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ScienceDirect
Applied Mathematics Letters 20 (2007) 1099–1101
Applied Mathematics Letters
www.elsevier.com/locate/aml

Solution of the distributional equation and Green's functions for scattering problems

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Received 16 May 2006; accepted 1 February 2007

Abstract

A general method is described to obtain Green's functions involved in nuclear scattering theory problems. By imposing specified boundary conditions on the general solution to the distribution equation, one can readily obtain the Green's function for the radial Schrödinger equation.

Keywords: Distributional equation; Green's functions; Scattering theory

Whitten and McCormick [1] discussed the calculational scheme for obtaining the Green's function for boundary-value problems. This method was extended by William Byrd [2] to obtain Green's functions corresponding to one- and two-point boundary conditions. Bagchi and Seyler [3] used this method to derive the Green's function for specific boundary conditions and discussed the usefulness of converting the radial Schrödinger equation into an integral equation. They discussed the boundary conditions which lead to the integral equations for the regular, Jost and physical s-wave scattering solutions. It is important to show that the Green's functions can be obtained readily by imposing boundary conditions on the solution of the distributional equation. Thus, once the general solution of the distributional equation is obtained, we can get the Green's functions for the initial or other kinds of boundary conditions which can determine the unknown constants in the general solution.

In order to demonstrate this method consider the operator for the radial Schrödinger equation ($\ell = 0$) case, acting on functions $U(r)$ in the interval $[0, \infty)$ and the boundary conditions for the regular, Jost and physical s-wave scattering solutions. In order to find the Green's functions, $G(r, r')$, we have to solve the distribution equation

$$\left(\frac{d^2}{dr^2} + K^2\right) G(r, r') = \delta(r - r') \tag{1}$$

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2007.02.001

under the specific boundary conditions. The general solution of Eq. (1) which follows is obtained by the Fourier transform technique and is given as

$$f(r,r') = \frac{1}{2K}\frac{|r-r'|}{r-r'}\sin K(r-r') + C_1(r)\cos Kr + C_2(r)\sin Kr. \tag{2}$$

The coefficients $C_1(r)$ and $C_2(r)$ can be obtained by imposing the boundary conditions.

The regular solution is defined as the solution of the equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + K^2\right)U(r) = V(r)U(r) \tag{3}$$

which satisfies the following conditions at the origin

$$\phi(K,0) = 0 \quad \text{and} \quad \phi'(K,0) = 1. \tag{4}$$

Imposing conditions (4) on Eq. (2) will give the Green's function for the regular solution as

$$G(r,r') = \begin{cases} 0 & r \le r' < \infty \\ \dfrac{\sin K(r-r')}{K} & 0 \le r' < r. \end{cases} \tag{5}$$

Jost solutions are defined as the solutions of Eq. (3) which satisfy the following boundary conditions

$$\lim_{r\to\infty} \quad \mathrm{e}^{\mp \mathrm{i}Kr} f^{\pm}(K,r) = 1. \tag{6}$$

Imposing this condition along with the continuity conditions for the Green's function at $r = r'$ we get

$$G^{\pm}(K,r,r') = \begin{cases} \mathrm{e}^{\pm \mathrm{i}Kr} - \dfrac{1}{K}\sin K(r-r') & r < r' \\ \mathrm{e}^{\pm \mathrm{i}Kr} & r > r'. \end{cases} \tag{7}$$

Using this Green's function, the differential equation (3) is converted to the following integral equation

$$f^{\pm}(K,r) = \mathrm{e}^{\pm \mathrm{i}Kr} - K^{-1}\int_r^{\infty} \sin K(r-r')V(r')f^{\pm}(K,r')\mathrm{d}r'. \tag{8}$$

However it should be pointed out that this method used in arriving at integral equation (8) is different from that of Bagchi and Seyler [3].

The physical solutions $\psi^{\pm}(K,r)$ are defined to be zero at the origin and to have the asymptotic form

$$\lim_{r\to\infty} \quad \psi^{+}(K,r) \to \frac{\mathrm{i}}{2}\left(\mathrm{e}^{-\mathrm{i}Kr} - S(K)\mathrm{e}^{\mathrm{i}Kr}\right). \tag{9}$$

The latter condition is usually put in the following form

$$\lim_{r\to\infty} \quad \psi^{+}(K,r) + \mathrm{i}K^{-1}\psi^{+}(K,r) \to 0. \tag{10}$$

Thus, imposing the above two conditions on Eq. (2) will yield the following Green's function for the physical solution:

$$G^{+}(K,r,r') = \begin{cases} -K^{-1}\sin Kr'\mathrm{e}^{\mathrm{i}Kr}, & 0 \le r' \le r \\ -K^{-1}\mathrm{e}^{\mathrm{i}Kr'}\sin Kr, & r \le r' \le \infty. \end{cases} \tag{11}$$

The above method of obtaining the Green's function is very general and can be used for any type of boundary condition which can determine the unknown constants uniquely in the solution of the distributional equation which we call the fundamental solution.

References

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