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The Extremal Length of a Network*

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I. INTRODUCTION

The extremal length of a plane region is defined by a minimax type limit of certain geometric quantities. Extremal length is invariant under conformal mapping and in this light it has been studied by Grötzsch, Beurling, and Ahlfors [1]. In this paper the definition of extremal length is extended to discrete systems. This permits defining the extremal length (and extremal width) between any two nodes of an electric network.

It is then proved that the extremal length between two nodes of a network is identical with the joint resistance between these nodes. The method of proof employs linear programming theory. In particular the *max-flow* equals *min-cut* theorem of Ford and Fulkerson [2] is used. In addition another theorem of this type is needed; we term it "max-potential equals min-work."

In a previous paper a planar network was considered and a dual associated network termed the "conjugate" was defined [3]. It was found that the joint resistance of the conjugate network is the reciprocal of the joint resistance of the original network. A simple proof of this result is given here by making use of the method of extremal length.

The last section of the paper gives an inequality relating paths and cuts in a network. This inequality results from the theorem that extremal width and extremal length are reciprocals.

II. RAYLEIGH'S RECIPROCAL THEOREM

Consider the problem of finding the electrical resistance of a plate in the shape of a curvilinear quadrilateral as shown in Fig. 1. The resistance ρ is required between edges A and B. The edges C and D are supposed insulated. The plate is of unit thickness and the top and bottom are insulated. If the

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plate is in the (x, y) plane let the specific resistivity be a scalar function of x and y, say $r(x, y)$. Then Ohm's law is

$$rw = -\operatorname{grad} u. \tag{1}$$

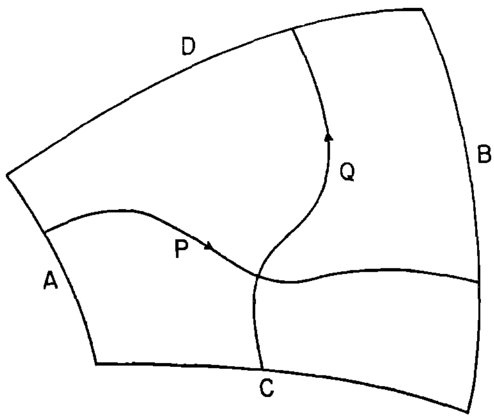

FIG. 1. A plate conductor.

Here w is the current density vector and u is the electric potential. Since $\operatorname{div} w = 0$,

$$\operatorname{div}(r^{-1} \operatorname{grad} u) = 0. \tag{2}$$

The boundary conditions are:

$$u = k_1 \quad \text{on} \quad A \qquad \text{and} \qquad u = k_2 \quad \text{on} \quad B, \tag{3}$$

$$\partial u/\partial n = 0 \qquad \text{on} \qquad B \quad \text{and} \quad C. \tag{4}$$

Then if this boundary value problem is solved for u, the resistance is defined as

$$\rho = \frac{\int_P \left[\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\right]}{\int_Q \left[-\frac{\partial u}{\partial y}\frac{dx}{r} + \frac{\partial u}{\partial x}\frac{dy}{r}\right]}. \tag{5}$$

Here P is a path going from edge A to edge B and Q is a path going from edge C to edge D.

The conjugate problem is to find the resistance of a plate of the same shape but between edges C and D. Now edges A and B are insulated. The specific resistance of the conjugate conductor is taken to be $r^* = 1/r$. Then the following result generalizes a theorem of Rayleigh. (He assumed $r(x, y)$ to be constant [4].)

THEOREM 1. *Conjugate conductors have reciprocal resistances.*

PROOF. Define a stream function $v(x, y)$ by the "Cauchy-Riemann equations"

$$\frac{\partial v}{\partial x} = -\frac{1}{r}\frac{\partial u}{\partial y}, \qquad \frac{\partial v}{\partial y} = \frac{1}{r}\frac{\partial u}{\partial x}. \tag{6}$$

Then by symmetry it is seen that v may be interpreted as the electrical potential for the conjugate problem in which the specific resistivity is $r^* = 1/r$. Then u becomes a stream function. Substituting (6) in (5) gives

$$\rho = \frac{\int_P \left[\frac{\partial v}{\partial y}\frac{dx}{r^*} - \frac{\partial v}{\partial x}\frac{dy}{r^*}\right]}{\int_Q \left[\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy\right]}. \tag{7}$$

But the right side of (7) is, by definition, $1/\rho^*$ and the theorem is proved.

It is worth noting that equations of the form (6) also appear in axially symmetric potential theory. For example consider the problem of finding the resistance of a body of revolution. Then in (6) r would be replaced by $r_1 = r(x, y)/y$ and y now denotes the distance to the axis of revolution.

Theorem 1 furnishes a tool for calculating upper and lower bounds for resistance (see [3] and [4]). The method applies to conductors having the form of a plane (or curved) surface. It also applies, as was seen above, to conductors of revolution.

III. DEFINITION OF EXTREMAL LENGTH

The theory of extremal length as developed by Grötzsch, Beurling, and Ahlfors begins as follows. Consider a rectangle of length L and width W, then their theory gives the extremal length λ as being simply L/W. More generally their theory gives the extremal length of a curvilinear quadrilateral R such as shown in Fig. 1. In fact if by a conformal mapping R is transformed into a rectangle of length L and width W then $\lambda = L/W$.

For the present purpose it is desirable to employ a similar but more general definition of extremal length than is employed by Ahlfors and Sario [1]. Thus the extremal length of the quadrilateral R between edges A and B is here taken to be

$$EL = \sup_W \inf_P \frac{\left(\int_P rW\, ds\right)^2}{\iint_R rW^2\, dx\, dy} \tag{8}$$

where $r(x, y)$ is the specific resistivity, taken to be positive and continuous. Here $W(x, y)$ denotes an arbitrary nonnegative continuous function. The line integral in the numerator is taken over a rectifiable path P joining A and B such as shown in Fig. 1. Then the infimum is taken for all such paths. Finally the supremum is taken over the class of functions W. This definition reduces to that of Ahlfors if we put $r = 1$.

The interest of extremal length from the point of view of applied mathematics is that it is precisely the resistance, that is $EL = \rho$. Moreover an optimizing function W is the current strength, that is $W = | w |$.

A related quantity of equal importance is the *extremal width*. The extremal width between edges A and B is here defined as

$$EW = \sup_W \inf_Q \frac{\left(\int_Q W\, ds\right)^2}{\iint_R r W^2\, dx\, dy}. \tag{9}$$

The line integral is taken over a rectifiable path Q separating edges A and B. Such a path Q is shown in Fig. 1. Then it may be shown that extremal length and extremal width are reciprocals,

$$(EL)\,(EW) = 1. \tag{10}$$

This is equivalent to Rayleigh's reciprocal theorem.

The considerations in this section and in the previous section are intended to motivate an analogous treatment of networks which follows. To clarify certain aspects of this analogy it is necessary to extend the definitions of extremal length and extremal width from two dimensions to three dimensions. Thus consider a simply connected three-dimensional body R. Now A and B will denote two separate parts of the boundary surface of R. Then we define the extremal length between A and B as

$$EL = \sup_W \inf_P \frac{\left(\int_P r W\, ds\right)^2}{\iiint_R r W^2\, dx\, dy\, dz} \tag{8a}$$

where P is a path connecting A and B. We define the extremal width between A and B as

$$EW = \sup_W \inf_Q \frac{\left(\iint_Q W\, dS\right)^2}{\iiint_R r W^2\, dx\, dy\, dz} \tag{9a}$$

where Q is a surface separating A and B.

Presumably EL and EW as defined by (8a) and (9a) also satisfy relation (10). This question will not be pursued but the considerations on networks which follow indicate that (10) is true.

IV. Network Definitions

An electric network is depicted as a graph diagram such as shown in Fig. 2. Each arc of the associated graph is assigned a direction. The directions are shown by arrows in Fig. 2. The electric current flowing through the jth arc in the positive direction is denoted by w_j; $j = 1, 2, \cdots, n$.

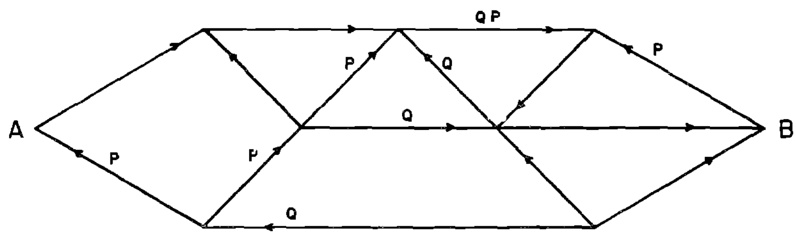

Fig. 2. An electric network.

Let $K_{\nu j}$ be the node-arc incidence matrix. Thus $K_{\nu j} = 1$ if the jth arc is directed toward the νth node, $K_{\nu j} = -1$ if the jth arc is directed away from the νth node, and $K_{\nu j} = 0$ if the jth arc and the νth node do not meet. Let

$$y_\nu = -\sum_{j=1}^{n} K_{\nu j} w_j. \tag{11}$$

Then it is said that there is a source of current y_ν at the νth node. Relation (11) may be regarded as a statement of the first law of Kirchhoff.

A *path* P between node A and node B is a set of arcs which form a simple curve. The arcs marked with a P form a path between A and B in Fig. 2.

A *cut* Q between node A and node B is a set of arcs which separate A from B and in addition no proper subset of Q separates A from B. By "separate" is meant that every path P between A and B has arcs in common with Q. The arcs marked with a Q form a cut between A and B in Fig. 2.

A circuit is a set of arcs forming a simple closed curve. A direction is assigned to each circuit. Then let $F_{\alpha j}$ be the incidence matrix between circuits and arcs. Thus $F_{\alpha j} = 1$ if the jth arc meets the αth circuit in the same direction, $F_{\alpha j} = -1$ if the jth arc meets the αth circuit in the opposite direction, and $F_{\alpha j} = 0$ if the jth arc does not meet the αth circuit.

Let

$$e_\alpha = \sum_{j=1}^{n} F_{\alpha j} r_j w_j \tag{12}$$

where r_j is the resistance of the jth arc ($r_j > 0$). Then it is said that there is an electromotive force e_α acting in the αth circuit. Relation (12) may be regarded as a statement of the second law of Kirchhoff.

Of especial concern in this paper are "passive flows between A and B." Such a flow satisfies two conditions:

I. The only sources are at A and B.

II. There are no electromotive forces in any circuit.

The last condition means $e_\alpha = 0$ and the first condition means $y_\nu = 0$ except at A and B. A passive flow between A and B results when a steady electric current enters a passive network at A and leaves at B, all other nodes being insulated. By "passive" is meant that there are no internal sources of power.

A flow w_j of strength I from A to B can be written in the form

$$w_j = IP_j + \sum_\alpha i_\alpha F_{\alpha j}. \tag{13}$$

Here P_j corresponds to a unit flow from A to B on a directed path P, so $P_j = 1\ (-1)$ if the jth arc meets P in the positive (negative) direction and $P_j = 0$ otherwise. The i_α are arbitrary. Multiply (13) by $r_j w_j$ and sum with respect to j. Making use of (12) this gives

$$\sum_1^n r_j w_j^2 = EI + \sum_\alpha e_\alpha i_\alpha. \tag{14}$$

Here E (the potential drop) is defined as

$$E = \sum_1^n P_j r_j w_j. \tag{15}$$

If w_j is a passive flow from A to B then $e_\alpha = 0$ so (14) becomes

$$\sum_1^n r_j w_j^2 = EI. \tag{16}$$

The quadratic form on the left will be denoted by $H(w)$.

Let w_j be a flow of strength I from A to B. Then for each cut Q between A and B

$$\sum_Q |w_j| \geq I.$$

This intuitively evident inequality is applied in the next Section; a formal proof is omitted.

The max-flow equals min-cut theorem may be stated as follows: *Let the flow capacity of the* jth *arc of a network be* $W_j \geq 0$. *Suppose that* $\Sigma_Q W_j \geq 1$ *for all cuts* Q *between notes* A *and* B. *Then there is a unit flow* w_j *from* A *to* B *such that* $|w_j| \leq W_j$.

V. The Extremal Width of a Network

The extremal width of a network relative to two nodes A and B is defined as

$$EW = \max_W \min_Q \frac{\left(\sum_Q W_j\right)^2}{\sum_1^n r_j W_j^2}. \tag{17}$$

Here r_j denotes the resistance of arc j. In (17) first the minimum is found over all cuts Q separating A and B. Then the maximum is found relative to arbitrary nonnegative numbers W_j.

Theorem 2. *The extremal width between two nodes of a network is equal to the reciprocal of the joint resistance between those nodes.*

Proof. We may put the definition of extremal width in the following equivalent form.

$$(EW)^{-1} = \min_W \sum_1^n r_j W_j^2 \tag{18}$$

subject to the constraints

$$\sum_Q W_j \geq 1 \text{ for all cuts.} \tag{19}$$

By the max-flow min-cut theorem there is a unit flow w_j from A to B (sources only at A and B) which satisfies the constraints $|w_j| \leq W_j$. Thus it follows that $\Sigma_Q |w_j| \geq 1$ for all cuts and

$$H(w) = \sum_1^n r_j w_j^2 \leq \sum_1^n r_j W_j^2.$$

So it may be assumed that there is an optimal solution W_j which satisfies $W_j = |w_j|$.

Thus the extremal width between A and B is the minimum of $H(w)$ for flows of unit strength from A to B. Let δw_j be a flow without sources; then at the minimum the variation of H is

$$\delta H = 2\sum_1^n r_j w_j \delta w_j = 0.$$

In particular, if δw_j is a flow around a closed circuit C_α this can be written

$$\sum_1^n r_j w_j F_{\alpha j} = 0.$$

This is condition II and it follows that the w_j correspond to a passive flow of current between A and B.

The joint resistance ρ between A and B is defined as

$$\rho = E/I \tag{20}$$

where I is the current entering at A and leaving at B in a passive flow from A to B. E is the potential drop from A to B. Substituting (20) in (16) gives

$$\rho I^2 = \sum_1^n r_j w_j^2. \tag{21}$$

Putting $I = 1$ in (21) we see that $\rho = H$. Comparing this with (18) shows that $\rho = (EW)^{-1}$ and the proof is complete.

VI. Max-Potential Equals Min-Work

With each network problem formulated in terms of current flows there is an associated problem formulated in terms of potentials. This principle is called "electrical duality" in some of the network literature. Theorem 3 to follow concerns the "electrical dual" to the max-flow equals min-cut theorem. Similar results are given by Dennis [5] and Minty [6].

Theorem 3. *Let frictional forces act so that an amount of work $c_j \geq 0$ is required to traverse the* jth *arc of a network. Let C denote the minimum net amount of work required to travel between two particular nodes, say A and B. By contrast consider conservative forces acting so that each node may be ascribed a potential. Suppose that the magnitude of the potential increase across the* jth *arc does not exceed c_j. Then the maximum potential difference between nodes A and B is C.*

PROOF. Let x_ν be the potential of the νth node; $\nu = 1, 2, \cdots, m$. Then the potential increase across the jth arc is

$$z_j = \sum_{\nu=1}^{m} x_\nu K'_{\nu j} \tag{22}$$

if $K'_{\nu j}$ denotes the potential increase across the jth arc when the νth node has potential 1 and all other nodes have potential 0. Clearly $K'_{\nu j} = K_{\nu j}$, the node-arc incidence matrix. Let node A correspond to $\nu = 1$ and let node B correspond to $\nu = 2$. In terms of the above definitions consider the following linear programming problem:

$$\text{maximize } x_2 - x_1 \text{ constrained by:} \tag{23}$$

$$\sum_{1}^{m} x_\nu K_{\nu j} \leq c_j \tag{24}$$

and

$$-\sum_{1}^{m} x_\nu K_{\nu j} \leq c_j. \tag{25}$$

With each maximizing problem in linear programming there is associated another problem termed the *dual*. Moreover there is a duality theorem which states that the maximum of the original problem is equal to the minimum of the dual problem. This duality theorem would lead to a proof of Theorem 3, however to fill in the details requires considerable space. Shorter proofs have been suggested to the author by A. Charnes, A. W. Tucker, G. Minty, and R. Gomory. The following direct proof follows the suggestion of Minty.

The definition of C may be expressed in the form

$$C = \min_P \sum_P c_j \tag{26}$$

where P is a path from A to B. The path P may be described as a sequence of successive arcs as follows

$$P \sim (e, f, \cdots, h). \tag{27}$$

On the other hand P defines a sequence of successive nodes,

$$P \sim (1, p, q, \cdots, t, 2). \tag{28}$$

If P is a minimizing path then

$$C = c_e + c_f + \cdots + c_h. \tag{29}$$

If the potentials x_ν satisfy the condition of the theorem then

$$c_e \geq | x_p - x_1 |, \cdots, c_h \geq | x_2 - x_t |. \tag{30}$$

Substituting these inequalities in relation (29) gives

$$\begin{aligned} C &\geq | x_p - x_1 | + | x_q - x_p | + \cdots + | x_2 - x_t | \\ &\geq (x_p - x_1) + (x_q - x_p) + \cdots + (x_2 - x_t). \end{aligned} \tag{31}$$

This last relation collapses into

$$C \geq x_2 - x_1. \tag{32}$$

In other words the maximum potential difference between A and B can not exceed C.

To show that relation (32) can become an equality the numbers x_ν are chosen in the following special way. Let x_ν denote the minimum net amount of work required to travel from node 1 to node ν. Suppose that nodes numbered i and j are the ends of an arc numbered k. If $x_i \geq x_j$ then it follows by the construction that

$$x_i \leq x_j + c_k. \tag{33}$$

This shows that the potentials x_ν satisfy the condition of the theorem. By construction

$$x_2 - x_1 = x_2 = C. \tag{34}$$

This completes the proof of Theorem 3.

VII. Extremal Length and Resistance

The extremal length of a network relative to two nodes A and B is defined as

$$EL = \max_W \min_P \frac{\left(\sum_P r_j W_j\right)^2}{\sum_1^n r_j W_j^2} \tag{35}$$

Here r_j denotes the resistance of arc j. First the minimum is found over all paths P joining A to B. Then the maximum is found as the parameters W_j vary over nonnegative values.

THEOREM 4. *The extremal length between two nodes of a network is equal to the joint resistance between those nodes.*

PROOF. We may put the definition of extremal length in the following equivalent form.

$$EL^{-1} = \min_W \sum_1^n r_j W_j^2 \tag{36}$$

subject to the constraints

$$\sum_P r_j W_j \geq 1 \text{ for all paths.} \tag{37}$$

Let $r_j W_j = c_j$ then (37) can be interpreted as stating that the minimum work from A to B exceeds 1. Then the max-potential min-work theorem may be applied. Thus there is an optimum potential x_ν of the nodes. Let z_j be the corresponding potential increases across arcs and define w_j by

$$w_j = z_j / r_j \tag{38}$$

By Theorem 3 it follows that $\sum_p | z_j | \geq 1$ so the constraints (37) are also satisfied by replacing W_j by $| w_j |$. Nevertheless

$$| w_j | \leq W_j. \tag{39}$$

It follows from (39) that

$$\sum_1^n r_j w_j^2 \leq \sum_1^n r_j W_j^2 \tag{40}$$

so it may be assumed that an optimal solution of the extremal length definition (35) is of the form $W_j = | w_j |$ where w_j comes from a potential, as is evidenced by the definition (38).

Thus the problem has been reduced to minimizing

$$H = \sum_j r_j^{-1} \left(\sum_\nu x_\nu K_{\nu j} \right)^2 \tag{41}$$

subject to the conditions

$$x_2 - x_1 = 1. \tag{42}$$

Then if $\nu \neq 1$ or 2 we have

$$0 = \frac{\partial H}{\partial x_\nu} = -2 \sum_j K_{\nu j} w_j. \tag{43}$$

This shows that at the minimum w_j defines a flow from A to B; this is Kirchhoff's first law.

Relation (38) clearly implies that w_j satisfies condition II for a passive network. Thus w_j satisfies the conditions for the passive flow of current from A to B.

It is seen from definition (15) for E, and (38) that

$$-E = x_2 - x_1 = 1. \tag{44}$$

Substitution of (44) in (16) shows that in the present case $\rho^{-1} = H$. It then follows from (36) that $\rho = EL$ and this completes the proof of Theorem 4.

COROLLARY 1. *Extremal length and extremal width are reciprocals.*
The proof follows from Theorems 2 and 4.

VIII. CONJUGATE NETWORKS

A graph is termed *planar* if it can be drawn on a plane in such a way that no arc crosses another. Each planar graph G is associated with another planar graph G^* termed the dual. The arcs of the primal graph G and the dual graph G^* have a one-to-one correspondence determined by the arcs crossing. The primal graph breaks up the plane into regions. There is exactly one node of the dual graph in each region of the primal graph.

LEMMA 1. *Let A and B be two nodes on an arc α of a planar graph G. Then Q is a cut between A and B if and only if the corresponding set Q^* in the dual graph G^* is a circuit containing the arc α^*.*

PROOF. First suppose Q^* is a circuit. Q^* crosses α at only one point so one end of α is inside Q^* and one end is outside. Thus any path P between the ends A and B must intersect Q^*. The arc where P intersects Q^* is an arc of Q. Thus Q is either a cut or else Q has a proper subset Q_1 which is a cut.

Since Q_1^* is a proper part of a circuit it has an end point. The end point is a node of G^* in a region R of G. Thus R has one and only one arc β of the cut Q_1 on its border. But this is impossible because then β would be effectively bypassed by the other arcs of the border of R, say β'. To see this it is noted that there is a path P having only β in common with Q_1. (This follows by the definition of a cut.) Let P' denote the set of arcs after deleting β from P. Then it is clear that the set $\beta' + P'$ contains a path. This path does not have arcs in common with Q_1. This is a contradiction, proving Q to be a cut.

Now suppose Q is an arbitrary cut. Then the argument just given proves that Q^* can not have an end point. It follows that Q^* must contain a circuit, say C^*. Then given an arc γ of Q there is a path P between A and B such that P and Q have only γ in common. In particular if γ is in the set C it follows that P crosses C^* once and only once. Thus one end of P is inside C^* and one end is outside. In other words C separates A and B. By the definition of a cut, $Q \equiv C$ and so Q^* is a circuit. This completes the proof of the lemma.

Consider a primal graph G and the dual graph G^*. Let α be an arbitrary arc of G with ends A and B and let α^* be the corresponding arc of G^* with ends C and D. The arc α is deleted from G and the arc α^* is deleted from G^*. This leads to graphs N and N^* which we term *conjugate*. Two networks N and N^* are said to be *conjugate* if they have conjugate graphs and if corresponding arcs have reciprocal resistance. Thus

$$r_j r_j^* = 1 \tag{45}$$

where r_j is the resistance of the jth arc of the network N and r_j^* is the resistance of the jth arc of the network N^*. Conjugate networks are shown in Fig. 3; N is in full lines and N^* is in dotted lines.

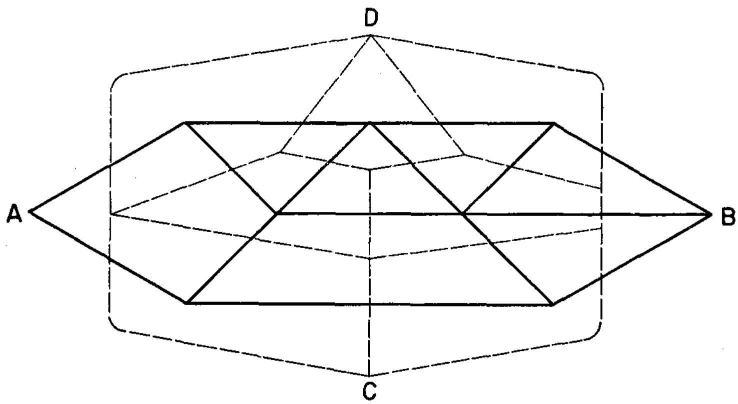

FIG. 3. Conjugate networks.

THEOREM 5. *Conjugate networks have reciprocal joint resistances.*

PROOF. The conductance of N between A and B is given by the extremal width so

$$\rho^{-1} = \max_W \min_Q \frac{\left(\sum_Q W_j\right)^2}{\sum_1^n r_j W_j^2}.$$

Now let $W_j = r_j^* W_j^*$ so

$$\rho^{-1} = \max_{W^*} \min_Q \frac{\left(\sum_Q r_j^* W_j^*\right)^2}{\sum_1^n r_j^* (W_j^*)^2}. \tag{46}$$

According to Lemma 1 the cuts Q of N correspond to the paths P^* from C to D. Thus the right side of (46) is the extremal length between C and D. But the extremal length is the resistance so $\rho^{-1} = \rho^*$ and the proof of the theorem is complete.

Theorem 5 is, of course, a discrete analogue of Rayleigh's reciprocal theorem. A different proof of this theorem was given in [3]. Another type of proof could be given by the Wang algebra [7].

IX. The Path-Cut Inequality

The relationship between extremal width and extremal length leads to an inequality relating paths and cuts.

THEOREM 6. *Let W_j and V_j denote arbitrary nonnegative numbers associated with the n arcs of a network. Then*

$$\sum_{j=1}^{n} W_j V_j \geq \left(\min_P \sum_P V_j\right) \left(\min_Q \sum_Q W_j\right). \tag{47}$$

Here the first minimum is carried out over paths P between two nodes and the second minimum is carried out over cuts Q separating the same two nodes.

PROOF. It follows from the definition of extremal width that

$$EW \geq \min_Q \frac{\left(\sum_Q W_j\right)^2}{\sum_1^n r_j W_j^2}. \tag{48}$$

Likewise the definition of extremal length gives

$$EL \geq \min_P \frac{\left(\sum_P r_j W_j\right)^2}{\sum_1^n r_j W_j^2}. \tag{49}$$

But $(EL)(EW) = 1$ so (48) and (49) give

$$\sum_1^n r_j W_j^2 \geq \left(\min_P \sum_P r_j W_j\right) \left(\min_Q \sum_Q W_j\right). \tag{50}$$

Suppose first that $V_j > 0$ and $W_j > 0$. Then let $r_j = V_j/W_j$ in (50) and (47) results. The general case can be obtained by taking the limit of both sides of (47) as some of the V_j and W_j approach zero.

COROLLARY 2. *Let the arcs of a network be colored red or blue. Then given any two nodes there is either a red path between them or else there is a blue cut separating them.*

PROOF. Let the arcs colored red have $W_j = 0$ and $V_j = 1$. Let the arcs colored blue have $W_j = 1$ and $V_j = 0$. Then the left side of (47) vanishes so it follows that one of the two factors on the right vanish. This implies the statement of the corollary.

A continuous analogy of inequality (47) for the plane is

$$\iint_R VW\,dx\,dy \geq \left(\inf_P \int_P V\,ds\right)\left(\inf_Q \int_Q W\,ds\right). \tag{51}$$

Here V and W are arbitrary nonnegative functions. Here R is a simply connected region of the plane and P denotes a path between two separate segments of the boundary and Q is a path separating the two segments of the boundary.

A continuous analogy of inequality (47) for space is

$$\iiint_R VW\,dx\,dy\,dz \geq \left(\inf_P \int_P V\,ds\right)\left(\inf_Q \iint_Q W dS\right). \tag{52}$$

Here V and W are arbitrary nonnegative functions. Here R denotes a simply connected body and P denotes a path between two separate parts of the boundary and Q is a surface separating these two parts of the boundary.

Relations (51) and (52) for continuous conductors are stated here without proof in order to throw light on the corresponding network relation (47). The proof for networks has made extensive use of finite linear programming theory. This suggests that continuous conductors be treated by infinite linear programming theory. (Certain problems of infinite programming are treated in [8] and [9].)

REFERENCES

1. AHLFORS, L. V., AND SARIO, L. "Riemann Surfaces," pp. 214-228. Princeton Univ. Press, 1960.
2. FORD, L., AND FULKERSON, D. R. Maximal flow through a network. *Can. J. Math.* 8, 399-404 (1956).
3. DUFFIN, R. J. Distributed and lumped networks. *J. Math. and Mech.* 8, 793-826 (1959).

4. LORD RAYLEIGH. On the approximate solution of certain problems relating to the potential. *Proc. London Math. Soc.* **7**, 70-75 (1876).
5. DENNIS, J. B. "Mathematical Programming and Electrical Networks," pp. 47-51. MIT Press and Wiley, New York, 1959.
6. MINTY, G. J. Monotone networks. *Proc. Roy. Soc.* **257**, 194-212 (1960).
7. DUFFIN, R. J. An analysis of the Wang algebra of networks. *Trans. Am. Math. Soc.* 114-131 (1959).
8. DUFFIN, R. J. Infinite programs. In "Linear Inequalities and Related Systems," Annals of Mathematical Studies No. 38, pp. 157-170 (1956).
9. DUFFIN, R. J. A variational problem relating to cooling fins. *J. Math. and Mech.* **8**, 47-56 (1959).
10. DANTZIG, G. B., AND FULKERSON, D. R. On the max-flow min-cut theorem of networks. *In* "Linear Inequalities and Related Systems," Annals of Mathematical Studies No. 38, pp. 215-221 (1956).