2-signalizers in almost simple groups

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This paper offers an exhaustive study of certain 2-signalizers in known non-sporadic finite simple groups. The main result of this paper is relevant to the Generation-2 proof of the Classification of Finite Simple Groups.

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1. Introduction

Let $q$ be a prime. The term $q$-signalizer was originally introduced by John Thompson to denote a $q'$-subgroup of a group $G$ normalized by a Sylow $q$-subgroup of $G$. With time the term came to mean a $q'$-subgroup of $G$ normalized by a “large” $q$-subgroup of $G$, where “large” has various meanings. For example, the $q$-subgroup in question contains an elementary abelian subgroup of order at least $q^3$, or contains the full centralizer of some $q$-element, etc. Control of the size and embedding of such signalizers has been a central theme of the Classification of Finite Simple Groups (CFSG). For a complete detailed discussion of the topic, e.g., the use of various $q$-signalizers, we refer a reader to a recent article on the subject by R. Solomon [S].

The usual goal of $q$-signalizer analysis is to obtain information about the $q$-local structure and then the general structure of $G$. It turns out that specific cases $q \in \{2, 3\}$ are especially important for the proof of CFSG: both the original proof and also the Second Generation proof of the Classification, a project of D. Gorenstein, R. Lyons and R. Solomon in the AMS Monographs Series, cf. [GLS1]. (We will refer to it as the GLS-project.) In the terminology of the GLS-project, the case $q = 2$ is used in the Special Odd Case, and the case $q = 3$ is to be used in the Even Case of the GLS-series. The exhaustive study of certain $3$-signalizers which will be used in the GLS-project has been done in [K]. In this paper we offer a corresponding study of certain 2-signalizers in finite simple groups. We begin by introducing the kind of 2-signalizers studied in this article.
Definition 1.1. Let $G$ be a finite group and $t \in \text{Aut}(G)$ of order 2. Take $p$ to be an odd prime. Let $X$ be a $p$-subgroup of $G$ such that

1. $[X, t] = X$; and
2. $C_G(t) \leq N_G(X)$.

Then $X$ is called a $C_G(t)$-signalizer.

The main result of this paper is a description of $C_G(t)$-signalizers in the known non-sporadic finite simple groups, i.e., non-sporadic $K$-groups in the GLS-terminology. The sporadic case can be done by careful inspection of the information in [ATLAS]. We believe that there are no examples of $C_G(t)$-signalizers with $G$ sporadic.

Theorem 1.2. Let $G$ be a finite simple non-sporadic $K$-group. Suppose that $G$ admits a non-trivial involutory automorphism $t$ and $X$ is a $C_G(t)$-signalizer. Then either $X = 1$, or one of the following conclusions holds:

1. $G \in \text{Lie}(p)$, $p \neq 2$, and there exists a maximal parabolic subgroup $P$ of $\text{Inndiag}(G)$ such that $X$ is contained in the abelian unipotent radical of $P$, and $t$ is contained in the center of a Levi complement of $P$;
2. $G \cong L_{2n+1}(q)$ with $q = 2^a$, $t$ is a graph automorphism of $G$ and $X \leq \text{Inndiag}(G)$ is isomorphic to a non-trivial subgroup of $\mathbb{Z}_{q-1}$;
3. $G \cong U_{2n+1}(q)$ with $q = 2^a$, $t$ is a graph automorphism of $G$ and $X \leq \text{Inndiag}(G)$ is isomorphic to a non-trivial subgroup of $\mathbb{Z}_{q+1}$.

The proof of this theorem is achieved via careful investigation of the known non-sporadic finite simple groups. The main subdivision is between alternating groups, groups of Lie type in odd characteristic, and groups of Lie type in characteristic 2. We assume the reader's familiarity with the basic theory of finite groups of Lie type. Throughout the paper, we will use the terminology and notation of the third volume of the GLS-series [GLS3].

The alternating group case is, as reader will see, easy to deal with and does not require deep knowledge of the alternating simple groups. The situation is dramatically different in the remaining cases. The crucial result we use to deal with the Lie-type groups in odd characteristic is the structure theorem for the centralizers of the involutory automorphisms in simple groups of Lie type defined in odd characteristic. (The complete list can be found in Chapter 4 of [GLS3], cf. Theorems 4.5.1, 4.5.2 of [GLS3].) The result that turns out to be important for the Lie-type groups in characteristic 2 is a paper of Aschbacher and Seitz [AS], which provides us with the full list of the conjugacy classes and centralizers of involutory automorphisms of the groups of Lie type over a field of order $2^a$, $a \geq 1$.

For expository purposes we shall subdivide the proof of the main result into three separate sections, each of which deals with a particular case: alternating (Section 2), odd Lie type (Section 3) and even Lie type (Section 4).

We now fix the notation.

Definition 1.3. For $G$ a group, $t$ an automorphism of $G$ order 2, $C := C_G(t)$, and $X$ a $C_G(t)$-signalizer, we say that $(G, t, X)$ is a $C$-signalizer triple.

And now, on with the proof.

2. Alternating groups

Proposition 2.1. Let $G = A_n$ with $n \geq 5$, $t \in \text{Aut}(G)$ and $X \leq G$ be such that $(G, t, X)$ is a $C$-signalizer triple. Then $X = 1$.

Proof. The proof of this proposition is by induction on $n$. If $G = A_5$ and $t$ is a transposition, then by inspection, $C \cong S_3$ is a maximal subgroup of $G$. If $G = A_6$ and $G(t) \cong \text{PGL}(2, 9)$, then $C \cong D_{10}$ does not normalize any non-trivial 3-subgroup of $G$. Hence $X = 1$ in both of these cases.
Since \( \text{Aut}(G) = S_6 \) for \( n \neq 6 \), and since every involution of \( \text{Aut}(A_6) \) lies either in \( \text{PGL}(2,9) \) or \( S_6 \), without loss of generality we may suppose that \( t = (12) \ldots \) is the product of \( l \) consecutive disjoint transpositions for some \( l \geq 1 \), with \( l = 2 \) if \( n = 5 \). Consider \( Z \cong E_4 \) where \( Z = \langle (34)(56), (35)(46) \rangle \) if \( t = (12) \), while \( Z = \langle (12)(34), (13)(24) \rangle \) otherwise. Note that \( [t, Z] = 1 \) in all cases. Since \( Z \leq C \), \( Z \) must act on \( X \), and as \( (2, p) = 1 \), \( X = \langle X_u = C_X(u) \mid u \in Z^\# \rangle \) (cf. [GLS2, 11.13]).

Suppose first that \( G = A_7 \). We may assume that \( X = O^2(C_G(u)) \) for all \( 1 \neq u \in Z \), with \( |X| = 3 \). If \( t = (12) \), then \( X = \langle (127) \rangle \) is not invariant under \( C \cong S_5 \). If \( t \in Z \), then \( [X, t] = 1 \), contrary to assumption. Finally, suppose that \( t = (12)(34)(56) \) and \( X = \langle (567) \rangle \). Then \( r = (135)(246) \in C \) and \( X \neq X^r \), a contradiction.

Suppose next that \( n \neq 7 \). Then, for all \( u \in Z^\# \), either \( C_G(u) \) is a 2-group (with \( n \leq 6 \)) or \( O^2(C_G(u)) = M, M \cong A_{n-4} \), independent of \( u \). Since \( X_u \) is a p-group with \( p \) odd, we must have \( n \geq 8 \) and \( X_u \leq M \) for all \( u \), implying \( X \leq M \). As \( [t, Z] = 1 \), \( M \) is \( t \)-invariant, and \( X \) still equals \( [X, t] \).

In particular, if \( X \neq 1 \), then \( t \) induces a non-trivial involutory automorphism on \( M \). If \( M \cong A_4 \) and \( X \neq 1 \), then \( |X| = 3 \) and \( 1 \neq C_{O^2(M)}(t) \leq N_M(X) \), contrary to the fact that \( N_M(X) = X \). Hence, we may assume that \( M \cong A_{n-4} \) with \( n - 4 \geq 5 \). As \( t \) induces a non-trivial involutory automorphism on \( M \), we conclude by induction that \( X = 1 \), completing the proof. \( \Box \)

3. \( \text{Lie}(r), r \) odd

We will now investigate the case in which \( G \) is a group of Lie type in odd characteristic. Let \( (G, t, X) \) be a \( C \)-signalizer triple, where \( G \) is a simple group with \( G \in \text{Lie}(r) \), \( r \neq 2 \). Clearly, there are two possibilities to consider:

\[ p = r \quad \text{and} \quad p \neq r. \]

First we dispose of the case when \( p = r \).

**Proposition 3.1.** If \((G, t, X)\) is a \( C \)-signalizer triple, where \( G \) is a simple group with \( G \in \text{Lie}(p) \), then \( t \) induces an inner-diagonal automorphism of \( G \), and there exists a maximal parabolic subgroup \( P \) of \( G(t) \) with abelian unipotent radical \( U \), such that \( X \) is contained in \( U \) and \( t \) is contained in the center of a Levi complement \( L_P \) of \( P \).

**Proof.** By the Borel–Tits Theorem (cf. [GLS3, 3.1.3]) and its Corollary [GLS3, 3.1.4], there exists a parabolic subgroup \( P_0 \) of \( G \) such that \( X \leq O_2(P_0) \) and \( N_G(X) \leq P_0 \). Hence, \( C \leq N_G(X) \leq P_0 \). If \( t \) is in the coset of a field or graph-field automorphism, then by [GLS3, 4.9.1] we may assume that \( t \) is a field or graph-field automorphism. However, then, by [BGL], \( C \) does not embed in a parabolic subgroup of \( G \). Hence \( t \) is an inner-diagonal automorphism or graph automorphism. By inspection of Table 4.5.1 of [GLS3], we see that \( t \) must induce an inner-diagonal automorphism of \( G \) of parabolic type. Indeed \( P := P_0(t) \) is a maximal parabolic subgroup of \( G(t) \). Moreover \( O^p(C) \) embeds in some Levi complement \( L_P \) of \( P \). In particular, \( O_2(C) = 1 \), whence \( t \) inverts \( U := O_2(P) \). As \( L_P \) acts faithfully on \( U \), it follows that \( t \) is in the center of \( L_P \), as claimed. \( \Box \)

**Remark 3.2.** For the remainder of this section, \((G, t, X)\) is a \( C \)-signalizer triple with \( G \) a simple group in \( \text{Lie}(r) \), \( p \neq r \). We set \( L = O^r(C) \).

Also, we exclude from consideration \( G \cong G_2(3)' \), as in this case \( G \cong L_2(8) \) and this situation will be dealt with in the next section.

**Lemma 3.3.** If \( x \in X \) and \( y \in C \) are such that \( y^r = 1 \neq y \) and \([x, t] \neq 1\), then \([x, y] \neq 1\).

**Proof.** Assume the contrary. Then there exist a \( p \)-element \( x \in X - C \), and \( y \in C \) of order \( r \) such that \([x, y] = 1\). Then \([x, t] \leq C_{G(t)}(y)\). Since \( t \) normalizes \( X \), \([x, t] := X_0(t) \) where \( X_0 \leq X \) and \([X_0, y] = 1\). As \( y \) is a non-trivial \( r \)-element of \( G \), \( R := F^r(C_G(y)) \) is an \( r \)-group by the Borel–Tits Theorem. Hence,
$X_0$ acts faithfully on $R$, and so on $R/\Phi(R)$ [GLS2, 11.1]. Set $\tilde{R} := R/\Phi(R)$. By 11.12 of [GLS2], $\tilde{R} = W \oplus C_{\tilde{R}}(X_0)$ with $W = [\tilde{R}, X_0]$ a faithful $X_0$-module.

Now, $t$ acts on $W$. By [GLS2, 11.3], the pre-image of $C_W(t)$ in $R$ is contained in $C_R(t)$. As $[C_R(t), X_0] \leq R \cap X = 1$, $X_0$ must act trivially on $C_W(t)$, whence $C_W(t) \leq W \cap C_R(X_0) = 1$. Therefore, $C_W(t) = 1$, and so $t$ acts as $-1$ on $W$. Since $X_0$ acts faithfully on $W$, it follows that $[t, X_0] = 1$, which contradicts the fact that $[x, t] \neq 1$. □

**Corollary 3.4.** If $m_r(C) \geq 2$, then $X = 1$.

**Proof.** Suppose that $m_r(C) \geq 2$. Then there exists $R \leq C, R \cong E_2$, which acts on $X$. Hence, by [GLS3, 11.13],

$$X = \langle C_X(y) \mid y \in R^\# \rangle.$$

Now Lemma 3.3 implies that for every $y \in R^\#$, $C_X(y) \leq C_X(t)$. Thus $X \leq C$, i.e., $[X, t] = 1$, which is a contradiction unless $X = 1$. □

Using the notation from Chapter 4 of [GLS3] for class of involutions, we define the following set of pairs $(G, t)$:

$$\mathcal{A} = \{ (L_2(r^a), t), (L_3^\pm(r), t_1), (L_3^\pm(r), \gamma_1), (PSp_4(r), t_2), (PSp_4(r), t'_2) \}.$$

**Lemma 3.5.** If $(G, t) \notin \mathcal{A}$, then $X = 1$.

**Proof.** If $t$ is either inner-diagonal, or a graph automorphism of $G$, Table 4.5.1 of [GLS3] gives a complete list for the possible isomorphism type of $L := O^+(C_G(t))$. Otherwise $t$ is either a field or a graph-field automorphism of $G$, in which situation, Proposition 4.9.1 of [GLS3] provides us with the possible structure of $L$. In all the cases, $L \in \text{Lie}(r)$, and now [GLS3, 3.3.3] allows us to evaluate $m_r(L)$. By an explicit calculation, if $(G, t) \notin \mathcal{A}$, $m_r(L) \geq 2$. Now Corollary 3.5 finishes the proof. □

It now remains to look at $(G, t) \in \mathcal{A}$.

**Lemma 3.6.** If $G \cong L_2(r^a), a \geq 1$, then $X = 1$.

**Proof.** As $L_2(5) \cong A_5$ and $L_2(9) \cong A_6$, we may suppose that $r^a \notin \{5, 9\}$. Suppose that $X \neq 1$. As $p \neq r$, both $X$ and $C_G(X)$ are cyclic groups. As $X$ is a cyclic $p$-group, $\text{Aut}(X)$ is also cyclic. If $t$ is a field automorphism of $G$, by Proposition 4.9.1 of [GLS3], $L \cong L_2(r^2)$. As $C_G(X)$ is cyclic, $L$ acts faithfully on $X$, which is a contradiction. Thus $t \in \text{Inddiag}(G)$. If $r^a > 11$, then $C$ is a maximal (dihedral) subgroup of $G$, whence $C = XC$ and $X = 1$. Finally, suppose that $r^a \in \{7, 11\}$. If $C < M < G$, then $M \cong S_4$ or $A_5$, and so $M \neq XC$. Thus, in all cases, $X = 1$, as claimed. □

**Lemma 3.7.** If $(G, t) \in \{(L_3^+(r), t_1), (L_3^-(r), \gamma_1), (PSp_4(r), t_2), (PSp_4(r), t'_2)\}$, then $X = 1$.

**Proof.** Suppose that $X \neq 1$. Using Table 4.5.1 of [GLS3] we obtain that in all these cases $L \cong L_2(r)$. Now Lemma 3.3 implies the existence of $y \in L$ of order $r$ with $[X, y] \neq 1$. Thus $[L, X] \neq 1$. Suppose that $r \neq 3$. Then $L$ is simple, whence $L$ acts faithfully on $X$, implying $m_r(X) \geq 2$. On the other hand $m_p(G) \leq 2$, as $G$ is a group of Lie rank 2. Therefore, $L$ embeds into $SL_2(p)$, and so $r \mid (p^2 - 1)$. But $m_p(G) = 2$, whence, $p \mid (r^2 - 1)$. As both $p$ and $r$ are odd primes, a numerical contradiction follows immediately. Hence we may assume that $G \cong L_3(3), U_3(3), \text{ or } PSp_4(3)$. Then $|X| = 13, 7, \text{ or } 5$, respectively, and $X$ is not $C$-invariant, a final contradiction. □

We may now combine the results of Lemmas 3.5, 3.6 and 3.7, to obtain the following result.
Lemma 4.3. Proof. As \( \text{Lemma 4.1. Preliminary lemmas and reductions} \)

Lemma 4.2. Hence \( \text{Lemma 4.2. Preliminary lemmas and reductions} \)

Proposition 3.8. Let \((G, t, X)\) be a \(C\)-signalizer triple, where \(G\) is a simple group with \(G \in \text{Lie}(r)\) with \(r \neq p\). Then \(X = 1\).

4. \(\text{Lie}(2)\)

The purpose of this last section is a proof of the following result.

Proposition 4.1. Let \((G, t, X)\) be a \(C\)-signalizer triple, where \(G\) is a simple group with \(G \in \text{Lie}(2)\). Then either \(X = 1\), or one of the following holds:

1. \(G \cong L_{2n+1}(q)\) with \(q = 2^a\), \(t\) is a graph automorphism of \(G\) and \(X \leq \text{Inndiag}(G)\) is isomorphic to a non-trivial subgroup of \(\mathbb{Z}_{q-1}\);
2. \(G \cong U_{2n+1}(q)\) with \(q = 2^a\), \(t\) is a graph automorphism of \(G\) and \(X \leq \text{Inndiag}(G)\) is isomorphic to a non-trivial subgroup of \(\mathbb{Z}_{q+1}\).

Combined with Propositions 2.1, 3.1 and 3.8, this will finish the proof of Theorem 1.2. For the remainder of the proof let us assume the following hypotheses:

- \((G, t, X)\) is a \(C\)-signalizer triple.
- \(G\) is a simple non-abelian group with \(G \in \text{Lie}(2) - \{A_1(4), A_2(2), B_2(2)', G_2(2)\}'\).

Note that we may exclude \(A_1(4), A_2(2), B_2(2)'\) and \(G_2(2)'\) as we already studied them in the previous sections \((A_1(4) \cong A_5, A_2(2) \cong L_2(7), B_2(2)' \cong A_6, G_2(2)' \cong U_3(3))\).

4.1. Preliminary lemmas and reductions

First we treat an easy case.

Lemma 4.2. If \(t\) is a field or a graph-field automorphism of \(G\), \(X = 1\).

Proof. By Theorem 1 of [BGL], \(C\) is a maximal subgroup of \(G\), except perhaps if \(C \cong L_2(2)\) or \(Sz(2)\). However, the latter occurs only if \(G \cong L_2(4)\), an excluded case. Then as \(C \leq XC \leq G\), it follows that \(X = 1\). \(\Box\)

Thus we may assume henceforth that either \(t \in G\) or \(t\) induces an involutory graph automorphism on \(G\). Fix a \(t\)-invariant Sylow 2-subgroup \(S\) of \(G\) and let \(\bar{S} = S(t)\). Now \(Z(S) = Z(\bar{S})\) is the center of a long root subgroup of \(G\).

Let us investigate the structure of \(X\).

Lemma 4.3. The following conditions hold:

1. \(C_X(t) = 1\), i.e., \(t\) inverts \(X\); and
2. \(X\) is abelian.

Proof. As \(X\) is \(C\)-invariant, \(C_X(t) \leq O_2^+(C)\). Since \(t\) is either an inner or a graph automorphism of \(G\), it follows by [AS, 2.4] and Theorem 4.9.2 of [GLS3] that \(F^+(C)\) is either a 2-group, or a simple group. Hence \(C_X(t) = 1\), whence \(X\) is abelian, as claimed. \(\Box\)

We assume henceforth that \(X \neq 1\). For \(s\) an involution in \(C_{G(t)}(t)\), we let \(X_s := C_X(s)\).

Lemma 4.4. For all \(z \in Z(S)\), \([z, X] = 1\). In particular, \(t \notin Z(S)\).
Proof. If \( t \notin G \), then as \( t \) is a graph automorphism and \( G \not\cong I_3(2) \), it follows that \( m_2(C) > 1 \). As noted before, \( Z(S) = Z(\tilde{S}) \). Thus, in all cases, we can choose \( V = (z, s) \subseteq C_5(t) \) with \( \tilde{V} \cong E_4 \). By [AS, 2.4], \( F^*(C_C(v)) \) is a 2-group for all \( v \in V^\# \). It follows that \( z \in O_2(C_C(v)) \), whence \( [X_v, z] \leq X \cap O_2(C_C(v)) = 1 \). Hence \( [z, X] = 1 \), as claimed. \( \square \)

Now let \( P \) be any parabolic overgroup of \( N_G(Z(S)) \) such that \( P = P^t \). We set \( U := O_2(P) = F^*(P) \).

An easy consequence of the preceding lemmas is the following result.

Corollary 4.5. The following conditions hold:

1. \( X \leq P \);
2. \( t \notin U \);
3. \( U \) centralizes every \( (X, t) \)-invariant abelian subgroup of \( U \);
4. \( U \) is non-abelian; and
5. if \( t \in G \), then \( G \) has more than one class of involutions.

Proof. Conclusions (1) and (5) follow immediately from Lemma 4.4. If \( t \in U \), then \( \{t, X\} \leq U \cap X = 1 \), a contradiction, proving (2). As \( U = F^*(P) \), \( [U, X] \neq 1 \). Hence (4) will follow from (3).

Finally, suppose that \( A \) is an \( (X, t) \)-invariant abelian subgroup of \( U \). Then \( A = C_A(X) \times [A, X] \) by [GLS2, 11.3]. As \( t \) normalizes \( A \) and \( t \) inverts \( X \), \( t \) acts on \( [A, X] \). Now, \( [C_{[A, X]}(t), X] \leq A \cap X = 1 \), whence

\[
C_{[A, X]}(t) \leq [A, X] \cap C_A(X) = 1.
\]

But \( [A, X] \) is a 2-group, whence, \( [A, X] = 1 \), proving (3). \( \square \)

We need one more crucial lemma before we actually deal with the proof.

Lemma 4.6. Suppose that \( U \) has nilpotence class 2. Set \( Z := Z(U) \) with \( |Z| = q \). Set \( \mathcal{U} = U/Z \) and \( W = [\mathcal{U}, X] \). Then \( |C_W(t)| = q \) and \( |W| = q^2 \). Moreover, for all \( z \in Z \), \( tz \in t^G \).

Proof. By the previous lemma, \( X \leq P \). Clearly, \( W \neq 1 \). Now, \( W \) can be thought of as a vector space over \( \mathbb{F}_q \) (cf. [GLS3, 3.2.4]). Moreover, as \( W \) is \( t \)-invariant and \( t \) inverts \( X \), \( t \) acts freely on \( W \). Thus \( \dim W = 2 \dim C_W(t) \).

Let \( Y \) be the full pre-image of \( C_W(t) \) in \( U \). For all \( y \in Y \) define \( \phi : Y \to Z \) by

\[
\phi(y) := [y, t] = y^{-1} y^t.
\]

Taking any elements \( y_1, y_2 \in Y \), we have

\[
\phi(y_1 y_2) = [y_1 y_2, t] = [y_1, t]^{y_2} [y_2, t] = [y_1, t][y_2, t] = \phi(y_1) \phi(y_2).
\]

i.e., \( \phi \) is a homomorphism.

If \( \phi(y) = 1 \), then \( y \in C \leq N_C(X) \), and so \( [y, X] \leq U \cap X = 1 \), i.e., \( y \in W \cap C_D(X) = 1 \). Hence, \( y \in Z \). Therefore, \( \ker(\phi) = Z \). Since \( Y \neq Z \) and \( C_W(t) \) is an \( \mathbb{F}_q \)-space, it follows that \( |C_W(t)| = q \) and \( |W| = q^2 \). Finally, since \( |C_W(t)| = q \), \( |Y| = q^2 \) and \( \phi \) is surjective. Therefore for all \( z \in Z \) there exists \( y \in Y \) with \( \phi(y) = z \). Hence, \( y^{-1} y^t = z \), and so \( t^x = zt = tz \), completing the proof. \( \square \)
4.2. Proof of the main theorem

First, we can now quickly handle the case when \( t \not\in G \).

Lemma 4.7. Suppose that \( t \not\in G \). Then \( G \cong A_n^\pm(q), n = 2l \) for some \( l \), and \( t \) acts as a graph automorphism. Moreover

\[
X \cong \begin{bmatrix} a1_{2l} & 0 \\ 0 & a^{-2l} \end{bmatrix},
\]

where \( a \in \mathbb{F}_{q^2}^\times \) and \( p \mid (q + (-1)^\epsilon) \), with \( \epsilon = 1 \) if \( G \cong A_n(q) = L_{n+1}(q) \), and \( \epsilon = 2 \) if \( G \cong A_n^-(q) = U_{n+1}(q) \).

Proof. By Theorem 4.9.2 of [GLS3], if \( G \not\cong A_n^\pm(q) \) with \( n \) even, then \( C \cong C_C(tz) \) for \( z \in Z(S) \), contrary to the preceding lemma. Hence, \( G \cong A_n^\pm(q) \), with \( n = 2l \) for some \( l \). Again, by Theorem 4.9.2 of [GLS3], \( G \) has a unique conjugacy class of graph automorphisms \( t^G \), and \( C \cong C_t(q) \). Since \( G \not\cong L_3(2) \) or \( U_3(2) \), \( C \) is simple. As there exists a non-trivial element of \( C \) centralizing \( X \) by Lemma 4.4, \([X, C] = 1\). Now \( C \) acts as the orthogonal group \( O_{n+1}(q) \) on the natural module \( V \) for \( G \), with 1-dimensional radical \( R \). Also \( C \) acts irreducibly on \( V/R \). It now follows easily that \( C_C(C) \) acts as scalars on \( V/R \), from which we easily obtain the desired conclusion. \( \dashv \)

Thus, henceforth, we may assume that \( t \in G \). Since \( t \in G \), all the statements of Corollary 4.5 hold. Moreover, we may choose \( P \) to be a maximal parabolic overgroup of \( N_C(Z(S)) \). We set \( O^2(P) = UM \), where \( U \) is the unipotent radical of \( P \) and \( M = O^2(L_P) \), where \( L_P \) is a Levi complement in \( P \). We tabulate the following information.

Lemma 4.8. If \( q = 2^a, a \geq 1 \), then the following table is correct.

| \( G \) | \( M \) | \( Cl(U) \) | \( |Z(U)| \) |
|---|---|---|---|
| \( A_1(q) \) | 1 | 1 | \( q \) |
| \( A_n(q), n \geq 2 \) | \( A_n-1(q) \) | 1 | \( |U| \) |
| \( A_n^\pm(q), n \geq 2 \) | \( A_n-2(q) \) | 2 | \( q \) |
| \( B_2(q) \) | 1 | 2 | \( q \) |
| \( C_2(q), n \geq 2 \) | \( C_n(q) \) | 2 | \( q \) |
| \( D_n^\pm(q), n \geq 4 \) | \( A_1(q) \times D_n^\pm(q) \) | 2 | \( q^\prime \) |
| \( D_n(q) \) | \( A_1(q^\prime) \) | 2 | \( q^\prime \) |
| \( G_2(q) \) | \( A_1(q) \) | 2 | \( q \) |
| \( F_4(q) \) | \( D_4(q) \) | 2 | \( q \) |
| \( 2F_4(q) \) | \( E_6(q) \) | 2 | \( q \) |
| \( 2F_4^*(q) \) | \( E_7(q) \) | 2 | \( q \) |
| \( 2F_4^*(q) \) | \( E_8(q) \) | 2 | \( q \) |

Proof. The result follows immediately by [GLS3, 3.2.6, 3.3.1] and direct computations. Otherwise, a reader can obtain this result using [AS]. \( \dashv \)

Invoking Lemma 4.8 and Corollary 4.5, we obtain the following result.

Lemma 4.9. If \( G \in \{L_n(q), n \geq 1, Sp_{2n}(q), F_4(q), Sz(q), 2F_4(q)\} \), then \( X = 1 \).

Proof. If \( G \cong L_n(q) \) or \( Sp_{2n}(q) \), then by Lemma 4.8, \( U \) is abelian, a contradiction. If \( G \cong F_4(q) \), then \( V := [Z(U), M] \) is the natural 6-dimensional module for \( M \cong Sp(6, q) \) [GLS3, 3.5.4]. Thus \( V \) is a faithful \( X(t) \)-module, contrary to Corollary 4.5(3).
If $G \cong S_z(q)$, then $t \in U$. If $G \cong 2^F_4(q)$, then by [AS, 18.6], $G$ has two conjugacy classes of involutions both of which have representatives in $U$, whence again $t \in U$. This contradicts Corollary 4.5(2).

Using Lemma 4.6 we can handle the unitary case.

**Lemma 4.10.** If $G = U_n(q)$ with $n \geq 3$, then $X = 1$.

**Proof.** By [GLS3, 3.2.5], $\overline{U} = U/Z(U)$ is an $F_q[P]$-module. Hence, $W = [X, \overline{U}]$ is an $F_q[X(t)]$-module. In particular, $|C_W(t)| \geq q^2$, which contradicts Lemma 4.6. □

We can now handle the remaining families by induction.

**Lemma 4.11.** If $G \in \{3D_4(q), G_2(q), D_{\pm n}(q), E_{\pm 6}(q), E_7(q), E_8(q)\}$, then $X = 1$.

**Proof.** By [AS, 12.4], we may assume that $t \in M \times Z$. Then using Lemma 4.6, we can place $t \in M$. As $X = [X, t]$, we may likewise assume that $X \subseteq M$. Hence, if $M$ is simple, then $(M, t, X)$ forms a $C_M(t)$-triple, and we are done by induction. In the $D_{\pm n}$ case, we have $M = M_1 \times M_2$ with $M_i$ simple, or else $G = D_4(q)$ and $M = M_1 \times M_2 \times M_3$ with $M_i \cong L_2(q)$ for all $i$. Then set $t = t_1t_2$ or $t = t_1t_2t_3$, with $t_i \in M_i$. If $X \neq 1$, then for some $i$, $M_i \geq X_i := [X, t_i] \neq 1$. For such a choice of $i$, $(M_i, t_i, X_i)$ forms a $C_{M_i}(t_i)$-triple, and again we are done by induction. □

We have now completed the proof of Proposition 4.1, and with that a proof of the main result.

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**References**


