cap \overline{\Xi}_1(\overline{\sigma})$ *is empty for* $i = 3$ *and* 4;

(g) *each vertex from* $\overline{\Xi}_1(\overline{\mu}_3) \cap \overline{\Xi}_3(\overline{\sigma})$ *is adjacent to a vertex from* $\overline{\Xi}_1(\overline{\mu}_3) \cap \overline{\Xi}_1(\overline{\sigma})$ *or to a vertex from* $\overline{\Xi}_1(\overline{\mu}_3) \cap \overline{\Xi}_2(\overline{\sigma})$.

Proof. The subdegrees, 2-point stabilizers $\overline{B}_i$ of the action of $\overline{C}$ on $\overline{\Xi}$ and $\langle \overline{\sigma}, \overline{\mu}_i \rangle$ are well known (cf. Lemma 49.8 in [As] or Lemma 2.22.1(ii) in [ILLSS]). The distribution diagram on Fig. 3 is taken from [PS]. This diagram and the structure of $\overline{B}_1$ show that the subgraph induced on $\overline{\Xi}_1(\overline{\sigma})$ is isomorphic to the graph Δ as in Lemma 1.4 and that $\overline{C}(\overline{\sigma})$ induces its full automorphism group. This means that $\overline{B}_1$ acts transitively on $\overline{\Xi}_1(\overline{\mu}_1) \cap \overline{\Xi}_i(\overline{\sigma})$ for $i = 1, 2, 3, 4$ and hence for every vertex $\overline{\gamma}$ at distance 2 from $\overline{\sigma}$ in $\overline{\Xi}$, $\overline{C}(\overline{\sigma}) \cap \overline{C}(\overline{\gamma})$ acts transitively on $\overline{\Xi}_1(\overline{\sigma}) \cap \overline{\Xi}_1(\overline{\gamma})$. Let χ_i be the permutational character of $\overline{C}(\overline{\sigma})$ on the cosets of $\overline{B}_i\overline{\sigma}$ for $i = 1$, 2, and 4. By Lemma 2.13.1 in [ILLSS] the inner product of χ_1 and χ_i is 5, 3, and 2 for $i = 1$, 2, and 4, respectively. This implies the transitivity statements in (c), (d), and (e). By [At] every action of $\overline{B}_3$ of degree 100 or 280 as well as every action of $\overline{B}_4$ of degree 2080 or 20,800 is primitive.

Let $\overline{\delta}_i \in \overline{\Xi}_1(\overline{\sigma}) \cap \overline{\Xi}_1(\overline{\mu}_i)$ for $i = 2, 3, 4$. Then since locally $\overline{\Xi}$ is Δ, the distance from $\overline{\sigma}$ to $\overline{\mu}_i$ in the subgraph induced on $\overline{\Xi}_1(\overline{\delta}_i)$ is i. Hence the subgraph induced by $\overline{\Xi}_1(\overline{\mu}_i) \cap \overline{\Xi}_1(\overline{\sigma})$ is empty for $i = 3$ and 4, while for $i = 2$ it is locally $K_{4,4}$ (compare Lemma 1.4(e)). It is well known and easy to check that $K_{4,4,4}$ is the only connected graph which is locally $K_{4,4}$ and the structure of the subgraph induced on $\overline{\Xi}_1(\overline{\mu}_2) \cap \overline{\Xi}_1(\overline{\sigma})$ follows. Finally, every transitive action of $\overline{B}_2$ of degree 40 is primitive and has $O_2(\overline{B}_2)$ in its kernel. Thus all statements except (g) are proved.

We will prove (g) with the roles of $\overline{\sigma}$ and $\overline{\mu}_3$ interchanged. For this we first determine the orbits of $\overline{B}_3$ on $\overline{\Xi}_1(\sigma)$. Let $A = \langle \overline{\sigma}, \overline{\mu}_3 \rangle$. Then $A \cong \mathrm{Alt}(5)$. Note that there exist exactly two elements $\rho \in \overline{\Xi} \cap A$ such that $\langle \rho, \overline{\sigma} \rangle \cong \mathrm{Alt}(4)$ and $\langle \rho, \overline{\mu}_3 \rangle \cong \mathrm{Alt}(5)$. Without loss $\overline{\mu}_4$ is one of these two. Put $J = N_{\overline{B}_3}(\overline{\mu}_4) = \overline{B}_4 \cap \overline{B}_3$. Then J is of index two in $\overline{B}_3$ and $J \sim J_2 \cdot 2$. Put $K := \overline{B}_4$. Then $K \sim G_2(4) \cdot 2$.

As the main step in determining the orbits of $\overline{B}_3$ on $\overline{\Xi}_1(\overline{\sigma})$ we compute the orbits of J by decomposing the orbits of K. By (e), K acting on $\overline{\Xi}_1(\overline{\sigma})$ has two orbits, $\Gamma_1 = \overline{\Xi}_1(\overline{\sigma}) \cap \overline{\Xi}_3(\overline{\mu}_4)$ and $\Gamma_2 = \overline{\Xi}_1(\overline{\sigma}) \cap \overline{\Xi}_1(\overline{\mu}_4)$ with lengths 20,800 and 2080, respectively, moreover if K_1 and K_2 are the respective stabilizers, then $K_1 \sim U_3(3)\colon 2 \times 2$ and $K_2 \sim 3 \cdot L_3(4) \cdot 2^2$. Consider the graph Σ with 416 vertices of valency 100 on which K acts as a rank 3 automorphism group (see [BvL]). Then the parameters of Σ are

$$v = 416, \qquad k = 100, \qquad l = 315, \qquad \lambda = 36, \qquad \mu = 20.$$

It follows from the list of maximal subgroups in K, that Γ_1 can be identified with the set of edges of Σ while J is the stabilizer in K of a vertex x of Σ. By well known properties of the action of K on Σ [BvL] the orbit of J on the edge-set of Σ containing an edge $\{y_1, y_2\}$ of Σ is uniquely determined by the pair $\{d_1, d_2\}$ where d_i is the distance from x to y_i in Σ. This and the parameters of Σ given above show that under the action of J the set of edges of Σ (identified with the set Γ_1) splits into four orbits Ω_1, Ω_2, Ω_3, and Ω_4 corresponding to the pairs of distances $\{0, 1\}$, $\{1, 1\}$, $\{1, 2\}$, and $\{2, 2\}$ and having lengths 100, 1800, 6300, and 12,600, respectively. Let $\Omega_5 = \overline{\Xi}_1(\overline{\sigma}) \cap \overline{\Xi}_1(\overline{\mu}_3)$ and $\gamma \in \Omega_5$. Note that J acts transitively on Ω_5 and $|\Omega_5| = 280$. By (a), γ commutes with $\overline{\mu}_3$. Thus $\gamma \leq J \leq K'$ and so $\gamma \in \Gamma_2$ and Ω_5 is an orbit for J on Γ_2. Let K_2 be the stabilizer of γ in K. Then $\gamma = O_3(K_2)$. By (f) all 280 vertices adjacent to γ in the subgraph induced on $\overline{\Xi}_1(\overline{\sigma})$ are in Γ_1 and by (a) these 280 vertices are fixed by γ. Let $\Sigma(\gamma)$ be the set of vertices in Σ fixed by γ. Comparing the permutation characters of K on σ with the permutational character of K on Γ_2, we see that K_2 has exactly two orbits on the vertex set of Σ. On one hand this means that under the action of J the set Γ_2 splits into two orbits namely Ω_5 and an orbit Ω_6 of length 1800. On the other hand $K_2/\gamma \sim L_3(4) \cdot 2^2$ acts transitively on $\Sigma(\gamma)$ and so $|\Sigma(\gamma)| = 280 \cdot (|J|/|K_2|) = 56$. Any transitive action of the latter group of degree 56 is the rank 3 action on the vertex set of the Gewirtz graph which is strongly regular with parameters

$$v = 56, \qquad k = 10, \qquad l = 45, \qquad \lambda = 0, \qquad \mu = 2.$$

Hence we conclude that K_2 acts transitively on the set of edges in Σ fixed by γ. Again since γ is adjacent in $\overline{\Xi}$ to exactly 280 vertices from Γ_1 there are 280 edges in the subgraph of Σ induced on $\Sigma(\gamma)$ and hence this subgraph is the Gewirtz graph rather than its complement.

Note that Ω_i, $1 \leq i \leq 6$ are the orbits for J on $\overline{\Xi}_1(\overline{\sigma})$. If $\overline{B}_3$ normalizes K' then K' centralizes $\langle \overline{\sigma}, \overline{\mu}_4^{\overline{B}_3} \rangle = A$ and so $K' \leq \overline{B}_3$, a contradiction. Since K' is generated by the elements of $\Gamma_2 = \overline{\Xi} \cap K$ we conclude that $\overline{B}_3$ does not normalize Γ_2. Thus some of the orbits of J must be fuzed by $\overline{B}_3$. Since J is normal in $\overline{B}_3$, only orbits with the same lengths can fuse. Thus $\Omega_2 \cup \Omega_6$ is a single orbit of $\overline{B}_3$. The distribution diagram of $\overline{\Xi}$ enables us to identify Ω_5, Ω_3, $\Omega_2 \cup \Omega_4 \cup \Omega_6$, and Ω_1 with $\overline{\Xi}_1(\overline{\sigma}) \cap \overline{\Xi}_i(\overline{\mu}_3)$ for $i = 1, 2, 3$, and 4, respectively. A vertex from Γ_1 is adjacent to γ in $\overline{\Xi}$ if and only if the corresponding edge of Σ is fixed by γ. The parameters of the Gewirtz graph imply that γ is adjacent to 10, 90, 180 vertices from Ω_i for $i = 1, 3$, and 4, respectively. Since every vertex from Γ_1 is adjacent to $28 = 280 \cdot |\Gamma_2|/|\Gamma_1|$ vertices from Γ_2 and every vertex from Ω_3 is adjacent to $4 = 90 \cdot |\Omega_5|/|\Omega_3|$ vertices of Ω_5, we observe that a vertex $v \in \Omega_3$ is adjacent to $24 = 28 - 4$ vertices from Ω_6. Since Ω_2 and Ω_6 are fuzed under $\overline{B}_3$ this means that v is also adjacent to 24 vertices from Ω_2. Hence every vertex from $\overline{\Xi}_1(\overline{\sigma}) \cap \overline{\Xi}_3(\overline{\mu}_3) = \Omega_2 \cup \Omega_4 \cup \Omega_6$ is adjacent to a vertex from $\overline{\Xi}_1(\overline{\sigma}) \cap \overline{\Xi}_1(\overline{\mu}_3) = \Omega_5$ or a vertex from $\overline{\Xi}_1(\overline{\sigma}) \cap \overline{\Xi}_2(\overline{\mu}_3) = \Omega_3$ (or both). ∎

Let Ψ be the image in $\overline{\Xi}$ of the subgraph Ψ in Ξ as in Lemma 4.4. Since none of the 2-point stabilizers of the action of $\overline{C}$ on $\overline{\Xi}$ involve $U_5(2)$, every vertex from the antipodal block containing σ maps onto $\overline{\sigma}$ and we have the following

LEMMA 5.2. *Let $\overline{\Psi}$ be the image of Ψ in $\overline{\Xi}$. Then $\overline{\Psi}$ is the antipodal folding of Ψ which is a strongly regular graph with parameters*

$$v = 672, \qquad k = 176, \qquad l = 495, \qquad \lambda = 40, \qquad \mu = 48.$$

The image $\overline{W}$ of $W = C_{M_\mu}(z)$ in $\overline{C}$ is isomorphic to $U_6(2) \cdot \mathrm{Sym}(3)$.

Since locally $\overline{\Psi}$ (as well as Ψ) is the commuting graph Θ of 3-central subgroups of order 3 in $U_5(2) \cdot 2$ which is strongly regular, it is easy to see that in terms of Lemma 5.1, $\overline{\Psi} \subseteq \{\overline{\sigma}\} \cup \overline{\Xi}_1(\overline{\sigma}) \cup \overline{\Xi}_2(\overline{\sigma})$.

Let $\varrho: \tilde{\mathcal{C}} \to \mathcal{C}^M$ be a covering of $\mathcal{C}^M$ such that there is a flat-transitive automorphism group of $\tilde{\mathcal{C}}$ which commutes with ϱ and whose induced action on $\mathcal{C}^M$ coincides with that of $C/\langle z\rangle$. In particular ϱ can be the restriction to a connected component of $\mathcal{C}^G$ of the morphism of $\mathcal{C}^G$ onto $\mathcal{C}^M$ induced by the homomorphism $\chi: G \to M$. In this case $\tilde{C}$ is the setwise stabilizer in $C_G(z)/\langle z\rangle$ of that connected component. Let $\tilde{R}$ be the kernel of the natural homomorphism of $\tilde{C}$ onto $\tilde{C} \sim C/R$. Let $\tilde{\overline{\Xi}}$ be the collinearity graph of $\tilde{\mathcal{C}}$ so that there is a natural morphism of $\tilde{\overline{\Xi}}$ onto $\overline{\Xi}$.

Let Ψ and W be as in Lemma 4.4. Let $\overline{\Psi}$ be the image of Ψ in $\overline{\Xi}$ and $\overline{W}$ be the image of W in $\overline{C}$. Let $\overline{\Psi}$ be a connected component of the preimage of Ψ under ϱ and let $\tilde{W}$ be the stabilizer of $\tilde{\Psi}$ in the preimage of $W/\langle z\rangle$ in $\tilde{C}$.

LEMMA 5.3. *In the above notation $\tilde{\Psi}$ is isomorphic to Ψ, $\tilde{W} \sim W/\langle z\rangle \sim$ $2^2 \cdot U_6(2) \cdot \mathrm{Sym}(3)$ and hence $\tilde{W} \cap \tilde{R}$ is elementary abelian of order 2^2.*

Proof. The result follows from Lemma 4.4 and the fact that Ψ is the collinearity graph of the geometry $\mathcal{U}$ which is simply connected by [Me]. ∎

Let $\overline{\mathcal{T}}(\overline{\sigma})$ be the set of images of $\overline{\Psi}$ under $\overline{C}$ which contain $\overline{\sigma}$. Equivalently we can define $\overline{\mathcal{T}}(\overline{\sigma})$ to be the set of images of $\overline{\Psi}$ under

$N_{\overline{C}}(\overline{\sigma})$. Let $\tilde{\sigma}$ be a preimage of $\overline{\sigma}$ in $\tilde{C}$. Let $\tilde{T}(\tilde{\sigma})$ be the set of connected subgraphs $\tilde{\Phi}$ such that $\tilde{\sigma} \in \tilde{\Phi}$ and $\tilde{\Phi}$ maps onto some $\overline{\Phi} \in \overline{T}(\overline{\sigma})$. If $\tilde{\Phi} \in \tilde{T}(\tilde{\sigma})$ and $\tilde{U} := \tilde{C}(\tilde{\Phi})$ is the setwise stabilizer of $\tilde{\Phi}$ in $\tilde{C}$, then by Lemma 5.3, $O_2(\tilde{U}) = \tilde{U} \cap \tilde{R}$ is of order 2^2. Let

$$\tilde{R}_\sigma = \left\langle O_2\left(\tilde{C}(\tilde{\Phi})\right) \mid \tilde{\Phi} \in \tilde{T}(\tilde{\sigma}) \right\rangle.$$

LEMMA 5.4. $\tilde{R}_\sigma = \tilde{R}$.

Proof. Let $\hat{\Xi}$ be the folding of $\tilde{\Xi}$ with respect to the orbits of $\tilde{R}_\sigma$. This means that the vertices of $\hat{\Xi}$ are the orbits of $\tilde{R}_\sigma$ on the vertex set of $\tilde{\Xi}$ with the induced adjacency relation. Notice that in the way it is defined $\hat{\Xi}$ is not necessary vertex-transitive although every automorphism from $\tilde{C}$ stabilizing $\tilde{\sigma}$ can be realized as an automorphism of $\hat{\Xi}$. Nevertheless eventually we will see that $\hat{\Xi}$ is equal to $\overline{\Xi}$ and in particular it is vertex-transitive. Since the vertices of $\overline{\Xi}$ can be considered as orbits of $\tilde{R}$ on $\tilde{\Xi}$ and $\tilde{R}_\sigma$ is contained in $\tilde{R}$, there is a covering $\omega: \hat{\Xi} \to \overline{\Xi}$ and $\tilde{R}_\sigma = \tilde{R}$ if and only if ω is an isomorphism. Let $\hat{\sigma}$ be the image of $\tilde{\sigma}$ in $\hat{\Xi}$. Since $\tilde{R}_\sigma$ is normalized by the stabilizer $\tilde{C}(\tilde{\sigma})$ of $\tilde{\sigma}$ in $\tilde{C}$, there is a subgroup $\hat{C}(\hat{\sigma})$ in the automorphism group of $\hat{\Xi}$ which stabilizes $\hat{\sigma}$ and maps isomorphically onto $\overline{C}(\overline{\sigma}) \sim 3 \cdot \mathrm{Suz} \cdot 2$. We will identify $\hat{C}(\hat{\sigma})$ and $\overline{C}(\overline{\sigma})$. For $\hat{\delta} \in \hat{\Xi}$ let $\hat{\Xi}_1(\hat{\delta})$ be the set of vertices adjacent to $\hat{\delta}$ in $\hat{\Xi}$. Since ω is a covering, the subgraph induced on $\hat{\Xi}_1(\hat{\delta})$ is isomorphic to Δ and if $\hat{\delta} = \hat{\sigma}$ then $\hat{C}(\hat{\sigma})$ induces the full automorphism group of this subgraph. Hence $\hat{C}(\hat{\sigma})$ has exactly three orbits, on the vertices at distance 2 from $\hat{\sigma}$. We denote these orbits by $\hat{\Xi}_i(\hat{\sigma})$, so that $\omega(\hat{\Xi}_i(\hat{\sigma})) = \overline{\Xi}_i(\overline{\sigma})$ for $2 \leq i \leq 4$. Let $\hat{\mu}_i \in \hat{\Xi}_i(\hat{\sigma})$ and $\hat{B}_i$ be the stabilizer of $\hat{\mu}_i$ in $\hat{C}(\hat{\sigma})$. We assume that there is a vertex $\hat{\mu}_1 \in \hat{\Xi}_1(\hat{\sigma})$, adjacent to $\hat{\mu}_i$ for $2 \leq i \leq 4$ and that $\hat{\mu}_3$ is adjacent to $\hat{\mu}_2$ and $\hat{\mu}_4$. Assuming also that $\omega(\hat{\mu}_i) = \overline{\mu}_i$, we can consider $\hat{B}_i$ as a subgroup in $\overline{B}_i$, $1 \leq i \leq 4$. Notice that $\hat{B}_i$ acts transitively on the set $\hat{\Xi}_1(\hat{\sigma}) \cap \hat{\Xi}_1(\hat{\mu}_i)$. Since ω is a covering, the subgraph induced by $\hat{\Xi}_1(\hat{\sigma}) \cap \hat{\Xi}_1(\hat{\mu}_2)$ is union of m disjoint copies of $K_{4,4,4}$ where $1 \leq m \leq 40$. For $\tilde{\Phi} \in \tilde{T}(\tilde{\sigma})$ the image $\hat{\Phi}$ of $\tilde{\Phi}$ in $\hat{\Xi}$ is isomorphic to $\overline{\Psi}$ as in Lemma 5.2 and is contained in $\{\hat{\sigma}\} \cup \hat{\Xi}_1(\hat{\sigma}) \cup \hat{\Xi}_2(\hat{\sigma})$. The parameters of $\overline{\Psi}$ imply that $m \geq 3$. Since $\overline{B}_2$ acts primitively on the 40 copies of $K_{4,4,4}$ as in Lemma 5.1(c) we have $m = 40$ and $\hat{B}_2 = \overline{B}_2$. By Lemma 5.1(c), $\hat{B}_2$ has three orbits on the vertices from $\hat{\Xi}_1(\hat{\mu}_2)$ with lengths 480, 5120, and 17,280, moreover, these orbits are contained in $\hat{\Xi}_i(\hat{\sigma})$ for $i = 1$, 2, and 3, respectively. In particular $\hat{B}_2 \cap \hat{B}_3$ has order divisible by 2^7. By Lemma 5.1(d) the stabilizer in B_3 of a vertex from $\hat{\Xi}_1(\hat{\sigma}) \cap \hat{\Xi}_1(\hat{\mu}_2)$ has order not divisible by 2^7 and so $\hat{B}_3 \cap \hat{B}_1$ is a maximal subgroup of $\overline{B}_3$ not containing $\hat{B}_2 \cap \hat{B}_3$. Thus $\hat{B}_3 = \overline{B}_3$. Arguing similarly $\hat{B}_3 \cap \hat{B}_4$ and $\hat{B}_1 \cap \hat{B}_4$ are two different maximal subgroups of $\overline{B}_4$ and so $\hat{B}_4 = \overline{B}_4$. Let $\hat{\rho}$ be a vertex adjacent to $\hat{\mu}_i$ for $i = 2$ or 4. By Lemma 5.1(c), $\hat{\rho}$ is conjugate under $\hat{C}(\hat{\sigma})$ to $\hat{\mu}_j$ for some $1 \leq j \leq 4$, except maybe in the case where $\hat{\rho}$ is adjacent to $\hat{\mu}_2$ and $\hat{\rho}$ maps onto an element of $\overline{\Xi}_2(\overline{\sigma})$. In the latter case we see from the distribution diagram of Δ that such a $\hat{\rho}$ can already be found in the residue of μ_1. Hence in any case a vertex adjacent to $\hat{\mu}_i$ for $i = 2$ or 4 is in $\hat{\Xi}_j(\hat{\sigma})$ for $1 \leq j \leq 4$. Suppose that there is a vertex $\hat{\nu}$ which is adjacent to $\hat{\mu}_3$ and whose distance from $\hat{\sigma}$ is 3. By Lemma 5.1(g) there must be a vertex in $\hat{\Xi}_1(\hat{\mu}_3) \cap \hat{\Xi}_j(\hat{\sigma})$ for $j = 1$, or 2 which is adjacent to $\hat{\nu}$. As we have seen above, this is impossible. Hence there are no vertices at distance 3 from $\hat{\sigma}$ and ω is an isomorphism.

COROLLARY 5.5. *$\overline{C}$ is the universal completion of the amalgam* $(\overline{C}_1, \overline{C}_2, \overline{C}_{12}, \overline{W})$.

6. CONSTRUCTION OF THE 2-LOCALS

As above let G denote the universal completion of the amalgam $(M_i \mid i = 1, 2, 3, 8)$ and χ be the homomorphism of G onto M which is identical on this amalgam. We will consider the M_i's as subgroups both in M and G. The group G acts flag-transitively on the universal cover G of M. The points, lines, planes, and quadrics in G and M are the cosets of M_1, M_2, M_3, and M_8 in G and M, respectively. We follow notation introduced in the beginning of Section 4, so that X stays for an arbitrary completion of the amalgam which has M as an quotient.

Let $\sigma = M_1$ viewed as a point stabilized by M_1, $d = M_8$ viewed as a quadric stabilized by M_8, z an involution from P_1^*, $C = C_M(z) \sim 2_+^{1+24} \cdot Co_1$, and $R = O_2(C)$. Our nearest goal is to construct in $C_G(z)$ a subgroup $\tilde{C}$ which maps isomorphically onto C. As above let Ξ be the graph on the set of points τ with $z \in P_\tau^*$ in which two points are adjacent if they are incident to a common quadric. We will obtain $\tilde{C}$ as the stabilizer in $C_G(z)$ of the connected component of Ξ containing σ. Let Ω be a maximal clique in Ξ containing σ, H be the setwise stabilizer of Ω in $C_X(z)$, and put $A = O_3(H)$. Then by Lemma 4.1, $H \sim \langle z \rangle \times 3^6 \cdot 2 \cdot \mathrm{Mat}_{12}$, moreover there is a unique point α collinear to σ, and inverted by z, such that $H = C_{M_\alpha}(z)$ and $O_3(H) = P_\alpha \cap H$. We use notation introduced in Lemma 4.2, so that A and B are orbits of $\overline{H} = H / \langle A, z \rangle$ on the set of subgroups of order 3 in A with lengths 12 and 132, respectively, while L is the unique orbit of length 12 of $\overline{H}$ on the set of hyperplanes of A. Then it is straightforward to identify A with the vertices in Ω.

Let $\{\sigma, \delta\}$ be the edge of Ω incident to d. Then $\langle \sigma, \delta \rangle = C_{Q_d}(z)$. Besides σ and δ there are two subgroups, say ρ and ρ' of order 3 in

$C_{Q_d}(z)$. These subgroups are of Fischer type, and lie in the orbit $\mathcal{B}$. Since $\rho \leq P_\alpha$ we can define M_ρ as in Lemma 2.2. Since $C_{M_d}(z) \sim 2.3^2 \cdot U_4(3) \cdot D_8$, we have $C_{M_d}(z) \cap M_\rho \sim 2.3^2 \cdot U_4(3) \cdot 2^2$. Moreover by Lemma 3.2, z is a 2-central involution in M_ρ and

$$C_{M_\rho}(z) \sim \left(3 \times 2_+^{1+12}\right) \cdot 3 \cdot U_4(3) \cdot 2^2.$$

Put $C_0 = C_{M_\rho}(z)$ and $R_0 = O_2(C_0)$. Recall the choice of μ and the definition of W before Lemma 4.4. In particular $\sigma, \delta \leq W$ and both σ and δ act non-trivially on $O_2(W)$. Thus one of ρ and ρ' centralizes $O_2(W)$. We choose notation so that ρ centralizes $O_2(W)$. Recall the definition of C_i, $i = 1, 2, 3, 12$ before Lemma 4.3, where we choose $\Omega_2 = \{\sigma, \delta\}$. So $C_1 = C_{M_\sigma}(z)$, $C_2 = C_{M_{\{\sigma,\delta\}}}(z)$, and $C_{12} = H = C_{M_\alpha}(z)$.

LEMMA 6.1. (a) $R = \prod_{L \in \mathcal{L}} C_R(L)$;
(b) $R_0 = \prod_{\rho \leq L \in \mathcal{L}} C_R(L)$.

Proof. The image in $C/R \cong Co_1$ of H is the full normalizer of the image of A which shows that $R_0 \leq R$ and $R_0 = C_R(\rho)$. Note that $[R/\langle z \rangle, A]$ is a non-trivial $GF(2)$-module for H of dimension at most 24. The restriction of this module to A is a direct sum of irreducible 2-dimensional modules and the kernel of such a summand is a hyperplane. The hyperplanes appearing as kernels form a union of orbits under $\overline{H}$. By Lemma 4.2 there are no orbits of length less than 12 and $\mathcal{L}$ is the only orbit of length 12. This implies (a). Since ρ acts fixed-point freely on R/R_0, we have (b). ∎

PROPOSITION 6.2. *C is the universal completion of the amalgam $(C_0, C_1, C_2, C_{12}, W)$ of subgroups of C.*

Proof. Let $\tilde{C}$ be the universal completion of the amalgam and as usual view the C_i and W has subgroups of $\tilde{C}$. By Lemma 4.4(b), $C_2 \cap W \sim 3^2 \cdot U_4(2) \cdot 2$ and so $C_2 \cap W$ normalizes no non-trivial 2-subgroup of $O^2(C_0/R_0)$. Thus $O_2(W) \leq R_0$.

Since $H \cap W \sim 3^{4+1} \cdot 2 \cdot \mathrm{Sym}(6)$ we conclude from Lemma 4.2 applied to the dual of A that $(H \cap W)A = N_H(A \cap W)$ and that there exists unique elements L_1 and L_2 in $\mathcal{L}$ with $L_1 \cap L_2 \leq A \cap W$. Let $U = \langle O_2(W)^A \rangle$. Then $U/\langle z \rangle$ is a subspace in $R_0/\langle z \rangle$ of dimension at least 4 centralized by $C_A(O_2(W))$. Thus by Lemma 6.1(b), $C_A(O_2(W))$ is the intersection of two members of $\mathcal{L}$. Hence $C_A(O_2(W)) = L_1 \cap L_2$, $U = C_{R_0}(L_1)C_{R_0}(L_2)$, $\rho \leq L_1 \cap L_2$, and $|U| = 2^5$.

Put $V = C_{R_0}(L_1)$. We conclude from Lemma 6.1(b) that $N_{H \cap C_0}(V) \sim 3^6 \cdot 2 \cdot \mathrm{Sym}(5)$. On the other hand $(H \cap W)^\infty$ is normal in $(H \cap W)A$ and so $(H \cap W)^\infty$ centralizes all conjugates of $O_2(W)$ under A. Thus

$(H \cap W)^{\infty} \leq N_H(V)$. It follows that $N_H(V) = \langle N_{H \cap C_0}(V), (H \cap W)^{\infty} \rangle \sim 3^6 \cdot 2 \cdot \mathrm{Mat}_{11}$. In particular, H acts doubly transitive on the 12 elements of V^H and since $VV^h \cong 2_+^{1+4}$ for $h \in H \cap C_0 \setminus N_H(V)$ we conclude that $\tilde{R} := \langle V^H \rangle \cong 2_+^{1+24}$.

We claim that $\tilde{R}$ is normal in $\tilde{C}$. By definition H normalizes $\tilde{R}$. Moreover, $R_0 = \langle V^{H \cap C_0} \rangle$. Let $t \in H \cap C_2 \setminus C_0$. As $C_0 \cap C_2$ is of index two in C_2, t normalizes $C_0 \cap C_2$. Also t permutes ρ and ρ' and we conclude that $\tilde{R} = R_0 R_0^t$ is normalized by $R_0, C_0 \cap C_2$, and t. Thus both $C_0 = R_0(C_0 \cap C_2)$ and $C_2 = (C_0 \cap C_2)\langle t \rangle$ normalize $\tilde{R}$. Since $C_1 = \langle C_1 \cap C_2, C_1 \cap H \rangle$, $\tilde{R}$ is indeed normal in $\tilde{C}$.

Note that $\tilde{C}/\tilde{R}$ is a completion of the amalgam

$$\left(C_1\tilde{R}/\tilde{R}, C_2\tilde{R}/\tilde{R}, C_{12}\tilde{R}/\tilde{R}, W\tilde{R}/\tilde{R}\right).$$

As $O_2(W) \leq \tilde{R}$, we can apply Corollary 5.5 and conclude that $\tilde{C}/\tilde{R} \cong \overline{C} \cong Co_1$. Thus $\tilde{C} \sim 2_+^{1+24} \cdot Co_1$ and since C is a quotient of $\tilde{C}$, we obtain $\tilde{C} \cong C$. ∎

In view of the preceding proposition our nearest goal is to find such an amalgam inside of G. The first part, namely finding the subgroups, is already accomplished. Indeed the groups $C_0, C_1, C_2, C_{12} = H$ and W had been defined for X, in particular for G and for M. It remains to show that the pairwise intersections are the same when regarded as subgroups of G and M, respectively. The fact that the pairwise intersections between C_1, C_2, H, and W are correct follows immediately from the definitions of these groups. Also $H \leq M_\alpha$ and $C_2 \leq M_d$. Since ρ is perpendicular to Q_α in Q_d we conclude from Lemma 2.2 that C_0 intersects C_2 and H correctly. Moreover, $N_{C_1}(\rho) \leq N_{M_\sigma}(\rho) \leq N_{M_\sigma}(\langle \sigma\delta \rangle) \leq M_d$ and so C_0 and C_1 intersect correctly. It remains to check the intersection $C_0 \cap W$. As $C_0 \leq M_\rho$ and $W \leq M_\mu$ this is accomplished by

LEMMA 6.3. $N_{M_\rho}(\mu) = M_\rho \cap M_\mu$.

Proof. Let $F = \rho\mu$. Then F is a non-degenerated 2-space of "plus"-type with respect to the M_d invariant quadratic form on Q_d. Hence $N_{M_d}(\rho, \mu) \sim 3^8 \cdot \Omega_6^-(3) \cdot 2$ and F/ρ is of type $3C$ in $M_\rho/\rho \cong M(24)$ (compare [At]). This shows that $N_{M_\rho}(\mu) = N_{M_d}(\rho, \mu) \leq N_{M_d}(\mu) \leq M_\mu$. ∎

COROLLARY 6.4. *Let $\tilde{C}$ be the subgroup of G generated by C_0, C_1, C_2, H, and W. Then $\tilde{C} \sim 2_+^{1+24} \cdot Co_1$ and $\tilde{C}$ is the normalizer of the connected component of Ξ containing σ.*

We now proceed finding the remaining terms E_4 and E_8 (cf. Section 3) of the 2-local geometry of M. Of the 3-local subgroups considered so far only the normalizers of Fischer type subgroups contain a conjugate of E_4. (This follows from the fact E_4 centralizes all subgroups of odd order in M which are normalized by E_4.) This is not enough to reconstruct N as a subgroup of G and we are forced to first locate a further 3-local subgroup of G containing E_4. By Lemma 4.2(g) there exists a 2-space F in A all of whose 1-spaces are in $\mathcal{B}$ and so of Fischer type. Moreover $N_H(F)/C_H(F) \cong GL_2(3)$ and there exists L_1, L_2 in $\mathcal{L}$ with $F \leq L_1 \cap L_2$. Choose F so that $\rho \leq F$ and let δ be a further Fischer type subgroup of F.

We are trying to locate subgroups of $N_G(F)$ and for this we will produce a quadric d' with $F \leq Q_{d'}$. Let z' be an involution in H so that $P_\alpha^* z = P_\alpha^* z'$, but $P_\alpha z \neq P_\alpha z'$. Then by Lemma 3.3(b), $z' = z^r$ for some $r \in M_\alpha$. Let $A' = A^r$ and $\Omega' = \Omega^r$. Since $\sigma_i = C_{A'}(L_i)$ has 12-conjugates under $H \cap H^r \sim 2^2 \cdot \mathrm{Mat}_{12}$, $\sigma_i \in \Omega'$. Thus $\{\sigma_1, \sigma_2\}$ is an edge in Ω' and there exists a unique quadric d' adjacent to α, σ_1, and σ_2. In $Q_{d'}$ we see that $Q_{d'} \cap P_\alpha = \sigma_1\sigma_2[Q_{d'} \cap P_\alpha, z']$ and $[Q_{d'} \cap P_\alpha, z']$ has order 3^5. As $[Q_\alpha, z'] = AQ_\alpha$ and $C_{AQ_\alpha}(\sigma_1\sigma_2)$ has order 3^5 we conclude that $C_{AQ_\alpha}(\sigma_1\sigma_2) = [Q_{d'} \cap P_\alpha, z']$. Hence $F \leq Q_{d'}$.

Since all 1-spaces in F are of Fischer type, F is a non-degenerate 2-space of "minus"-type in $Q_{d'}$ and $C_{M_{d'}}(F) \sim 3^8 \cdot \Omega_6^+(3)$. Since $C_{M_{d'}}(F) \leq M_\rho$ we conclude [At] that F/ρ is of type $3A$ in M_ρ/ρ, which means that $C_{M_\rho}(F) \sim 3^2 \cdot P\Omega_8^+(3)$. Let g be a point incident to d' such that Q_g is perpendicular to F in $Q_{d'}$. Then Q_g is centralized by a Sylow 3-subgroup of $C_{M_{d'}}(F)$. Hence $Q_g F/F$ is 3-central in $C_{M_{d'}}(F)/F$ and so also 3-central in $C_{M_\rho}(F)$. Thus $C_{M_\rho}(F) \cap N_{M_\rho}(Q_g)$ is a maximal subgroup of $C_{M_\rho}(F)$ different from $C_{M_{d'}}(F)$. Hence

$$C_{M_\rho}(F) = \left\langle C_{M_{d'}}(F), C_{M_\rho}(F) \cap N_{M_\rho}(Q_g)\right\rangle \leq \left\langle N_{M_{d'}}(\delta), N_{M_g}(\delta)\right\rangle \leq M_\delta.$$

Put $T = C_{M_\rho}(F)$. We conclude that $T = C_{M_\delta}(F)$ and so $N_H(F)$ normalizes T. Put $M_F = TN_H(F)$. Then $M_F \sim (3^2 \times P\Omega_8^+(3)) \cdot GL_2(3)$ and in particular, M_F maps isomorphically onto the full normalizer of F in M.

Note that $C_{M_F}(z) = N_H(F)C_T(z) \subseteq HC_{M_\rho}(z) \subseteq \tilde{C}$. As z centralizes F, $z \in O^3(T) = T' \cong P\Omega_8^+(3)$. As $N_H(F)$ induces the full group of outer automorphisms on T' and by [At], T' has a unique class of involutions invariant under all automorphisms, z is 2-central in T'. In particular, there exists a pure Conway foursgroup E in T' with $z \in E \leq O_2(C_{T'}(z)) \leq O_2(C_{M_\rho}(z)) = R_0 \leq \tilde{R}$. Let t be an involution in E distinct from z. Then $t = z^g$ for some $g \in T' \leq M_\rho \cap M_F$. Put $\tilde{C}_t = \tilde{C}^g$. Then by conjugation of the corresponding statements for z we get $C_{M_\rho}(t) \leq \tilde{C}_t$ and $C_{M_F}(t) \leq \tilde{C}_t$.

LEMMA 6.5. $C_{\tilde{C}}(E) \leq \tilde{C}_t$.

Proof. Put $C_E = C_{\tilde{C}}(E)$. Then $C_E \sim 2^{2+11+22} \cdot \mathrm{Mat}_{24}$. Moreover $C_E \cap M_\rho = N_{C_E}(\rho)$ and so modulo $O_2(C_E), C_E \cap M_\rho$ has shape $3 \cdot \mathrm{Sym}(6)$. Similarly modulo $O_2(C_E)$ the intersection $C_E \cap M_F$ is of shape $3^2 \cdot GL_2(3)$. By [At] no proper subgroup of Mat_{24} has two such subgroups and thus $C_E = \langle C_E \cap M_\rho, C_E \cap M_F \rangle O_2(C_E)$. Since ρ has fixed points on any composition factor for C_E on $O_2(C_E)$ this implies

$$C_E = \langle C_E \cap M_\rho, C_E \cap M_F \rangle \leq \tilde{C}_t. \quad \blacksquare$$

Let E_8 be a pure Conway type eight subgroup of T' such that $E_8 \leq O_2(C_{T'}(x))$ for all $1 \neq x \in E_8$ and $E \leq E_8$. Put $E_4 = E$ and for $i = 4, 8$ put $C_{E_i} = \bigcap_{1 \neq x \in E_i} \tilde{C}_x$. Then by Lemma 6.5, $C_{E_i} = C_{\overline{C}}(E_i)$. Moreover $N_T(E_i)$ normalizes C_{E_i} and induces on E_i its full automorphism group. Put $\tilde{N} = C_{E_4} N_T(E_4)$ and $\tilde{L} = C_{E_8} N_T(E_8)$. Then χ maps the amalgam $(\tilde{C}, \tilde{N}, \tilde{L})$ isomorphically onto the amalgam (C, N, L) as in Section 3. Let $\tilde{M}$ be the group generated by $\tilde{C}$, $\tilde{N}$, and $\tilde{L}$. Then by Lemma 3.1, χ maps $\tilde{M}$ isomorphically onto M. Thus to complete the proof of Lemma 1.6 it remains to show that $G = M$. For this note first that M_ρ is generated by its intersection with $\tilde{C}$ and $\tilde{N}$. Moreover, M_1 and M_8 are both generated by their intersections with M_ρ and $\tilde{C}$. Finally M_1 and M_8 generate G and so $G = \overline{M}$ and Lemma 1.6 is proved.

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