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A combinatorial spanning tree model for knot Floer homology

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Abstract

We iterate Manolescu's unoriented skein exact triangle in knot Floer homology with coefficients in the field of rational functions over $\mathbb{Z}/2\mathbb{Z}$. The result is a spectral sequence which converges to a stabilized version of δ-graded knot Floer homology. The (E_2, d_2) page of this spectral sequence is an algorithmically computable chain complex expressed in terms of spanning trees, and we show that there are no higher differentials. This gives the first combinatorial spanning tree model for knot Floer homology.

Keywords: Knot Floer homology; Heegaard Floer homology; Khovanov homology; Spanning tree; Exact triangle

1. Introduction

Knot Floer homology is an invariant of oriented links in the 3-sphere, originally defined by Ozsváth–Szabó [36] and by Rasmussen [46] using Heegaard diagrams and holomorphic disks. This invariant comes in several flavors. The simplest is a bigraded vector space over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$,

$$\widehat{\mathrm{HFK}}(L) = \bigoplus_{m,a} \widehat{\mathrm{HFK}}_m(L, a),$$

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from which one can recover the Seifert genus of L [35] and determine whether L is fibered [9,32]. In addition, knot Floer homology categorifies the Alexander polynomial:

$$\sum_{m,a}(-1)^{m+(|L|-1)/2}\operatorname{rk}\widehat{\mathrm{HFK}}_m(L,a)\cdot t^a=(t^{-1/2}-t^{1/2})^{|L|-1}\cdot\Delta_L(t), \tag{1.1}$$

where $|L|$ is the number of components of L.

In 2006, Manolescu, Ozsváth, and Sarkar [29] and Sarkar and Wang [49] discovered algorithms for computing knot Floer homology via Heegaard diagrams in which the counts of holomorphic disks are completely combinatorial. The following year, Ozsváth and Szabó [42] gave an algebro-combinatorial formulation of knot Floer homology using a singular cube of resolutions construction which takes as input a marked braid-form projection of a knot. The purpose of this article is to give an entirely novel combinatorial description of the δ-*graded* knot Floer homology groups,

$$\widehat{\mathrm{HFK}}_\delta(L)=\bigoplus_{a-m=\delta}\widehat{\mathrm{HFK}}_m(L,a), \tag{1.2}$$

in terms of *spanning trees*. Before launching into this description, we provide some background and motivation.

Let $\mathcal{D}$ be a connected planar projection of L, and color its complementary regions black and white in a checkerboard fashion, so that the unbounded region of $\mathbb{R}^2\setminus\mathcal{D}$ is colored white. One forms the *black graph* $B(\mathcal{D})$ by placing a vertex in each black region and connecting two vertices by an edge for every crossing of $\mathcal{D}$ that joins the corresponding regions. A *spanning tree* of $B(\mathcal{D})$ is a connected, acyclic subgraph of $B(\mathcal{D})$ that contains all vertices of $B(\mathcal{D})$. The Alexander and Jones polynomials of L can be expressed as sums of monomials associated to such trees. When L is a knot, for example,

$$\Delta_L(t)=\sum_{s\in\mathcal{T}(B(\mathcal{D}))}(-1)^{M(s)}\cdot t^{A(s)}, \tag{1.3}$$

where $\mathcal{T}(B(\mathcal{D}))$ is the set of spanning trees of $B(\mathcal{D})$, and $A(s)$ and $M(s)$ are integers [16].

Since knot Floer homology encodes the Alexander polynomial, one expects that it should also admit a formulation in terms of spanning trees. Indeed, in [33], Ozsváth and Szabó associate to $\mathcal{D}$ a doubly pointed Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z,w)$ for L for which generators of the chain complex $\widehat{\mathrm{CFK}}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z,w)$ are in 1-to-1 correspondence with spanning trees of $B(\mathcal{D})$, with the bigrading given by the quantities $A(s)$ and $M(s)$ in (1.3). Using this Heegaard diagram, they prove that the knot Floer homology of an alternating knot is determined by its Alexander polynomial and signature. However, despite numerous efforts, no one has managed to find a combinatorial description of the differential on this complex, largely because there is no general algorithm for counting the relevant holomorphic disks.

In this article, we introduce a complex for knot Floer homology whose differential is combinatorial and can be described explicitly in terms of spanning trees. Our construction starts with an oriented, connected planar projection $\mathcal{D}$ for L. We choose m marked points on the edges of $\mathcal{D}$ so that every edge contains at least one such point. Let $\mathcal{F}=\mathbb{F}(T)$, the field of rational functions in a single variable T with coefficients in $\mathbb{F}$. In Section 2, we define a graded chain complex $(C^\Omega(\mathcal{D}),\partial^\Omega)$, where $C^\Omega(\mathcal{D})$ is a direct sum of 2^{m-1}-dimensional vector spaces over $\mathcal{F}$, one for each spanning tree of $B(\mathcal{D})$, and ∂^Ω can be described explicitly in terms of the planar embedding of $B(\mathcal{D})$, the marked points, and a generic function Ω from the crossings of $\mathcal{D}$ to the integers.

Our main theorem is the following.

Theorem 1.1. *The homology of* $(C^{\Omega}(\mathcal{D}), \partial^{\Omega})$ *is isomorphic as a graded* $\mathcal{F}$*-vector space to* $\widehat{\mathrm{HFK}}(L) \otimes_{\mathbb{F}} V^{\otimes(m-|L|)} \otimes_{\mathbb{F}} \mathcal{F}$ *with respect to the* δ*-grading on* $\widehat{\mathrm{HFK}}(L)$*, where* V *is a two-dimensional vector space over* $\mathbb{F}$ *supported in grading zero.*

Our construction makes use of Manolescu's unoriented skein exact triangle [27], which relates the knot Floer homology of L with those of its two resolutions at a crossing. Under mild technical assumptions, one can iterate Manolescu's triangle in the manner of Ozsváth–Szabó [39]. The result is a cube of resolutions spectral sequence $\mathcal{S}_{\mathbb{F}}$ that converges to $\widehat{\mathrm{HFK}}(L) \otimes V^{\otimes(m-|L|)}$ and whose E_1 page is a direct sum,

$$\bigoplus_{I\in\{0,1\}^n} \widehat{\mathrm{HFK}}(L_I) \otimes V^{\otimes(m-|L_I|)},$$

over complete resolutions L_I of $\mathcal{D}$. The d_1 differential of $\mathcal{S}_{\mathbb{F}}$ can be described explicitly. Unfortunately, however, $E_2(\mathcal{S}_{\mathbb{F}})$ is not an invariant of L (see Remark 7.7).

To skirt this issue, we perform the above iteration instead over $\mathcal{F}$, using a system of twisted coefficients determined by Ω. With these coefficients, the knot Floer homologies of disconnected resolutions vanish, and the E_1 page of the resulting spectral sequence, $\mathcal{S}^{\Omega}_{\mathcal{F}}$, is a direct sum of vector spaces associated to *connected* resolutions, which are themselves in 1-to-1 correspondence with spanning trees of $B(\mathcal{D})$. This page is isomorphic to the complex $C^{\Omega}(\mathcal{D})$, and $d_1(\mathcal{S}^{\Omega}_{\mathcal{F}})$ is identically zero since no edge in the cube of resolutions of $\mathcal{D}$ can join two connected resolutions. We identify the differential $d_2(\mathcal{S}^{\Omega}_{\mathcal{F}})$ with ∂^{Ω} and, based on a grading argument, show that $\mathcal{S}^{\Omega}_{\mathcal{F}}$ collapses at its E_3 page. This proves Theorem 1.1.

For the remainder of this section, we shall denote the homology $H_*(C^{\Omega}(\mathcal{D}), \partial^{\Omega})$ by $\mathrm{HS}_*(L, m)$.

Although the δ-grading on knot Floer homology contains less information than the bigraded theory (e.g. one generally needs the bigrading to determine Seifert genus), it is still a rather powerful invariant with several applications. Below, we briefly recast some of these in terms of $\mathrm{HS}(L, m)$. Recall that the *homological width* of L is

$$w(L) = 1 + \max\{\delta \mid \widehat{\mathrm{HFK}}_{\delta}(L) \neq 0\} - \min\{\delta \mid \widehat{\mathrm{HFK}}_{\delta}(L) \neq 0\}.$$

If $w(L) = 1$, we say that L is *thin*. One of the most useful features of our theory is that it measures width. Indeed, by Theorem 1.1,

$$w(L) = 1 + \max\{\delta \mid \mathrm{HS}_{\delta}(L, m) \neq 0\} - \min\{\delta \mid \mathrm{HS}_{\delta}(L, m) \neq 0\}.$$

Note that, when L is thin, its bigraded knot Floer homology is completely determined by $\mathrm{HS}(L, m)$ and $\Delta_L(t)$. Theorem 1.1 and the results of Ozsváth-Szabó [35], Ghiggini [9], and Ni [31] therefore imply the following.

Corollary 1.2. (1) *L is the k-component unlink if and only if* $w(L) = k$ *and* $\mathrm{rk}_{\mathcal{F}}\, \mathrm{HS}(L, m) = 2^{m-1}$.

(2) *L is the figure-eight knot if and only if L is thin and* $\Delta_L(t) = -t^{-1} + 3 - t$.

(3) *L is the left- or right-handed trefoil if and only if* $\mathrm{rk}_{\mathcal{F}}\, \mathrm{HS}(L, m) = 3 \cdot 2^{m-1}$ *and* $\mathrm{HS}(L, m)$ *is supported in the grading* -1 *or* $+1$*, respectively.*

Moreover, when L is a thin *knot*, its genus is simply the degree of $\Delta_L(t)$ [35], and it is fibered if and only if $\Delta_L(t)$ is monic [9,32]. In addition, the concordance invariant $\tau(L)$, whose absolute value is a lower bound for the smooth four-ball genus of L, is equal to the unique grading in which $\mathrm{HS}(L, m)$ is supported [34].

It would be interesting to find a refinement of our construction which captures the full bigrading on $\widehat{\mathrm{HFK}}$. However, the fact that the δ-grading is especially natural from our vantage hints that our theory may be well suited to certain applications, which we now describe.

The *reduced Khovanov homology* of a link $L \subset S^3$ is a bigraded vector space over $\mathbb{F}$,

$$\widetilde{\mathrm{Kh}}(L) = \bigoplus_{i,j} \widetilde{\mathrm{Kh}}^{i,j}(L),$$

which categorifies the Jones polynomial of L. In spite of their disparate origins, Khovanov homology and knot Floer homology possess intriguing similarities. For instance, although the bigrading on Khovanov homology behaves quite differently from that on knot Floer homology, one can collapse the former into a single grading,

$$\widetilde{\mathrm{Kh}}^{\delta}(L) = \bigoplus_{j/2-i=\delta} \widetilde{\mathrm{Kh}}^{i,j}(L),$$

and all available evidence points to the following conjecture, first formulated by Rasmussen [45] in the case of knots.

Conjecture 1.3. *For any link $L \subset S^3$,*

$$2^{|L|-1-\eta(L)} \cdot \mathrm{rk}_{\mathbb{F}}\, \widetilde{\mathrm{Kh}}^{\delta}(L) \geq \mathrm{rk}_{\mathbb{F}}\, \widehat{\mathrm{HFK}}_{\delta}(L),$$

where $\eta(L)$ is the rank of the Alexander module of L over $\mathbb{Z}[H_1(S^3 \setminus L; \mathbb{Z})]$.

A proof of this conjecture would imply that Khovanov homology detects not only the unknot, a fact recently established by Kronheimer and Mrowka [19] using instanton Floer homology, but also the trefoils and unlinks.[1]

Our new description for knot Floer homology bears an intriguing resemblance to recent work by Roberts [44] and Jaeger [15] that provides a spanning tree model for reduced Khovanov homology. Specifically, Roberts defines a complex $(C'(\mathcal{D}), \partial')$ whose generators (over a field $\mathcal{F}'$ of rational functions in several variables) correspond to spanning trees, with the same grading as in our complex $C^{\Omega}(\mathcal{D})$. Moreover, the component of our differential ∂^{Ω} from the summand corresponding to a spanning tree T to the summand corresponding to T' is nonzero precisely when the same is true in ∂'. Jaeger then proves that, when L is a knot, the homology of $(C'(\mathcal{D}), \partial')$ is precisely $\widetilde{\mathrm{Kh}}(L) \otimes \mathcal{F}'$ with its δ-grading.[2] Because of this similarity, we hope that our new model for knot Floer homology may shed some light on Conjecture 1.3. For a simple example in this vein, see Corollary 2.10 below.

Many of the ideas in this paper can be traced to work of Ozsváth and Szabó [39], who discovered a spectral sequence relating $\widetilde{\mathrm{Kh}}(L)$ to the Heegaard Floer homology of $\Sigma(\overline{L})$, the double cover of S^3 branched along the mirror of L. Generalizations and applications of

[1] Using [19], Hedden and Ni showed that the total rank of $\widetilde{\mathrm{Kh}}$ detects the 2-component unlink [13] and that $\widetilde{\mathrm{Kh}}$, equipped with some additional algebraic structure, detects all unlinks [12].

[2] Note that Champanerkar and Kofman [5] and Wehrli [55] independently discovered a different spanning tree model for Khovanov homology. However, the differential on this complex is not known explicitly in terms of spanning trees; to compute it, one must effectively compute the entire Khovanov complex. An advantage of their model, however, is that it provides the entire bigrading on $\widetilde{\mathrm{Kh}}$, not just the δ-grading.

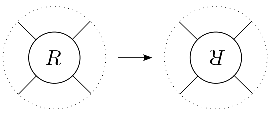
this spectral sequence have made for an active area of research in recent years; see, e.g., [2,4,8,14,43,19]. In forthcoming work, Ozsváth, Szabó, and the first author define an analogous construction with twisted coefficients, the result of which is a spectral sequence $\mathcal{S}$, converging to the twisted Heegaard Floer homology of $\Sigma(\overline{L})$, whose E_2 page is a spanning tree complex that formally resembles both our complex $C^{\Omega}(\mathcal{D})$ and Roberts' $C'(\mathcal{D})$.[3] In contrast with our setup, it is not clear whether $\mathcal{S}$ collapses at its E_3 page. However, the similarities between $(E_2(\mathcal{S}), d_2(\mathcal{S}))$ and $(C^{\Omega}(\mathcal{D}), \partial^{\Omega})$ suggest that one might hope to prove a relationship between $\widehat{\mathrm{HFK}}(L)$ and $\widehat{\mathrm{HF}}(\Sigma(L))$, as was also proposed by Greene [11]. Available evidence suggests the following.

Conjecture 1.4. *For any link $L \subset S^3$,*

$$\mathrm{rk}_{\mathbb{F}}\, \widehat{\mathrm{HFK}}_{*+(|L|-1)/2}(L) \geq 2^{|L|-1-\eta(L)} \cdot \mathrm{rk}_{\mathbb{F}}\, \widehat{\mathrm{HF}}_*(\Sigma(L)),$$

where the two gradings above are the mod-2 δ- and Maslov gradings, respectively.

A third potential application of our construction has to do with *mutation*, an operation on planar link diagrams in which one removes a 4-strand tangle and reglues it after a half-rotation, as in the figure below. Mutation leaves all classical link polynomials unchanged and preserves the homeomorphism type of the branched double cover. Moreover, Wehrli [56] and Bloom [3] have shown that it preserves reduced Khovanov homology (with coefficients in $\mathbb{F}$).



In contrast, mutation can change the bigraded knot Floer homology of a knot since it need not preserve Seifert genus [38]. Somewhat surprisingly, however, the computations in [1] support the following conjecture.

Conjecture 1.5. *If L' is obtained from L by mutation, then $\widehat{\mathrm{HFK}}_\delta(L) \cong \widehat{\mathrm{HFK}}_\delta(L')$.*

Indeed, if Conjectures 1.3 and 1.4 hold, then mutation cannot have too drastic an effect on these δ-graded groups. Moreover, since the Alexander polynomial is mutation invariant, a proof of this conjecture would imply that, for thin knots, mutation preserves genus, fiberedness, and the τ invariant.

Our model provides a reasonable starting point from which to approach Conjecture 1.5, since $(C^{\Omega}(\mathcal{D}), \partial^{\Omega})$ is formulated largely in terms of black graph data, much of which is preserved by mutation. In particular, spanning trees of $B(\mathcal{D})$ are in 1-to-1 correspondence with spanning trees of $B(\mathcal{D}')$ for any mutant $\mathcal{D}'$ of $\mathcal{D}$.

One of the most compelling features of our construction is that the complex $(C^{\Omega}(\mathcal{D}), \partial^{\Omega})$ is largely determined by formal properties; very little direct computation is required. This suggests that our approach might be used to give an axiomatic characterization of knot Floer homology or to prove that $\widehat{\mathrm{HFK}}$ is isomorphic to other knot homology theories, such as Kronheimer and Mrowka's monopole knot homology [18]. It is known (or soon will be) that monopole knot homology agrees with knot Floer homology, as a result of nearly 1000 pages of work of Taubes [50–54], Kutluhan–Lee–Taubes [20–24] and Colin–Ghiggini–Honda [6,7], combined with work of Lekili [25]. Still, it would be nice to prove this equivalence (and to find a

[3] Kriz and Kriz [17] have proven that the homology of $(E_2(\mathcal{S}), d_2(\mathcal{S}))$ is a link invariant.

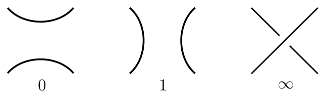


Fig. 1. The 0-, 1- and ∞- resolutions of the crossing on the right.

combinatorial formulation of monopole knot homology) without resorting to their SW $=$ ECH $=$ HF machinery. The key will be to define an analogue of Manolescu's exact triangle in the monopole setting; if done correctly, almost everything should follow from purely formal considerations.

Finally, it is worth mentioning some advantages of our model over the other combinatorial formulations of knot Floer homology. For an n-crossing projection with $2n$ marked points, the dimension of our complex (over $\mathcal{F}$) is $s(\mathcal{D}) \cdot 2^{2n-1}$, where $s(\mathcal{D}) \leq 2^n$ is the number of spanning trees of $B(\mathcal{D})$, whereas the dimension of the Manolescu–Ozsváth–Sarkar grid complex is on the order of $n!$ (albeit over the simpler field $\mathbb{F}$). Thus, our theory should be more computable for large knots. Furthermore, in contrast with Ozsváth and Szabó's singular braid model [42], our construction does not require a braid projection, and it applies to arbitrary links rather than just knots. (Of course, the main drawback is that our complex computes only $\widehat{\mathrm{HFK}}(L)$ with its δ-grading, not the more robust version $\mathrm{HFK}^-(L)$ or the bigrading on $\widehat{\mathrm{HFK}}(L)$.)

Organization. In Section 2, we define the complex $(C^{\Omega}(\mathcal{D}), \partial^{\Omega})$. In Section 3, we provide background on knot Floer homology with twisted coefficients and we introduce an action on knot Floer homology defined by counting disks which pass over basepoints. In Section 4, we compute the twisted knot Floer homologies of unknots and unlinks in terms of this action. In Section 5, we iterate Manolescu's exact triangle with twisted coefficients in $\mathcal{F}$. The result of this iteration is a filtered cube of resolutions complex that computes knot Floer homology. In Section 6, we determine the δ-grading shifts of the maps in this filtered complex and show that the associated spectral sequence $\mathcal{S}^{\Omega}_{\mathcal{F}}$ collapses at its E_3 page. In Section 7, we compute the (E_2, d_2) page of $\mathcal{S}^{\Omega}_{\mathcal{F}}$ and show that it is isomorphic to $(C^{\Omega}(\mathcal{D}), \partial^{\Omega})$, proving Theorem 1.1.

2. Definition of the complex

Fix an oriented, connected planar projection $\mathcal{D}$ of L. Let $c_1, \ldots, c_n$ denote the crossings of $\mathcal{D}$, and let $\mathbf{p} = \{p_1, \ldots, p_m\}$ be a set of marked points on the edges of $\mathcal{D}$ so that every edge is marked, and so that p_1 lies on an outermost edge of $\mathcal{D}$. Let $n_+(\mathcal{D})$ and $n_-(\mathcal{D})$ denote the numbers of positive and negative crossings in $\mathcal{D}$, respectively. Additionally, we fix an arbitrary orientation on the edges of $B(\mathcal{D})$.

The 0- and 1-resolutions of $\mathcal{D}$ at a crossing c_j are the diagrams obtained from $\mathcal{D}$ by smoothing c_j according to the convention in Fig. 1. Taking the ∞-resolution of c_j means leaving the crossing unchanged. For each $I = (I_1, \ldots, I_n) \in \{0, 1\}^n$, let $\mathcal{D}_I$ be the *complete resolution* of $\mathcal{D}$ gotten by replacing c_j with its I_j-resolution. $\mathcal{D}_I$ is a planar unlink, and we shall orient its components as the boundaries of the black regions. (This orientation is not, in general, consistent with any orientation on L.) Let $|\mathcal{D}_I|$ denote the number of components of $\mathcal{D}_I$, and let $|I| = I_1 + \cdots + I_n$.

For $j = 1, \ldots, n$, let e_j denote the edge of $B(\mathcal{D})$ which corresponds to the crossing c_j. Given a spanning subgraph $\gamma \subset B(\mathcal{D})$ – i.e., a subgraph containing all vertices of $B(\mathcal{D})$ – one obtains a complete resolution of $\mathcal{D}$ by smoothing each crossing c_j in such a way as to join the black regions incident to c_j if and only if e_j is contained in γ; see Fig. 2(b). Let γ_I denote the

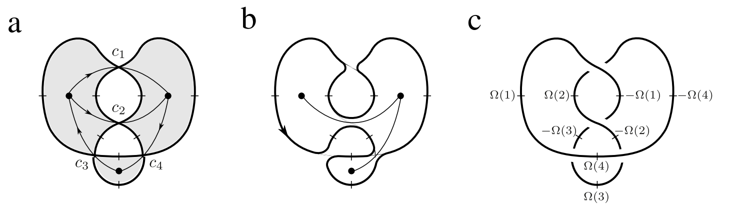


Fig. 2. (a) A pointed diagram $\mathcal{D}$ for the unknot, along with its black graph and a choice of orientations on the edges of $B(\mathcal{D})$. The marked points are indicated by dashes. (b) A spanning tree of $B(\mathcal{D})$ and the corresponding resolution of $\mathcal{D}$. (c) The values r_i associated to the marked points, as in Definition 2.2.

subgraph corresponding to the resolution $\mathcal{D}_I$. It is not hard to see that $\mathcal{D}_I$ is connected if and only if γ_I is a spanning tree.

In order to work with twisted coefficients, we need to specify certain cohomology classes via the following definition.

Definition 2.1. A *system of weights* is a tuple $\mathbf{r} = (r_1, \dots, r_m)$ satisfying $r_1 + \cdots + r_m = 0$, which are associated with the marked points $p_1, \dots, p_m$. Given $\mathbf{r}$, let $\omega_\mathbf{r} \in H^2(S^3 \setminus L; \mathbb{Z})$ be the cohomology class whose evaluation on the boundary torus of a tubular neighborhood of each component L_j of L equals the sum of the weights on L_j. (Note that the sum of these tori equals zero in homology, so the condition that $r_1 + \cdots + r_m = 0$ is needed.) A system of weights is called *generic* if, for every $I \in \{0,1\}^n$ for which the resolution $\mathcal{D}_I$ is disconnected, the sum of the weights on each component of $\mathcal{D}_I$ is nonzero.

We shall often make use of systems of weights coming from the following construction.

Definition 2.2. A function $\Omega\colon \{1, \dots, n\} \to \mathbb{Z}$ is called *generic* if the values $\Omega(1), \dots, \Omega(n)$ do not satisfy any nontrivial linear relation with coefficients in $\{-1, 0, 1\}$. (For instance, the function $\Omega(i) = 2^i$ is generic.) Such a function (generic or not) determines a system of weights $\mathbf{r}_\Omega = (r_1, \dots, r_m)$ by the following construction. For each $j = 1, \dots, n$, view the crossing c_j so that the oriented edge e_j points from left to right. If p_{i_1}, p_{i_2}, p_{i_3}, and p_{i_4} are the closest marked points to c_j on the four edges of $\mathcal{D}$ incident to c_j, starting in the upper right and going counterclockwise, define $r_{i_2} = \Omega(j)$ and $r_{i_4} = -\Omega(j)$. This convention determines $2n$ of the integers $r_1, \dots, r_m$. Define the remaining 1s to be zero. (See Fig. 2(c) for an example.) Additionally, we call i_1 and i_3 the *special indices* associated to c_j.

Lemma 2.3. *If* $\mathbf{r} = \mathbf{r}_\Omega$ *for a function* $\Omega\colon \{1, \dots, n\} \to \mathbb{Z}$*, then* $\omega_\mathbf{r} = 0$ *in* $H^2(S^3 \setminus L; \mathbb{Z})$*. Moreover, if* Ω *is generic in the sense of Definition 2.2, then* $\mathbf{r}$ *is generic in the sense of Definition 2.1.*

Proof of Lemma 2.3. The first statement is true because, for each $j = 1, \dots, n$, the two marked points with weights $\pm\Omega(j)$ lie on the same component of L, so the sum of the weights on each component is 0.

For the second statement, let $I \in \{0,1\}^n$ be such that $\mathcal{D}_I$ is disconnected, and call its components $\mathcal{D}_I^1, \dots, \mathcal{D}_I^{\ell_I}$. Let $i \in \{1, \dots, \ell_I\}$, and suppose that $p_{a_1}, \dots, p_{a_k}$ are the marked points on $\mathcal{D}_I^i$. Suppose, toward a contradiction, that

$$r_{a_1} + \cdots + r_{a_k} = 0. \tag{2.1}$$

By Definition 2.2, the nonzero terms on the left-hand side of (2.1) are distinct elements of the set $\{\pm\Omega(1), \ldots, \pm\Omega(n)\}$, so (2.1) gives a linear relation among $\Omega(1), \ldots, \Omega(n)$ with coefficients in $\{-1, 0, 1\}$. Because the diagram $\mathcal{D}$ is connected, there is some crossing c_j which connects $\mathcal{D}_I^i$ with some other component of $\mathcal{D}_I$. By Definition 2.2, one of the two marked points with weight $\pm\Omega(j)$ is on $\mathcal{D}_I^i$ and one is not. Therefore, the coefficient of $\Omega(j)$ in (2.1) is nonzero, which contradicts the genericity of Ω. □

Henceforth, we fix a generic system of weights $\mathbf{r}$, not necessarily arising from Definition 2.2.

Let $\mathbb{F}[\mathbb{Z}]$ denote the mod-2 group ring of the integers, which we think of as the ring of Laurent polynomials in T with coefficients in $\mathbb{F}$. As in the Introduction, let $\mathcal{F} = \mathbb{F}(T)$ denote the field of rational functions in T over $\mathbb{F}$; this equals the fraction field of $\mathbb{F}[\mathbb{Z}]$. Let $\mathcal{Y}$ denote the vector space over $\mathcal{F}$ generated freely by $y_1, \ldots, y_m$.

Let $\mathcal{R}(\mathcal{D})$ denote the set of $I \in \{0,1\}^n$ for which $\mathcal{D}_I$ is connected. For each $I \in \mathcal{R}(\mathcal{D})$, let $\sigma_I \in \mathfrak{S}_m$ be the permutation of $\{1, \ldots, m\}$ such that $\sigma_I(1) = 1$ and such that the marked points are ordered $p_{\sigma_I(1)}, \ldots, p_{\sigma_I(m)}$ according to the orientation on $\mathcal{D}_I$. Let $\mathcal{Y}_I$ be the quotient of $\mathcal{Y}$ by the relation

$$\sum_{i=1}^{m} T^{r_{\sigma_I}(1)+\cdots+r_{\sigma_I}(i)} y_{\sigma_I(i)} = 0, \tag{2.2}$$

so that $\dim_{\mathcal{F}}(\mathcal{Y}_I) = m - 1$. That is, the power of T in the coefficient of y_j is the sum of the weights of the marked points on the oriented segment of $\mathcal{K}_I$ from p_1 to p_j, including the endpoints. Note that the coefficient of $y_{\sigma_I(m)}$ in (2.2) is always 1, since $\sum_{i=1}^m r_i = 0$.

For $I, I'' \in \mathcal{R}(\mathcal{D})$, we say that I'' is a *double successor* of I if it is obtained from I by changing two 0s to 1s. For every such pair I, I'', we shall define a linear map

$$d_{I,I''} \colon \Lambda^*(\mathcal{Y}_I) \to \Lambda^*(\mathcal{Y}_{I''}),$$

as follows. Suppose j_1 and j_2 are the two coordinates in which I and I'' differ, and let I^1 (resp. I^2) be the tuple obtained from I by changing its j_1th (resp. j_2th) coordinate from a 0 to a 1. Without loss of generality, let us assume that σ_I is the identity. Choose $1 \le a < b < c < d \le m$ so that a, c are the special indices associated to c_{j_1} and b, d are the special indices associated to c_{j_2}. In particular, this establishes which crossing is c_{j_1} and which is c_{j_2}; see Fig. 4 for an example.

In $\mathcal{D}_{I^1}$, the marked points on one component are $p_1, \ldots, p_a, p_{c+1}, \ldots, p_m$, and those on the other are $p_{a+1}, \ldots, p_c$, ordered according to the orientation of $\mathcal{D}_{I^1}$. Likewise, the marked points on the two components of $\mathcal{D}_{I^2}$ are $p_1, \ldots, p_b, p_{d+1}, \ldots, p_m$ and $p_{b+1}, \ldots, p_d$. Let

$$A = \sum_{i=1}^{a} r_i, \qquad B = \sum_{i=a+1}^{b} r_i, \qquad C = \sum_{i=b+1}^{c} r_i, \qquad D = \sum_{i=c+1}^{d} r_i.$$

The weights of the components of $\mathcal{D}_{I^1}$ and $\mathcal{D}_{I^2}$ that do not contain p_1 are $B + C$ and $C + D$, respectively. The genericity of $\mathbf{r}$ guarantees that these two numbers are nonzero.

In defining the map $d_{I,I''}$, there are two cases to consider: either

$$\gamma_{I^1} = \gamma_I \cup e_{j_1} \quad \text{or} \quad \gamma_{I^1} = \gamma_I \setminus e_{j_1}.$$

We shall distinguish these cases with a number $\nu = \nu_{I,I''} \in \{0,1\}$, defined to be 1 in the first case and 0 in the second.

Definition 2.4. The map $d_{I,I''}$ is the sum

$$d_{I,I''} = d^{1,1}_{I,I''} + d^{1,2}_{I,I''} + d^{2,1}_{I,I''} + d^{2,2}_{I,I''},$$

where $d^{k,l}_{I,I''}\colon \Lambda^*(\mathcal{Y}_I) \to \Lambda^*(\mathcal{Y}_{I''})$ are the $\mathcal{F}$-linear maps defined by the rules (omitting the subscripts for convenience)

$$d^{1,1}(1) = 0 \tag{2.3}$$

$$d^{1,2}(1) = \frac{T^{\nu C}}{1+T^{C+D}} \tag{2.4}$$

$$d^{2,1}(1) = \frac{T^{B+\nu C}}{1+T^{B+C}} \tag{2.5}$$

$$d^{2,2}(1) = \frac{T^{-A+\nu C}}{(1+T^{B+C})(1+T^{C+D})} \sum_{i=1}^{m} T^{r_1+\dots+r_i} y_i, \tag{2.6}$$

and for any monomial x in $y_1, \dots, y_m$, any $i = 1, \dots, m$, and any $k, l \in \{1, 2\}$,

$$d^{k,l}(xy_i) = \begin{cases} d^{1,l}(x)y_i + d^{2,l}(x) & \text{if } k = 1 \text{ and } i \in \{a, c\} \\ d^{k,1}(x)y_i + d^{k,2}(x) & \text{if } l = 1 \text{ and } i \in \{b, d\} \\ d^{k,l}(x)y_i & \text{otherwise.} \end{cases} \tag{2.7}$$

To be more precise, viewing $\Lambda^*(\mathcal{Y}_I)$ and $\Lambda^*(\mathcal{Y}_{I''})$ as modules over the exterior algebra $\Lambda^*(\mathcal{Y})$, $d^{2,2}$ is defined to be the $\Lambda^*(\mathcal{Y})$-module homomorphism determined by (2.6). Since the right-hand side of (2.6) is a multiple of the defining relator for $\mathcal{Y}_I$ given in (2.2), $d^{2,2}$ is well defined. Next, $d^{1,2}$ and $d^{2,1}$ are defined on all monomials by induction on degree using (2.4), (2.5), and (2.7). To check that these are well defined – i.e., that they vanish on multiples of the defining relator for $\mathcal{Y}_I$ – note that the values of $d^{1,2}(1)$ and $d^{2,1}(1)$ are chosen such that

$$d^{1,2}\left(\sum_{i=1}^{m} T^{r_1+\dots+r_i} y_i\right) = d^{1,2}(1) \sum_{i=1}^{m} T^{r_1+\dots+r_i} y_i + (T^A + T^{A+B+C}) d^{2,2}(1) = 0 \tag{2.8}$$

$$\begin{aligned} d^{2,1}\left(\sum_{i=1}^{m} T^{r_1+\dots+r_i} y_i\right) &= d^{2,1}(1) \sum_{i=1}^{m} T^{r_1+\dots+r_i} y_i + (T^{A+B} + T^{A+B+C+D}) d^{2,2}(1) \\ &= 0. \end{aligned} \tag{2.9}$$

Induction using (2.7) then shows that $d^{1,2}$ and $d^{2,1}$ vanish on any expression of the form

$$y_{i_1} \cdots y_{i_k} \sum_{i=1}^{m} T^{r_1+\dots+r_i} y_i,$$

as required. Finally, $d^{1,1}$ is defined on all monomials by induction on degree using (2.3) and (2.7), and the proof of well-definedness goes through in the same way.

Remark 2.5. The map $d^{1,1}$ decreases degree (of polynomials in the y_i) by 1, $d^{1,2}$ and $d^{2,1}$ preserve degree, and $d^{2,2}$ increases degree by 1. Knowing just this, the total map $d_{I,I''}$ is determined up to an overall scalar by (2.2) and (2.7), since the value of $d^{2,2}(1)$ is forced to be a multiple of the relator on $\mathcal{Y}_I$, and the values of $d^{1,2}(1)$ and $d^{2,1}(1)$ are forced in order for

(2.8) and (2.9) to hold. In particular, the maps in the two cases distinguished by ν differ only by an overall factor of T^C. (Compare Section 7.2.)

We now define the complex $(C^{\mathbf{r}}(\mathcal{D}), \partial^{\mathbf{r}})$ as follows.

Definition 2.6. Define

$$C^{\mathbf{r}}(\mathcal{D}) = \bigoplus_{I \in \mathcal{R}(\mathcal{D})} \Lambda^*(\mathcal{Y}_I),$$

where $\Lambda^*(\mathcal{Y}_I)$ is supported in the grading $(|I| - n_-(\mathcal{D}))/2$, and let $\partial^{\mathbf{r}}$ be the direct sum of the maps $d_{I,I''}\colon \Lambda^*(\mathcal{Y}_I) \to \Lambda^*(\mathcal{Y}_{I''})$. If $\mathbf{r} = \mathbf{r}_\Omega$ as in Definition 2.2, we denote $(C^{\mathbf{r}}(\mathcal{D}), \partial^{\mathbf{r}})$ by $(C^\Omega(\mathcal{D}), \partial^\Omega)$, as in the Introduction.

The fact that $\partial^{\mathbf{r}}$ squares to zero will be established at the end of Section 7, when we identify $(C^{\mathbf{r}}(\mathcal{D}), \partial^{\mathbf{r}})$ with the E^2 page of the cube of resolutions spectral sequence that we construct below. A more general version of Theorem 1.1 is then as follows.

Theorem 2.7. *The homology of $(C^{\mathbf{r}}(\mathcal{D}), \partial^{\mathbf{r}})$ is isomorphic as a graded $\mathcal{F}$-vector space to*

$$\underline{\widehat{\mathrm{HFK}}}(L, \omega_{\mathbf{r}}; \mathcal{F}) \otimes_{\mathcal{F}} (V^{\otimes(m-|L|)} \otimes_{\mathbb{F}} \mathcal{F}),$$

where $\underline{\widehat{\mathrm{HFK}}}(L, \omega_{\mathbf{r}}; \mathcal{F})$ denotes the twisted knot Floer homology of L with perturbation $\omega_{\mathbf{r}}$, equipped with its δ-grading. (See Proposition 3.4 for a precise definition of this invariant.)

When $\mathbf{r} = \mathbf{r}_\Omega$, we have $\omega_{\mathbf{r}} = 0$, so $\underline{\widehat{\mathrm{HFK}}}(L, \omega_{\mathbf{r}}; \mathcal{F})$ is simply the untwisted knot Floer homology, tensored with $\mathcal{F}$, giving Theorem 1.1.

Example 2.8. Let $\mathcal{D}$ be the diagram for the two-component unlink L shown in Fig. 5, whose cube of resolutions is precisely Fig. 4 with $a = 1$, $b = 2$, $c = 3$, and $d = m = 4$. The connected resolutions of $\mathcal{D}$ correspond to $I = (0, 0)$ and $I'' = (1, 1)$; note that $\nu_{I,I''} = 1$. For ease of notation, define $r = r_1 = A$, $s = r_2 = B$, $t = r_3 = C$, and $u = r_4 = D$. The defining relations on $\mathcal{Y}_I$ and $\mathcal{Y}_{I''}$ give

$$\begin{aligned} &T^r y_1 + T^{r+s} y_2 + T^{r+s+t} y_3 + T^{r+s+t+u} y_4 = 0 \quad \text{in } \mathcal{Y}_I \\ &T^r y_1 + T^{r+u} y_4 + T^{r+t+u} y_3 + T^{r+s+t+u} y_2 = 0 \quad \text{in } \mathcal{Y}_{I''}. \end{aligned}$$

We shall use these relations and the fact that $r+s+t+u = 0$ to eliminate y_4 wherever it appears. For conciseness, we define

$$\lambda = \frac{1}{1+T^{t+u}} \quad \text{and} \quad \mu = \frac{T^{s+t}}{1+T^{s+t}},$$

so that

$$\begin{aligned} d^{2,2}(1) &= T^t \lambda\mu (y_1 + T^s y_2 + T^{s+t} y_3 + T^{s+t+u} y_4) \\ &= T^t \lambda\mu (y_1 + T^s y_2 + T^{s+t} y_3 + T^{s+t+u} (T^t y_3 + T^{s+t} y_2 + T^{r+s+t} y_1)) \\ &= T^t \lambda\mu ((1 + T^{s+t}) y_1 + T^s (1 + T^{s+2t+u}) y_2 + T^{s+t} (1 + T^{t+u}) y_3) \\ &= T^t \lambda\mu (\mu^{-1} y_1 + T^{2s+t} (\mu^{-1} + \lambda^{-1}) y_2 + T^{s+t} \lambda^{-1} y_3) \\ &= T^t (\lambda y_1 + T^s (\lambda + \mu) y_2 + \mu y_3). \end{aligned}$$

Using the inductive procedure described above, we can see that the values of the four functions $d^{1,1}$, $d^{1,2}$, $d^{2,1}$, and $d^{2,2}$ on a basis for $\Lambda^*(Y_I)$ are as follows.

x	$d^{2,2}(x)$
1	$T^t\lambda\, y_1 + T^{s+t}(\lambda+\mu)\, y_2 + T^t\mu\, y_3$
y_1	$T^{s+t}(\lambda+\mu)\, y_1y_2 + T^t\mu\, y_1y_3$
y_2	$T^t\lambda\, y_1y_2 + T^t\mu\, y_2y_3$
y_3	$T^t\lambda\, y_1y_3 + T^{s+t}(\lambda+\mu)\, y_2y_3$
y_1y_2	$T^t\mu\, y_1y_2y_3$
y_1y_3	$T^{s+t}(\lambda+\mu)\, y_1y_2y_3$
y_2y_3	$T^t\lambda\, y_1y_2y_3$
$y_1y_2y_3$	0

x	$d^{1,2}(x)$	$d^{2,1}(x)$
1	$T^t\lambda$	μ
y_1	$T^{s+t}(\lambda+\mu)\, y_2 + T^t\mu\, y_3$	$\mu\, y_1$
y_2	$T^t\lambda\, y_2$	$T^t\lambda\, y_1 + T^{s+t}(1+\lambda)\, y_2 + T^t\mu\, y_3$
y_3	$T^t\lambda\, y_1 + T^{s+t}(\lambda+\mu)\, y_2 + T^t(\lambda+\mu)\, y_3$	$\mu\, y_3$
y_1y_2	$T^t\mu\, y_2y_3$	$T^{s+t}(1+\lambda)\, y_1y_2 + T^t\mu\, y_1y_3$
y_1y_3	$T^{s+t}(\lambda+\mu)\, y_1y_2 + T^t\mu\, y_1y_3 + T^{s+t}(\lambda+\mu)\, y_2y_3$	$\mu\, y_1y_3$
y_2y_3	$T^t\lambda\, y_1y_2 + T^t(\lambda+\mu)\, y_2y_3$	$T^t\lambda\, y_1y_3 + T^{s+t}(1+\lambda)\, y_2y_3$
$y_1y_2y_3$	$T^t\mu\, y_1y_2y_3$	$T^{s+t}(1+\lambda)\, y_1y_2y_3$

x	$d^{1,1}(x)$
1	0
y_1	μ
y_2	$T^t\lambda$
y_3	μ
y_1y_2	$T^{s+t}(1+\lambda)\, y_2 + T^t\mu\, y_3$
y_1y_3	$\mu\, y_1 + \mu\, y_3$
y_2y_3	$T^t\lambda\, y_1 + T^{s+t}(1+\lambda)\, y_2 + T^t(\lambda+\mu)\, y_3$
$y_1y_2y_3$	$T^{s+t}(1+\lambda)\, y_1y_2 + T^t\mu\, y_1y_3 + T^{s+t}(1+\lambda)\, y_2y_3$

If the weights are determined by a generic function $\Omega\colon \{1,2\} \to \mathbb{Z}$ as in Definition 2.2, we have $r=-t$ and $s=-u$, while $s \neq \pm t$ and both s and t are nonzero. In this case, some linear algebra shows that d has rank 4, with kernel generated by the following four elements of $\Lambda^*(\mathcal{Y}_I)$:

$$\begin{aligned}
&(1+T^s+T^t+T^{s+t}) + (T^{-t}+T^s)\, y_1 + (T^{s-t}+T^{s+t})\, y_2 + (T^s+T^t)\, y_3\\
&(T^s+T^t) + (1+T^s+T^t+T^{s+t})\, y_1 + (T^{s-t}+T^{s+t})\, y_1y_2 + (T^s+T^t)\, y_1y_3\\
&(1+T^{s+t}) + (1+T^s+T^t+T^{s+t})\, y_2 + (T^{-t}+T^s)\, y_1y_2 + (T^s+T^t)\, y_2y_3\\
&(1+T^{s+t})\, y_1 + (T^s+T^t)\, y_2 + (1+T^s+T^t+T^{s+t})\, y_1y_2 + (T^s+T^t)\, y_1y_2y_3.
\end{aligned}$$

Thus, $H_*(C(\mathcal{D}), \partial^\Omega)$ has dimension 8, supported in gradings $\pm 1/2$, which agrees with the fact that $\widehat{\mathrm{HFK}}(L)$ is two dimensional, supported in δ-gradings $\pm 1/2$. On the other hand, if the weights are chosen such that $r \neq -t$ or $s \neq -u$, while $r+s+t+u=0$, it is not hard to show that d is an isomorphism, so the homology vanishes. This is consistent with the fact that, by Proposition 4.2, the twisted knot Floer homology group $\underline{\widehat{\mathrm{HFK}}}(L, \omega_{\mathbf{r}}; \mathcal{F})$ vanishes since the cohomology class $\omega_{\mathbf{r}}$ is nonzero in this case.

The preceding example can be generalized to show that the maps that make up ∂^{Ω} are almost always isomorphisms, as follows.

Lemma 2.9. *Let $\mathcal{D}$ be a diagram with $n \geq 3$ crossings for a knot or a nonsplit link, and let $\mathbf{r}$ be the system of weights coming from a generic function $\Omega\colon \{1, \ldots, n\}$. For any double successor pair I, I'', the map $d^{I,I''}\colon \Lambda^*(\mathcal{Y}_I) \to \Lambda^*(\mathcal{Y}_{I''})$ is an isomorphism.*

Proof. Without loss of generality, assume that $\gamma_{I^1} = \gamma_I \cup e_{j_1}$ and $\gamma_{I^2} = \gamma_I \setminus e_{j_2}$ and that σ_I is the identity permutation. Just as in Example 2.8, the mapping cone of $d^{I,I''}$ can be identified with the complex associated to a two-crossing diagram of the two-component unlink Q with the same m marked points, using the same choice of weights $\mathbf{r}$. By Theorem 2.7 and Proposition 4.2, it suffices to show that the associated cohomology class $\omega_{\mathbf{r}}$ has nonzero value on a generator of $H_2(S^3 \setminus Q; \mathbb{Z}) \cong \mathbb{Z}$.

Suppose, toward a contradiction, that

$$r_1 + \cdots + r_a + r_{b+1} + \cdots + r_c + r_{d+1} + \cdots + r_m = 0. \tag{2.10}$$

Note that the left-hand side of this equation automatically equals

$$-r_{a+1} - \cdots - r_b - r_{c+1} - \cdots - r_d.$$

Just as in the proof of Lemma 2.3, (2.10) is a linear relation among $\Omega(1), \ldots, \Omega(n)$ with coefficients in $\{-1, 0, 1\}$, and we must show that at least one of these coefficients is nonzero, which will contradict the genericity of Ω. Because $\mathcal{D}$ represents a nonsplit link, there is some crossing c_{j_3} in $\mathcal{D}$ whose trace connects the two components of Q. Therefore, the marked points with weights $\pm\Omega(j_3)$ are on different components of S. It follows that the sum on the left-hand side of (2.10) includes a non-canceling $\pm\Omega(j_3)$ term. □

As a corollary, we may describe a family of knots whose δ-graded knot Floer homology and reduced Khovanov homology are isomorphic. For a projection $\mathcal{D}$, let $\Gamma_{\mathcal{D}}$ denote the directed graph with vertices corresponding to $\mathcal{R}(\mathcal{D})$ and with an edge from I to I'' whenever I'' is a double successor of I.

Corollary 2.10. *Let K be a knot, and suppose that K admits a projection $\mathcal{D}$ such that $\Gamma_{\mathcal{D}}$ is a disjoint union of trees. Then $\widehat{\mathrm{HFK}}(K)$ and $\widetilde{\mathrm{Kh}}(K)$, equipped with their δ-gradings, are isomorphic.*

Proof. Say that $\mathcal{D}$ has n crossings, and put exactly one marked point on each of the $2n$ edges of $\mathcal{D}$. Choose a generic function $\Omega\colon \{1, \ldots, n\} \to \mathbb{Z}$ and consider the complex $(C(\mathcal{D}), \partial^{\Omega})$. If $\Gamma_{\mathcal{D}}$ is a disjoint union of trees, then we may inductively find bases for the vector spaces $\Lambda^*(\mathcal{Y}_I)$ with respect to which each map $d_{I,I''}$ is represented by the $2^{2n-1} \times 2^{2n-1}$ identity matrix. Thus, $(C(\mathcal{D}), \partial^{\Omega})$ splits as a direct sum of 2^{2n-1} copies of $X \otimes_{\mathbb{F}} \mathcal{F}$, where X is a complex generated freely over $\mathbb{F}$ by $\mathcal{R}(\mathcal{D})$ in which the differential of $I \in \mathcal{R}(\mathcal{D})$ is equal to the sum of the double successors of I. (Although we could define X in this manner for any link projection, in general the differential may not square to zero.)

The same argument can be used to show that Roberts' spanning tree complex $(C'(\mathcal{D}), \partial')$ is isomorphic to $X \otimes_{\mathbb{F}} \mathcal{F}'$, where $\mathcal{F}'$ is the field of rational functions in multiple indeterminates over which $C'(\mathcal{D})$ is defined [44]. By the universal coefficient theorem, we have

$$H_*(C^{\Omega}(\mathcal{D}), \partial^{\Omega}) \cong \bigoplus^{2^{2n-1}} H_*(X) \otimes_{\mathbb{F}} \mathcal{F} \quad \text{and} \quad H_*(C'(\mathcal{D}), \partial') \cong H_*(X) \otimes_{\mathbb{F}} \mathcal{F}'.$$

Since these homology groups are isomorphic to $\widehat{\mathrm{HFK}}(K) \otimes_{\mathbb{F}} V^{\otimes 2n-1} \otimes_{\mathbb{F}} \mathcal{F}$ and $\widetilde{\mathrm{Kh}}(K) \otimes_{\mathbb{F}} \mathcal{F}'$, respectively, the result follows. □

Via the Gordon–Litherland signature formula [10], Corollary 2.10 can be used to give a new proof of the fact that, for an alternating knot K, $\widetilde{\mathrm{Kh}}(K)$ and $\widehat{\mathrm{HFK}}(K)$ are both thin and supported in δ-grading $-\sigma(K)/2$.

3. Background on knot Floer homology

In this section, we review the construction of knot Floer homology with twisted coefficients and multiple basepoints, and we describe the maps on knot Floer homology induced by counting pseudo-holomorphic polygons. In Section 3.3, we describe some additional algebraic structure which comes from counting disks that pass over basepoints. We shall assume throughout that the reader has some familiarity with knot Floer homology; for a more basic treatment, see [36,41] and [37, Section 8].

3.1. Multiple basepoints and twisted coefficients

Recall that a *multi-pointed* Heegaard diagram is a tuple $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbb{O}, \mathbb{X})$, where

- Σ is an Riemann surface of genus g,
- $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_{g+m-1}\}$ and $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_{g+m-1}\}$ are sets of pairwise disjoint, simple closed curves on Σ which span g-dimensional subspaces of $H_1(\Sigma; \mathbb{Z})$, and
- $\mathbb{O} = (O_1, \ldots, O_m)$ and $\mathbb{X} = (X_1, \ldots, X_m)$ are tuples of basepoints such that every component of $\Sigma \setminus \boldsymbol{\alpha}$ and $\Sigma \setminus \boldsymbol{\beta}$ contains exactly one point of $\mathbb{O}$ and one of $\mathbb{X}$.

The sets $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ specify handlebodies U_α and U_β with $\partial U_\alpha = \Sigma = -\partial U_\beta$. Let Y denote the 3-manifold with Heegaard decomposition $U_\alpha \cup_\Sigma U_\beta$. $\mathcal{H}$ determines an oriented link $L \subset Y$ according to the following procedure. Fix m disjoint, oriented, embedded arcs in $\Sigma \setminus \boldsymbol{\alpha}$ from points in $\mathbb{O}$ to points in $\mathbb{X}$, and form $\xi_1^\alpha, \ldots, \xi_m^\alpha$ by pushing their interiors into U_α. Similarly, define pushoffs $\xi_1^\beta, \ldots, \xi_m^\beta$ in U_β of oriented arcs in $\Sigma \setminus \boldsymbol{\beta}$ from points in $\mathbb{X}$ to points in $\mathbb{O}$. L is the union

$$L = \xi_1^\alpha \cup \cdots \cup \xi_m^\alpha \cup \xi_1^\beta \cup \cdots \cup \xi_m^\beta.$$

The tuple $\mathbb{X}$ also determines an ordered marking $\mathbf{p} = (p_1, \ldots, p_m)$ on L.

The pair $\mathcal{L} = (L, \mathbf{p})$ is called an *m-pointed link*, and we say that $\mathcal{H}$ is a *compatible* Heegaard diagram for $\mathcal{L}$. More generally, an m-pointed link is an oriented link together with a marking $\mathbf{p}$ such that every component contains some p_i. We consider two such links $(L, \mathbf{p})$ and $(L', \mathbf{p}')$ to be equivalent if there is an orientation-preserving diffeomorphism of Y sending L to L' and $\mathbf{p}$ to $\mathbf{p}'$. A standard Morse-theoretic argument shows that every pointed link arises from a Heegaard diagram as above, and that compatible Heegaard diagrams for equivalent pointed links can be connected via a sequence of index one/two (de)stabilizations, and isotopies and handleslides avoiding $\mathbb{O} \cup \mathbb{X}$.

Following [37], we view

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_{g+m-1} \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g+m-1}$$

as tori in the symmetric product $\mathrm{Sym}^{g+m-1}(\Sigma)$. For $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we denote by $\pi_2(\mathbf{x}, \mathbf{y})$ the set of homotopy classes of Whitney disks from $\mathbf{x}$ to $\mathbf{y}$. For $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ and $a \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, let $a(\phi)$ be the algebraic intersection number

$$\#(\phi \cap (\{a\} \times \mathrm{Sym}^{g+m-2}(\Sigma))).$$

Label the regions of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ by $D_1, \dots, D_k$ and choose a point z_i in each D_i. The *domain* of ϕ is the formal $\mathbb{Z}$-linear combination

$$D(\phi) = \sum_{i=1}^{k} z_i(\phi) D_i.$$

More generally, we refer to any linear combination

$$D = \sum_{i=1}^{k} a_i D_i$$

as a domain, and we define $a(D)$ to be a_i if a and z_i are in the same component of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$.

A *periodic domain* is a domain whose boundary is a union of closed curves in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Periodic domains form a group $\Pi_{\alpha\beta}$ under addition. The subgroup $\Pi^0_{\alpha\beta}$ of $\Pi_{\alpha\beta}$ consisting of periodic domains which avoid $\mathbb{O} \cup \mathbb{X}$ is isomorphic to $H_2(Y \setminus L; \mathbb{Z})$. The diagram $\mathcal{H}$ is said to be *admissible* if every nontrivial element of $\Pi^0_{\alpha\beta}$ has both positive and negative coefficients.

To define a system of twisted coefficients, we fix a collection $\mathbb{A}$ of points in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ together with a function $\omega : \mathbb{A} \to \mathbb{Z}$, and we let

$$\langle \omega, \phi \rangle = \sum_{a \in \mathbb{A}} a(\phi)\omega(a) \tag{3.1}$$

for any $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. The map $\langle \omega, \cdot \rangle$ restricts to a linear functional on $\Pi^0_{\alpha\beta}$ and therefore determines a cohomology class $[\omega] \in H^2(Y \setminus L; \mathbb{Z})$.

Now, suppose that $\mathcal{H}$ is admissible and let $\mathcal{M}$ be a module over $\mathbb{F}[\mathbb{Z}]$. The twisted knot Floer complex with coefficients in $\mathcal{M}$ is defined as

$$\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega; \mathcal{M}) = \mathbb{F}[\mathbb{Z}]\langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle \otimes_{\mathbb{F}[\mathbb{Z}]} \mathcal{M},$$

with differential given by

$$\partial(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi)=1 \\ O_i(\phi)=X_i(\phi)=0\ \forall i}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot T^{\langle \omega, \phi \rangle} \mathbf{y}.$$

Here, $\mu(\phi)$ is the Maslov index of ϕ and $\mathcal{M}(\phi)$ is the moduli space of pseudo-holomorphic representatives of ϕ.

Henceforth, we shall assume that L is null-homologous. Define

$$O(\phi) = O_1(\phi) + \cdots + O_m(\phi), \qquad X(\phi) = X_1(\phi) + \cdots + X_m(\phi),$$
$$P(\phi) = O(\phi) + X(\phi).$$

If $\mathbf{x}$ represents a torsion Spin^c structure on Y, then it has an *Alexander* grading $A(\mathbf{x}) \in \mathbb{Z}$ and a *Maslov* grading $M(\mathbf{x}) \in \mathbb{Q}$. Following [28,45], we define the δ-grading of $\mathbf{x}$ to be $\delta(\mathbf{x}) = a(\mathbf{x}) - m(\mathbf{x})$. If $\mathbf{x}$ and $\mathbf{y}$ represent the same torsion Spin^c structure on Y, then their gradings are related as follows:

$$M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2O(\phi) \tag{3.2}$$

$$A(\mathbf{x}) - A(\mathbf{y}) = X(\phi) - O(\phi) \tag{3.3}$$

$$\delta(\mathbf{x}) - \delta(\mathbf{y}) = P(\phi) - \mu(\phi), \tag{3.4}$$

for any $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$.

Remark 3.1. Note that the relative δ-grading in (3.4) does not depend on which basepoints are in $\mathbb{O}$ and which are in $\mathbb{X}$, which is to say, on the orientation of L. In contrast, the relative Maslov and Alexander gradings and the absolute δ-grading *do* generally depend on the orientation of L.

Remark 3.2. For $\mathcal{M} = \mathbb{F}[\mathbb{Z}]/(T-1) \cong \mathbb{F}$, the complex $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega; \mathcal{M})$ does not depend on the marking $(\mathbb{A}, \omega)$. We refer to this as the *untwisted* knot Floer complex, $\widetilde{\mathrm{CFK}}(\mathcal{H})$.[4]

The following is a straightforward adaptation of [42, Lemma 2.2].

Lemma 3.3. *For markings $(\mathbb{A}, \omega)$ and $(\mathbb{A}', \omega')$ such that $[\omega] = [\omega']$ in $H^2(Y \setminus L; \mathbb{Z})$, the complexes $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega; \mathcal{M})$ and $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega'; \mathcal{M})$ are isomorphic.*

Proof of Lemma 3.3. For each relative Spin^c structure $\underline{\mathfrak{s}}$ on $Y \setminus L$, fix some generator $\mathbf{x}_{\underline{\mathfrak{s}}} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ which represents $\underline{\mathfrak{s}}$. For any other generator $\mathbf{x}$ representing $\underline{\mathfrak{s}}$, there exists a Whitney disk $\phi \in \pi_2(\mathbf{x}_{\underline{\mathfrak{s}}}, \mathbf{x})$ which avoids $\mathbb{X} \cup \mathbb{O}$. Let

$$\epsilon_{\underline{\mathfrak{s}}}(\mathbf{x}) = \langle \omega', \phi \rangle - \langle \omega, \phi \rangle. \tag{3.5}$$

Since $[\omega] = [\omega']$, $\langle \omega, D \rangle = \langle \omega', D \rangle$ for all periodic domains $D \in \Pi^0_{\alpha,\beta}$, which implies that the quantity in (3.5) does not depend on our choice of ϕ. Finally, let

$$f : \underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega; \mathcal{M}) \to \underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega'; \mathcal{M})$$

be the linear map which sends a generator $\mathbf{x}$ representing $\underline{\mathfrak{s}}$ to $f(\mathbf{x}) = T^{\epsilon_{\underline{\mathfrak{s}}}(\mathbf{x})} \cdot \mathbf{x}$. It is easy to check that f is a chain map, and it is obviously an isomorphism. □

Suppose that $\mathcal{H}$ and $\mathcal{H}'$ are compatible Heegaard diagrams for $\mathcal{L}$, with markings $(\mathbb{A}, \omega)$ and $(\mathbb{A}', \omega')$, respectively. As mentioned above, $\mathcal{H}$ and $\mathcal{H}'$ are related by a sequence of index one/two (de)stabilizations, and isotopies and handleslides avoiding $\mathbb{O} \cup \mathbb{X}$. These Heegaard moves induce a bijection ρ between periodic domains of $\mathcal{H}$ and those of $\mathcal{H}'$ (which restricts to a bijection between periodic domains that avoid $\mathbb{X} \cup \mathbb{O}$).

Proposition 3.4. *If $\langle \omega, P \rangle = \langle \omega', \rho(P) \rangle$ for all periodic domains P in $\Pi^0_{\alpha,\beta}$, then the complexes $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega; \mathcal{M})$ and $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}', \omega'; \mathcal{M})$ are quasi-isomorphic. Therefore, the homology*

$$\underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}, [\omega]; \mathcal{M}) = H_*(\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega; \mathcal{M}), \partial)$$

depends only on the m-pointed link $\mathcal{L}$ and $[\omega]$. (When each component of $\mathcal{L}$ has a single basepoint, we denote this group by $\underline{\widehat{\mathrm{HFK}}}(L, [\omega]; \mathcal{M})$.)

[4] When $\mathcal{H}$ is a grid diagram for a link L in S^3, $\widetilde{\mathrm{CFK}}(\mathcal{H})$ is just the complex $\widetilde{CL}(\mathcal{H})$ defined in [30].

Proof of Proposition 3.4. It is not always possible to perform the above Heegaard moves while avoiding $\mathbb{A}$ — an isotopy might get "stuck" on a point of $\mathbb{A}$ as in Fig. 6(a). Modifying the marking as in Fig. 6(b) does not change the associated cohomology class, but allows one to proceed with the isotopy in the complement of the new marking. In this way, the triple $(\mathcal{H}', \mathbb{A}', \omega')$ may be obtained from $(\mathcal{H}, \mathbb{A}, \omega)$ via a combination of marking changes which preserve the cohomology class, and Heegaard moves which avoid the basepoints and the markings. These marking changes induce isomorphisms, by Lemma 3.3. Moreover, the standard Heegaard Floer arguments [37] show that these Heegaard moves induce quasi-isomorphisms, and that the chain homotopy type of $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega; \mathcal{M})$ is invariant under changes of almost-complex structure. □

When $[\omega] = 0$, as for a knot $L \subset S^3$, we may choose $\mathbb{A}$ to be the empty set. Therefore,

$$\underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}, 0; \mathcal{M}) \cong \widetilde{\mathrm{HFK}}(\mathcal{L}) \otimes_{\mathbb{F}} \mathcal{M}, \tag{3.6}$$

where $\widetilde{\mathrm{HFK}}(\mathcal{L})$ denotes the homology of $\widetilde{\mathrm{CFK}}(\mathcal{H})$. Moreover, it is well known that

$$\widetilde{\mathrm{HFK}}(\mathcal{L}) \cong \widehat{\mathrm{HFK}}(L) \otimes_{\mathbb{F}} V^{\otimes(m-|L|)}, \tag{3.7}$$

where V is a two-dimensional vector space over $\mathbb{F}$ supported in the (m, a)-bigradings $(0, 0)$ and $(-1, -1)$ (see, e.g., [30] for links in S^3). Combining the isomorphisms in (3.6) and (3.7), we see that

$$\underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}, 0; \mathcal{M}) \cong \widehat{\mathrm{HFK}}(L) \otimes_{\mathbb{F}} V^{\otimes(m-|L|)} \otimes_{\mathbb{F}} \mathcal{M}. \tag{3.8}$$

Furthermore, it is not hard to see that a twisted version holds as well:

$$\underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}, \omega; \mathcal{M}) \cong \underline{\widehat{\mathrm{HFK}}}(L, \omega; \mathcal{M}) \otimes_{\mathbb{F}} V^{\otimes(m-|L|)}. \tag{3.9}$$

We shall generally suppress $\mathcal{M}$ from our notation unless we wish to emphasize the module we are working over. When we state a result about $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$ or $\underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}, [\omega])$, we shall mean that it holds with coefficients in any $\mathcal{M}$. Also, we shall often use $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ to denote $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$, as long as Σ, $\mathbb{O}$, $\mathbb{X}$, and $(\mathbb{A}, \omega)$ are clear from the context.

3.2. Pseudo-holomorphic polygons

A multi-pointed Heegaard *multi-diagram* is a tuple

$$\mathcal{H} = (\Sigma, \boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^n, \mathbb{O}, \mathbb{X})$$

for which each subtuple $(\Sigma, \boldsymbol{\eta}^i, \boldsymbol{\eta}^j, \mathbb{O}, \mathbb{X})$ is a multi-pointed Heegaard diagram of the sort described in Section 3.1. Fix a marking $(\mathbb{A}, \omega)$ on $\mathcal{H}$. For distinct indices $i_1, \dots, i_k$ and intersection points $\mathbf{x}_1 \in \mathbb{T}_{\eta^{i_1}} \cap \mathbb{T}_{\eta^{i_2}}, \dots, \mathbf{x}_{k-1} \in \mathbb{T}_{\eta^{i_{k-1}}} \cap \mathbb{T}_{\eta^{i_k}}$ and $\mathbf{x}_k \in \mathbb{T}_{\eta^{i_1}} \cap \mathbb{T}_{\eta^{i_k}}$, we denote by $\pi_2(\mathbf{x}_1, \dots, \mathbf{x}_k)$ the set of homotopy classes of Whitney k-gons connecting them. For $\phi \in \pi_2(\mathbf{x}_1, \dots, \mathbf{x}_k)$ and $a \in \Sigma \setminus (\boldsymbol{\eta}^1 \cup \dots \cup \boldsymbol{\eta}^n)$, let $a(\phi)$ denote the intersection of ϕ with $\{a\} \times \mathrm{Sym}^{g+m-2}(\Sigma)$, and define the pairing $\langle \omega, \phi \rangle$ as in (3.1).

A *multi-periodic domain* is a formal $\mathbb{Z}$-linear combination of the regions in $\Sigma \setminus (\boldsymbol{\eta}^1 \cup \dots \cup \boldsymbol{\eta}^n)$ whose boundary is a union of curves among the sets $\boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^n$. Let $\Pi_{\eta^1 \dots \eta^n}$ denote the group of multi-periodic domains, and let $\Pi^0_{\eta^1 \dots \eta^n}$ denote the subgroup of $\Pi_{\eta^1 \dots \eta^n}$ consisting of multi-periodic domains that avoid $\mathbb{O} \cup \mathbb{X}$. As before, we say that $\mathcal{H}$ is admissible if every nontrivial element of $\Pi^0_{\eta^1 \dots \eta^n}$ has both positive and negative coefficients.

Suppose that $\mathcal{H}$ is admissible, and let $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}^{i_s}, \boldsymbol{\eta}^{i_t})$ denote the complex associated to $(\Sigma, \boldsymbol{\eta}^{i_s}, \boldsymbol{\eta}^{i_t}, \mathbb{O}, \mathbb{X})$ and $(\mathbb{A}, \omega)$. For $k \geq 3$, we define a map

$$F_{\eta^{i_1}\dots\eta^{i_k}} : \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}^{i_1}, \boldsymbol{\eta}^{i_2}) \otimes \cdots \otimes \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}^{i_{k-1}}, \boldsymbol{\eta}^{i_k}) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}^{i_1}, \boldsymbol{\eta}^{i_k})$$

by

$$F_{\eta^{i_1}\dots\eta^{i_k}}(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_{k-1}) = \sum_{\mathbf{x}_k \in \mathbb{T}_{\eta^{i_1}} \cap \mathbb{T}_{\eta^{i_k}}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k) \\ \mu(\phi) = 3-k \\ O_i(\phi) = X_i(\phi) = 0 \ \forall i}} \#(\mathcal{M}(\phi)) \cdot T^{\langle \omega, \phi \rangle} \mathbf{x}_k. \tag{3.10}$$

Here, $\mathcal{M}(\phi)$ is the moduli space pseudo-holomorphic representatives of ϕ, where the conformal structure on the source is allowed to vary. For a k-gon, this set of conformal structures forms an associahedron of dimension $k-3$, so $\mathcal{M}(\phi)$ has expected dimension zero when $\mu(\phi) = 3-k$.

These $F_{\eta^{i_1}\dots\eta^{i_k}}$ are chain maps when $k = 3$. Counting the ends of the one-dimensional moduli spaces $\mathcal{M}(\phi)$, for all Whitney k-gons ϕ with $\mu(\phi) = 4-k$ and $O_i(\phi) = X_i(\phi) = 0$ for all i, one obtains the $\mathcal{A}_\infty$ relation

$$\sum_{1 \leq s < t \leq k} F_{\eta^{i_1}\dots\eta^{i_s}\eta^{i_t}\dots\eta^{i_k}}(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_{s-1} \otimes F_{\eta^{i_s}\dots\eta^{i_t}}(\mathbf{x}_s \otimes \cdots \otimes \mathbf{x}_{t-1}) \otimes \mathbf{x}_t \otimes \cdots \otimes \mathbf{x}_k) = 0, \tag{3.11}$$

where $F_{\eta^{i_s}\eta^{i_t}}$ is understood to mean the differential on the complex $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}^{i_s}, \boldsymbol{\eta}^{i_t})$.

3.3. The basepoint action

Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbb{O}, \mathbb{X})$ be an admissible multi-pointed Heegaard diagram with marking $(\mathbb{A}, \omega)$. For each $i = 1, \dots, m$, let

$$\Psi_i : \underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega) \to \underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$$

be the map given by

$$\Psi_i(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1 \\ O_j(\phi) = 0 \forall j \\ X_j(\phi) = 0 \forall j \neq i \\ X_i(\phi) = 1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot T^{\langle \omega, \phi \rangle} \mathbf{y}. \tag{3.12}$$

Counting the ends of the moduli spaces $\mathcal{M}(\phi)/\mathbb{R}$, for all Whitney disks ϕ satisfying the basepoint conditions in (3.12) but with $\mu(\phi) = 2$, we find that

$$\Psi_i \circ \partial + \partial \circ \Psi_i = 0.$$

Therefore, Ψ_i is a chain map and it induces a map ψ_i on homology. Similar degeneration arguments show that $\psi_i^2 = 0$ and that $\psi_i \psi_j = \psi_j \psi_i$. Thus, we have an action of the exterior algebra $\Lambda^*(\mathbb{F}[\mathbb{Z}]\langle \psi_1, \dots, \psi_m \rangle)$ on $H_*(\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega), \partial)$. Moreover, a straightforward generalization of [37, Lemma 6.2] shows that ψ_i does not depend on our choices of analytic data. Note that ψ_i lowers Alexander and Maslov gradings by 1 and therefore preserves the δ-grading.

The following is an immediate analogue of Lemma 3.3.

Lemma 3.5. *Suppose* $(\mathbb{A}, \omega)$ *and* $(\mathbb{A}', \omega')$ *are markings on* $\mathcal{H}$ *such that* $\langle \omega, P \rangle = \langle \omega', P \rangle$ *for every periodic domain* P *of* $\mathcal{H}$. *Then there is an isomorphism from* $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$ *to* $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega')$ *which commutes with the action of* $\Lambda^*(\mathbb{F}[\mathbb{Z}]\langle \psi_1, \dots, \psi_m \rangle)$. $\square$

These ψ_i interact nicely with the maps defined by counting higher polygons, as follows. Given an admissible multi-diagram $\mathcal{H} = (\Sigma, \boldsymbol{\eta}^1, \dots, \boldsymbol{\eta}^n, \mathbb{O}, \mathbb{X})$, we let

$$\Psi_i^{\eta^{i_1} \dots \eta^{i_k}} : \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}^{i_1}, \boldsymbol{\eta}^{i_2}) \otimes \dots \otimes \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}^{i_{k-1}}, \boldsymbol{\eta}^{i_k}) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}^{i_1}, \boldsymbol{\eta}^{i_k})$$

be the map which counts pseudo-holomorphic k-gons that pass once over X_i and avoid all other basepoints, in analogy with (3.11). When $k = 2$, $\Psi_i^{\eta^{i_1}\eta^{i_2}}$ is just the map on $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}^{i_2}, \boldsymbol{\eta}^{i_2})$ defined in (3.12). These maps fit into an $\mathcal{A}_\infty$ relation,

$$\begin{aligned}
&\sum_{1 \le s < t \le k} F_{\eta^{i_1} \dots \eta^{i_s} \eta^{i_t} \dots \eta^{i_k}} (\mathbf{x}_1 \otimes \dots \otimes \mathbf{x}_{s-1} \otimes \Psi_j^{\eta^{i_s} \dots \eta^{i_t}} (\mathbf{x}_s \otimes \dots \otimes \mathbf{x}_{t-1}) \otimes \mathbf{x}_t \otimes \dots \otimes \mathbf{x}_k) \\
&\quad + \sum_{1 \le s < t \le k} \Psi_j^{\eta^{i_1} \dots \eta^{i_s} \eta^{i_t} \dots \eta^{i_k}} (\mathbf{x}_1 \otimes \dots \otimes \mathbf{x}_{s-1} \\
&\quad \otimes F_{\eta^{i_s} \dots \eta^{i_t}} (\mathbf{x}_s \otimes \dots \otimes \mathbf{x}_{t-1}) \otimes \mathbf{x}_t \otimes \dots \otimes \mathbf{x}_k) \\
&\quad = 0.
\end{aligned} \tag{3.13}$$

When $k = 3$, writing $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = (\boldsymbol{\eta}^{i_1}, \boldsymbol{\eta}^{i_2}, \boldsymbol{\eta}^{i_3})$, this becomes

$$\begin{aligned}
&F_{\alpha\beta\gamma}(\Psi_j^{\alpha\beta}(\mathbf{x}) \otimes \mathbf{y}) + F_{\alpha\beta\gamma}(\mathbf{x} \otimes \Psi_j^{\beta\gamma}(\mathbf{y})) + \Psi_j^{\alpha\gamma}(F_{\alpha\beta\gamma}(\mathbf{x} \otimes \mathbf{y})) \\
&\quad + \Psi_j^{\alpha\beta\gamma}(\partial_{\alpha\beta}(\mathbf{x}) \otimes \mathbf{y}) + \Psi_j^{\alpha\beta\gamma}(\mathbf{x} \otimes \partial_{\beta\gamma}(\mathbf{y})) + \partial_{\alpha\gamma}(\Psi_j^{\alpha\beta\gamma}(\mathbf{x} \otimes \mathbf{y})) = 0.
\end{aligned}$$

In particular, if $\mathbf{y}$ is a cycle in $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\beta}, \boldsymbol{\gamma})$ and y is its homology class, then the maps f_y and $f_{\psi_j^{\beta\gamma}(y)}$, induced on homology by $F_{\alpha\beta\gamma}(\cdot \otimes \mathbf{y})$ and $F_{\alpha\beta\gamma}(\cdot \otimes \Psi_i^{\beta\gamma}(\mathbf{y}))$, satisfy

$$f_y(\psi_j^{\alpha\beta}(x)) + \psi_j^{\alpha\gamma}(f_y(x)) + f_{\psi_j^{\beta\gamma}(y)}(x) = 0 \tag{3.14}$$

for any $x \in H_*(\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\beta}), \partial_{\alpha\beta})$.

Proposition 3.6. *Suppose* $\mathcal{H}'$ *is obtained from* $\mathcal{H}$ *via an isotopy, handleslide, or index one/two (de)stabilization in the complement of* $\mathbb{A} \cup \mathbb{O} \cup \mathbb{X}$. *Then the induced isomorphism*

$$\Phi : H_*(\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega), \partial) \to H_*(\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}', \omega), \partial)$$

satisfies $\Phi \circ \psi_i = \psi_i \circ \Phi$.

Proof of Proposition 3.6. The isomorphism on knot Floer homology associated to a handleslide is defined by counting pseudo-holomorphic triangles. Consider, for example, the set $\boldsymbol{\beta}' = \{\beta_1', \dots, \beta_{g+m-1}'\}$, where β_1' is obtained by handlesliding β_1 over some β_i, and β_j' is the image of β_j under a small Hamiltonian isotopy for $j = 2, \dots, g+m-1$. Since this handleslide takes place in the complement of $\mathbb{A}$, there is a unique top-dimensional generator $\Theta^{\beta\beta'}$ of $H_*(\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\beta}, \boldsymbol{\beta}'), \partial_{\beta\beta'})$, and the associated isomorphism Φ is just the map $f_{\Theta^{\beta\beta'}}$. It is easy to see that each X_i is in the same region of $\Sigma \setminus (\boldsymbol{\beta} \cup \boldsymbol{\beta}')$ as some O_j. Therefore, the map $\psi_i^{\beta\beta'}$ is identically zero, and (3.14) implies that

$$f_{\Theta\beta\beta'}(\psi_i^{\alpha\beta}(x)) + \psi_i^{\alpha\beta'}(f_{\Theta\beta\beta'}(x)) = 0.$$

Handleslides among the $\boldsymbol{\alpha}$ curves are treated in the same manner.

The isomorphism on knot Floer homology associated to an isotopy may also be defined by counting pseudo-holomorphic triangles [26,43] (though it was not originally defined in this way). The above reasoning then proves Proposition 3.6 in this case.

The proof of Proposition 3.6 for index one/two (de)stabilization is immediate. □

Now, suppose that $\mathcal{H}$ and $\mathcal{H}'$ are compatible diagrams for the pointed link $\mathcal{L}$, with markings $(\mathbb{A}, \omega)$ and $(\mathbb{A}', \omega')$, respectively. As before, $\mathcal{H}$ and $\mathcal{H}'$ are related by a sequence of Heegaard moves which avoid the basepoints. Let ρ denote the induced bijection between the periodic domains of $\mathcal{H}$ and those of $\mathcal{H}'$. The combination of Proposition 3.6 and Lemma 3.5 implies the following immediate analogue of Proposition 3.4.

Proposition 3.7. *If $\langle \omega, P\rangle = \langle \omega', \rho(P)\rangle$ for every periodic domain P of $\mathcal{H}$, then there is a quasi-isomorphism from $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$ to $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}', \omega')$ which commutes with the action of $\Lambda^*(\mathbb{F}[\mathbb{Z}]\langle \psi_1, \dots, \psi_m\rangle)$.* □

In particular, the actions of $\psi_1, \dots, \psi_m$ on $H_*(\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega), \partial)$ satisfy the same linear relations as those on $H_*(\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}', \omega'), \partial)$.

4. Unknots and unlinks

In this section, we prove a few results about the twisted knot Floer homologies of unknots and unlinks that will be useful later on. We start with a result about gradings. According to Remark 3.1, the absolute δ-grading on the chain complex $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$ for a pointed link $\mathcal{L} = (L, \mathbf{p})$ generally depends on the orientation of L. The lemma below says that this is not the case if L is an unlink.

Lemma 4.1. *If L is an unlink in S^3, then $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$ has a canonical absolute δ-grading, independent of the orientation of L.*

Proof of Lemma 4.1. Let $\mathfrak{o}$ and $\mathfrak{o}'$ be two orientations of L, and let $\delta_{\mathfrak{o}}$ and $\delta_{\mathfrak{o}'}$ denote the corresponding absolute δ-gradings on the untwisted complex $\widetilde{\mathrm{CFK}}(\mathcal{H})$. Since any two k-component oriented unlinks are isotopic as oriented links, the δ-gradings on $\widetilde{\mathrm{HFK}}(\mathcal{L})$ induced by $\delta_{\mathfrak{o}}$ and $\delta_{\mathfrak{o}'}$ are the same (this homology is non-trivial). Suppose that $\mathbf{x}$ is a cycle in $\widetilde{\mathrm{CFK}}(\mathcal{H})$ which generates the maximal δ-grading of $\widetilde{\mathrm{HFK}}(\mathcal{L})$ with respect to $\delta_{\mathfrak{o}}$. Since the relative δ-gradings induced by $\delta_{\mathfrak{o}}$ and $\delta_{\mathfrak{o}'}$ are the same, $\mathbf{x}$ generates the maximal δ-grading of $\widetilde{\mathrm{HFK}}(\mathcal{L})$ with respect to $\delta_{\mathfrak{o}'}$ as well. As this maximal δ-grading is independent of orientation, $\delta_{\mathfrak{o}}(\mathbf{x}) = \delta_{\mathfrak{o}'}(\mathbf{x})$, which implies that $\delta_{\mathfrak{o}} = \delta_{\mathfrak{o}'}$. □

For the proposition below, let $\mathcal{L} = (L, \mathbf{p})$ be a pointed unlink in S^3 with k components, and denote the marked points on the ith component of L by $p_1^i, \dots, p_{s_i}^i$, according to its orientation.

Proposition 4.2. *If $k > 1$ and $[\omega] \neq 0$, then $\underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}, [\omega]; \mathcal{F}) = 0$.*

Proof of Proposition 4.2. Fig. 7 shows an admissible multi-pointed Heegaard diagram $\mathcal{H} = (S^2, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbb{O}, \mathbb{X})$ for $\mathcal{L}$, with the points of $\mathbb{X}$ labeled just like those of $\mathbf{p}$. Let $\mathbb{A} = \{a_1, \dots, a_{k-1}\}$ as shown. For each $i = 1, \dots, k-1$, there is a unique periodic domain P_i which is bounded by the curves $\beta^i_{s_i}, \alpha^i_1, \dots, \alpha^i_{s_i}$ and contains a_i, obtained as the difference of the light and dark regions in Fig. 7. These domains correspond to generators of $H_2(S^3 \setminus L; \mathbb{Z})$; thus, we may obtain any cohomology class $[\omega]$ by defining $\omega(a_i)$ to be the evaluation of the desired class on P_i. Thus, the above choice of $\mathbb{A}$ suffices. Since $[\omega] \neq 0$, we may assume, without loss of generality, that $\omega(a_1) \neq 0$.

A generator $\mathbf{x}$ of $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$ consists of a choice of d^i_j or e^i_j for each $i = 1, \dots, k-1$ and $j = 1, \dots, s_i$ as well as a choice of b^k_j or c^k_j for each $j = 1, \dots, s_k - 1$. In particular, the rank of the *untwisted* complex $\widetilde{\mathrm{CFK}}(\mathcal{H})$ is $2^{s_1+\dots+s_k-1}$ over $\mathbb{F}$, which agrees with the rank of its homology. Therefore, the pseudo-holomorphic disks which count for the differential on $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$ come in canceling pairs. Their domains are the heavily shaded bigons and the lightly shaded punctured bigons in Fig. 7, with vertices at $d^i_{s_i}$ and $e^i_{s_i}$.

Let C_e denote the subcomplex of $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega; \mathcal{F})$ consisting of intersection points which contain $e^1_{s_1}$, and let C_d be its quotient complex. Let $\tau : C_d \to C_e$ be the map which, on generators, replaces $e^1_{s_1}$ with $d^1_{s_1}$; note that τ is an isomorphism of vector spaces. The discussion above implies that $(\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega; \mathcal{F}), \partial)$ is isomorphic to the mapping cone of $(1 + T^{\omega(a_1)}) \cdot \tau$. Since $1 + T^{\omega(a_1)} \neq 0$ and $\mathcal{F}$ is a field, we have $H_*(\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega; \mathcal{F}), \partial) = 0$. □

Next, we describe the structure of $H_*(\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega), \partial)$ as a module over $\Lambda^*(\mathbb{F}[\mathbb{Z}]\langle \psi_1, \dots, \psi_m \rangle)$ for a particular class of Heegaard diagrams and markings compatible with the unknot.

Proposition 4.3. *Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbb{O}, \mathbb{X})$ be a Heegaard diagram for an m-pointed unknot in S^3, such that O_i and X_i are in the same component of $\Sigma \setminus \boldsymbol{\alpha}$, and X_i and O_{i+1} are in the same component of $\Sigma \setminus \boldsymbol{\beta}$. Let $(\mathbb{A}, \omega)$ be a marking on $\mathcal{H}$ such that* (1) *all points of $\mathbb{A}$ are contained in a single component of $\Sigma \setminus \boldsymbol{\alpha}$, and* (2) *for each $i = 1, \dots, m$, the component of $\Sigma \setminus \boldsymbol{\beta}$ containing O_i contains a single point $a_i \in \mathbb{A}$ with $\omega(a_i) = r_i$. Let $\mathcal{Y}$ denote the module over $\mathbb{F}[\mathbb{Z}]$ generated by $y_1, \dots, y_m$ modulo the relation*

$$\sum_{j=1}^{m} T^{r_1+\dots+r_j} y_j = 0. \tag{4.1}$$

Then $\underline{\widetilde{\mathrm{HFK}}}(\mathcal{H}, \omega)$ can be identified with $\Lambda^*(\mathcal{Y}) \otimes_{\mathbb{F}[\mathbb{Z}]} \mathcal{M}$, such that each map ψ_i is given by multiplication by y_i.

(Compare the definition of $\mathcal{Y}_I$ in Section 2.)

Proof of Proposition 4.3. It suffices to take $\mathcal{M} = \mathbb{F}[\mathbb{Z}]$. Since $\Pi_{\alpha,\beta}$ is generated by the components of $\Sigma \setminus \boldsymbol{\alpha}$ and $\Sigma \setminus \boldsymbol{\beta}$ (see [28] or Section 5.2), hypotheses (1) and (2) determine the evaluations of ω on all periodic domains. By Proposition 3.7, we may assume that $\mathcal{H}$ and $(\mathbb{A}, \omega)$ are the diagram and marking shown in Fig. 8.

Generators of the complex $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$ consist of a choice of c_j or b_j for each $j = 1, \dots, m-1$; therefore, $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$ has rank 2^{m-1} over $\mathbb{F}[\mathbb{Z}]$. It is easy to see that the differential vanishes, so we may identify $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$ with its homology. For $j = 1, \dots, m-1$, consider the linear operator τ_j on $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$ defined on generators by

$$\tau_j(\mathbf{x}) = \begin{cases} \mathbf{x} \setminus \{b_j\} \cup \{c_j\} & b_j \in \mathbf{x} \\ 0 & b_j \notin \mathbf{x}. \end{cases}$$

The only domain of $\mathcal{H}$ that counts for ψ_1 is the small bigon containing X_1 with vertices at b_1 and c_1. For $j = 2, \dots, m-1$, the only domains that count for ψ_j are the two small bigons containing

X_j with vertices at b_{j-1} and c_{j-1}, and b_j and c_j. Similarly, the only domain that counts for ψ_m is the small bigon containing X_m with vertices at b_{m-1} and c_{m-1}. Therefore,

$$\begin{aligned} &\psi_1(\mathbf{x}) = T^{r_2}\tau_1(\mathbf{x}),\\ &\psi_j(\mathbf{x}) = \tau_{j-1}(\mathbf{x}) + T^{r_{j+1}}\tau_j(\mathbf{x}) \quad (j = 2, \dots, m-1),\\ &\psi_m(\mathbf{x}) = \tau_{m-1}(\mathbf{x}), \end{aligned} \tag{4.2}$$

which implies that

$$\sum_{j=1}^{m} T^{r_1+\dots+r_j}\psi_j(\mathbf{x}) = 0. \tag{4.3}$$

Let $\mathbf{x}_0$ denote the generator consisting of all the intersection points $\{b_j\}$. There is a well-defined linear map

$$\rho : \Lambda^*(\mathcal{Y}) \to \underline{\widetilde{\mathrm{HFK}}}(\mathcal{H}, \omega)$$

taking 1 to $\mathbf{x}_0$ and $y_{i_1} \cdots y_{i_k}$ to $(\psi_{i_1} \circ \dots \circ \psi_{i_k})(\mathbf{x}_0)$. Moreover, by (4.2), every element of $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$ can be obtained from $\mathbf{x}_0$ by a composition of the ψ_i maps, so ρ is surjective. As both $\Lambda^*(\mathcal{Y})$ and $\underline{\widetilde{\mathrm{CFK}}}(\mathcal{H}, \omega)$ are both free $\mathbb{F}[\mathbb{Z}]$-modules of rank 2^{m-1}, ρ is an isomorphism. □

5. A cube of resolutions for $\underline{\widetilde{\mathrm{HFK}}}$

In this section, we show that Manolescu's unoriented skein exact triangle [27] holds with twisted coefficients in any $\mathbb{F}[\mathbb{Z}]$-module $\mathcal{M}$, and can be iterated in the manner of Ozsváth and Szabó [39].

5.1. A Heegaard multi-diagram for a link and its resolutions

Fix a connected projection $\mathcal{D}$ of L. Let $c_1, \dots, c_n$ denote the crossings of $\mathcal{D}$, and let $\mathbf{p} = \{p_1, \dots, p_m\}$ be a set of markings on the edges of $\mathcal{D}$ so that every edge is marked and p_1 is assigned to an outermost edge, as in Section 2. This marking specifies an m-pointed link $\mathcal{L} = (L, \mathbf{p})$. For $I \in \{0, 1, \infty\}^n$, let $\mathcal{D}_I$ denote the diagram obtained from $\mathcal{D}$ by taking the I_j-resolution of c_j, as prescribed in Fig. 1. $\mathcal{D}_I$ is called a *partial resolution* of $\mathcal{D}$, and it represents an m-pointed link $\mathcal{L}_I = (L_I, \mathbf{p})$. In this subsection we construct an admissible multi-pointed Heegaard multi-diagram which encodes all partial resolutions of $\mathcal{D}$, following [33,27].

Let U_β denote the closure of a regular neighborhood of $\mathcal{D}$, and let $U_\alpha = S^3 \setminus \operatorname{int} U_\beta$. This determines a genus-$(n+1)$ Heegaard splitting $S^3 = U_\alpha \cup_\Sigma U_\beta$, where Σ is the oriented boundary of U_α. The handlebody U_α is specified by curves $\alpha_1, \dots, \alpha_{n+1}$ that are the intersections of Σ with the bounded regions of $\mathbb{R}^2 \setminus \mathcal{D}$.

Near each marked point p_i, let μ_i be the boundary of a meridional disk of U_β. Let ζ_i be an short arc on the upper half of Σ meeting μ_i once transversally. Orient the edge of $\mathcal{D}$ containing p_i as the boundary of the black region that it abuts, and orient ζ_i in the same direction. Let a_i and X_i be the initial and final points of ζ_i, and let O_i be a point on ζ_i between a_i and $\zeta_i \cap \mu_i$. For $i = 2, \dots, m$, let α_{p_i} be the boundary of a disk that contains X_i and O_i but not a_i, chosen such that α_{p_i} and μ_i meet transversally in a pair of points. (See Fig. 9(c).) We refer to the configuration $\alpha_{p_i} \cup \mu_i \cup \{O_i, X_i\}$ as a *ladybug*. Set $\mathbb{O} = \{O_1, \dots, O_m\}$, $\mathbb{X} = \{X_1, \dots, \mathbb{X}_m\}$, $\mathbb{A} = \{a_1, \dots, a_m\}$, and $\mathbb{P} = \mathbb{O} \cup \mathbb{X}$.

As shown in Fig. 9, the component of $\Sigma \setminus (\mu_1 \cup \dots \cup \mu_m)$ corresponding to c_j is a sphere with four punctures. If we view c_j with the incident black regions on the left and right and the white regions on the top and bottom, and the adjacent marked points labeled p_{i_1}, p_{i_2}, p_{i_3}, and p_{i_4} just as in Definition 2.2, this component contains the basepoints X_{i_1}, O_{i_2}, X_{i_3}, and O_{i_4}, as well as the marked points a_{i_2} and a_{i_4}. As will be seen below, the positions of X_{i_1}, a_{i_2}, X_{i_3}, and a_{i_4} will motivate the conventions of Definition 2.2. Let β_j, γ_j, and δ_j be curves on Σ as shown in Fig. 9(a–b).

For each $I \in \{0, 1, \infty\}^n$, let

$$\boldsymbol{\eta}(I) = \{\eta_{c_1}(I), \dots, \eta_{c_n}(I), \eta_{p_1}(I), \dots, \eta_{p_m}(I)\},$$

where $\eta_{p_i}(I)$ is a small Hamiltonian translate of μ_i, and $\eta_{c_j}(I)$ is a small Hamiltonian translate of β_j, γ_j, or δ_j, according to whether I_j is 0, 1, or ∞, respectively. We choose these curves so that $\eta_{p_i}(I)$ and $\eta_{p_i}(I')$ (resp. $\eta_{c_j}(I)$ and $\eta_{c_j}(I')$) meet transversely in exactly two points for each $I \neq I'$, and so that no three curves intersect in the same point. Let

$$\tilde{\boldsymbol{\alpha}} = \{\alpha_1, \dots, \alpha_{n+1}, \alpha_{p_2}, \dots, \alpha_{p_m}\}.$$

The Heegaard diagram $\tilde{\mathcal{H}}_I = (\Sigma, \tilde{\boldsymbol{\alpha}}, \boldsymbol{\eta}(I), \mathbb{P})$ then specifies the unoriented m-pointed link $\mathcal{D}_I$. Moreover, for any orientation $\mathfrak{o}$ of $\mathcal{D}_I$, one can partition $\mathbb{P}$ into subsets $\mathbb{O}_{I,\mathfrak{o}}$ and $\mathbb{X}_{I,\mathfrak{o}}$ of equal size so that $(\Sigma, \tilde{\boldsymbol{\alpha}}, \boldsymbol{\eta}(I), \mathbb{O}_{I,\mathfrak{o}}, \mathbb{X}_{I,\mathfrak{o}})$ encodes $\mathcal{D}_I$ with orientation $\mathfrak{o}$. In particular, if $I \in \{0, 1\}^n$ and $\mathfrak{o}$ is the orientation that $\mathcal{D}_I$ inherits as the boundary of the black regions, then $\mathbb{O}_{I,\mathfrak{o}} = \mathbb{O}$ and $\mathbb{X}_{I,\mathfrak{o}} = \mathbb{X}$. (On the other hand, if $I \notin \{0, 1\}^{\infty}$, there is no orientation $\mathfrak{o}$ on $\mathcal{D}_I$ for which this statement holds.) The multi-diagram

$$\tilde{\mathcal{H}} = (\Sigma, \tilde{\boldsymbol{\alpha}}, \{\boldsymbol{\eta}(I)\}_{I \in \{0,1,\infty\}^n}, \mathbb{P})$$

thus encodes all unoriented partial resolutions of $\mathcal{D}$. Note, however, that we cannot partition $\mathbb{P}$ to describe orientations on all the resolutions $\mathcal{D}_I$ (for $I \in \{0, 1, \infty\}^n$) simultaneously. (We do not need to distinguish between $\mathbb{O}$ and $\mathbb{X}$ again until the end of Section 5.3.)

In order to define systems of twisted coefficients, fix a system of weights $\mathbf{r} = (r_1, \dots, r_m)$ as in Definition 2.1, which at this point need not be generic. Define $\omega_{\mathbf{r}} \colon \mathbb{A} \to \mathbb{Z}$ by $\omega_{\mathbf{r}}(a_i) = r_i$.

Note that $\tilde{\mathcal{H}}_I$ (and, hence, $\tilde{\mathcal{H}}$) is inadmissible when $\mathcal{D}_I$ has more than one component. Following [27], we may achieve admissibility by stretching the tips of the $\tilde{\boldsymbol{\alpha}}$ curves used in the ladybugs until they reach regions containing O_1 or X_1. We require that these isotopies avoid the points in $\mathbb{A} \cup \mathbb{P}$. Let $\boldsymbol{\alpha}$ denote the resulting set of curves, and let $\mathcal{H}_I$ and $\mathcal{H}$ denote the corresponding admissible diagrams. We shall henceforth use $\underline{\widetilde{\mathit{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$ to denote the complex $\underline{\widetilde{\mathit{CFK}}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\eta}(I), \mathbb{P}, \omega_{\mathbf{r}})$, with its relative δ-grading given by (3.4), and coefficients in an arbitrary $\mathbb{F}[\mathbb{Z}]$-module $\mathcal{M}$. For $I \neq I'$, we shall likewise use $\underline{\widetilde{\mathit{CFK}}}(\boldsymbol{\eta}(I), \boldsymbol{\eta}(I'))$ to denote the relatively δ-graded complex $\underline{\widetilde{\mathit{CFK}}}(\Sigma, \boldsymbol{\eta}(I), \boldsymbol{\eta}(I'), \mathbb{P}, \omega_{\mathbf{r}})$. We close this section with a description of the latter.

Each curve in $\boldsymbol{\eta}(I)$ intersects one curve in $\boldsymbol{\eta}(I')$ in exactly two points and is disjoint from all others; therefore, $|\mathbb{T}_{\eta(I)} \cap \mathbb{T}_{\eta(I')}| = 2^{n+m}$. For $I_j = I'_j$, the curves $\eta_{c_j}(I)$ and $\eta_{c_j}(I')$ are related by a small Hamiltonian isotopy, so the two points of $\eta_{c_j}(I) \cap \eta_{c_j}(I')$ differ in their δ-grading contributions by 1; let $\theta_{c_j}^{I,I'}$ be the point with the smaller contribution. Similarly, let $\theta_{p_i}^{I,I'}$ denote the intersection point of $\eta_{p_i}(I) \cap \eta_{p_i}(I')$ with the smaller δ-grading contribution, for $i = 1, \dots, m$. Suppose I and I' differ in $\epsilon(I, I')$ entries. Then there are $2^{\epsilon(I,I')}$ generators in $\mathbb{T}_{\eta(I)} \cap \mathbb{T}_{\eta(I')}$ that use all of the points $\theta_{c_j}^{I,I'}$ and $\theta_{p_i}^{I,I'}$; we denote these generators by $\Theta_1^{I,I'}, \dots, \Theta_{2^{\epsilon(I,I')}}^{I,I'}$, indexed arbitrarily.

Lemma 5.1. *There is an isomorphism of relatively graded* $\mathbb{F}[\mathbb{Z}]$*-modules,*

$$\widetilde{\underline{\mathrm{CFK}}}(\boldsymbol{\eta}(I), \boldsymbol{\eta}(I')) \cong (H^1(S^1; \mathbb{F}))^{\otimes(m+n-\epsilon(I,I'))} \otimes_{\mathbb{F}} V^{\otimes \epsilon(I,I')} \otimes_{\mathbb{F}} \mathcal{M},$$

and the summand of $\widetilde{\underline{\mathrm{CFK}}}(\boldsymbol{\eta}(I), \boldsymbol{\eta}(I'))$ *in the minimal* δ*-grading is generated by the points* $\Theta_1^{I,I'}, \dots, \Theta_{2^{\epsilon(I,I')}}^{I,I'}$*. Moreover, the differential* ∂ *on* $\widetilde{\underline{\mathrm{CFK}}}(\boldsymbol{\eta}(I), \boldsymbol{\eta}(I'))$ *is zero.*

Proof of Lemma 5.1. It suffices to take $\mathcal{M} = \mathbb{F}[\mathbb{Z}]$. Both modules above are free of rank 2^{n+m}; we must simply show that the generators $\Theta_1^{I,I'}, \dots, \Theta_{2^{\epsilon(I,I')}}^{I,I'}$ have the same δ-gradings. For this, suppose $\Theta_a^{I,I'}$ and $\Theta_b^{I,I'}$ differ near a single crossing c_j. As in [28, Lemma 11], there is class $\phi \in \pi_2(\Theta_a^{I,I'}, \Theta_b^{I,I'})$ whose domain is an annulus containing a single point of $\mathbb{P}$, with one boundary component comprised of segments of the curves $\eta_{c_j}(I)$ and $\eta_{c_j}(I')$, and the other boundary component equal to $\eta_{p_i}(I)$ (or $\eta_{p_i}(I')$) for some p_i near c_j. By Lipshitz's formula for the Maslov index [26], $\mu(\phi) = 1$, so

$$\delta(\Theta_a^{I,I'}) - \delta(\Theta_b^{I,I'}) = P(\phi) - \mu(\phi) = 0.$$

Finally, the only regions of $\Sigma \setminus (\boldsymbol{\eta}(I) \cup \boldsymbol{\eta}(I'))$ that do not contain basepoints are thin bigons bounded by the curves $\eta_{c_j}(I)$ and $\eta_{c_j}(I')$ with $I_j = I'_j$ or the curves $\eta_{p_i}(I)$ and $\eta_{p_i}(I')$. These bigons come in pairs, so the differential ∂ is zero. □

5.2. Periodic domains

In this subsection, we describe the multi-periodic domains of $\mathcal{H}$, generalizing the results of Manolescu–Ozsváth [28, Section 3.1].

Let Π_α and $\Pi_{\eta(I)}$ denote the groups of $\mathbb{Z}$-linear combinations of the components of $\Sigma \setminus \boldsymbol{\alpha}$ and $\Sigma \setminus \boldsymbol{\eta}(I)$, respectively. Since $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\eta}(I))$ is a Heegaard diagram for S^3, the curves in $\boldsymbol{\alpha}$ and $\boldsymbol{\eta}(I)$ span $H_1(\Sigma; \mathbb{Z})$. Therefore,

$$\Pi_{\alpha,\eta(I)} = \Pi_\alpha + \Pi_{\eta(I)}, \tag{5.1}$$

by [28, Corollary 7]; that is, any periodic domain of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\eta}(I))$ is a sum of components of $\Sigma \setminus \boldsymbol{\alpha}$ with components of $\Sigma \setminus \boldsymbol{\eta}(I)$. Note that the latter are either annuli or pairs of pants.

Now, consider distinct tuples I, I'. For $i = 1, \dots, m$, there is a periodic domain $D_{p_i}^{\eta(I),\eta(I')}$ with boundary $\eta_{p_i}(I) - \eta_{p_i}(I')$, formed as the sum of two thin bigons with opposite signs. Likewise, for $I_j = I'_j$, there is a periodic domain $D_{c_j}^{\eta(I),\eta(I')}$ with boundary $\eta_{c_j}(I) - \eta_{c_j}(I')$. The lemma below is an analogue of [28, Lemma 9].

Lemma 5.2. *The group* $\Pi_{\eta(I),\eta(I')}$ *is spanned by* $\Pi_{\eta(I)}$*,* $\Pi_{\eta(I')}$*, and periodic domains of the forms* $D_{c_j}^{\eta(I),\eta(I')}$ *and* $D_{p_i}^{\eta(I),\eta(I')}$*.*

Proof of Lemma 5.2. Let D be a domain in $\Pi_{\eta(I),\eta(I')}$ and suppose that, for some $I_j \neq I'_j$, $\eta_{c_j}(I)$ appears with non-zero multiplicity in the boundary of D. There is a pair of pants in $\Pi_{\eta(I)}$ bounded by $\eta_{c_j}(I)$ and two curves, $\eta_{p_i}(I)$ and $\eta_{p_{i'}}(I)$. Adding some multiple of this pair of pants, we obtain a domain whose boundary does not contain any multiple of $\eta_{c_j}(I)$. Iterating this sort of procedure, we can write D as the sum of domains in $\Pi_{\eta(I)}$ and $\Pi_{\eta(I')}$ with a domain D' whose boundary consists of curves of the forms $\eta_{p_i}(I)$, $\eta_{p_i}(I')$ and $\eta_{c_{j'}}(I)$, $\eta_{c_{j'}}(I')$ for $I_{j'} = I'_{j'}$. We may then write D' as the sum of a domain in $\Pi_{\eta(I)}$ with domains of the forms $D_{p_i}^{\eta(I),\eta(I')}$ and $D_{c_{j'}}^{\eta(I),\eta(I')}$. This proves Lemma 5.2. □

The following result generalizes [28, Lemma 10].

Lemma 5.3. *Suppose $I^0, \dots, I^k \in \{0, 1, \infty\}^n$ is a sequence of distinct tuples, and $k \geq 1$. Then*

$$\Pi_{\alpha,\eta(I^0),\dots,\eta(I^k)} = \Pi_\alpha + \Pi_{\eta(I^0),\eta(I^1)} + \cdots + \Pi_{\eta(I^{k-1}),\eta(I^k)}.$$

Proof of Lemma 5.3. We claim that

$$\Pi_{\alpha,\eta(I^0),\dots,\eta(I^k)} = \Pi_{\alpha,\eta(I_0),\dots,\eta(I^{k-1})} + \Pi_{\eta(I^{k-1}),\eta(I^k)}$$

for $k \geq 1$. This claim, together with (5.1), implies Lemma 5.3 by induction. Let D be a domain in $\Pi_{\alpha,\eta(I^0),\dots,\eta(I^k)}$ and suppose that, for some $I_j^k \neq I_j^{k-1}$, the curve $\eta_{c_j}(I^k)$ appears with nonzero multiplicity in the boundary of D. As above, there is a pair of pants in $\Pi_{\eta(I^k)}$ bounded by $\eta_{c_j}(I^k)$ and two curves, $\eta_{c_{p_i}}(I^k)$ and $\eta_{c_{p_{i'}}}(I^k)$. Adding some multiple of this pair of pants, we obtain a domain whose boundary does not contain any multiple of $\eta_{c_j}(I^k)$. Iterating this procedure, and adding domains of the forms $D_{p_i}^{\eta(I^{k-1}),\eta(I^k)}$ and $D_{c_{j'}}^{\eta(I^{k-1}),\eta(I^k)}$, we obtain a domain in $\Pi_{\alpha,\eta(I_0),\dots,\eta(I^{k-1})}$. Reversing this process proves the claim. □

We shall use the following proposition in many places throughout this paper; compare with [28, Lemma 11].

Proposition 5.4. *Suppose ϕ and ϕ' are two Whitney polygons for which $D(\phi) - D(\phi')$ is a multi-periodic domain of $\mathcal{H}$. Then*

$$P(\phi) - \mu(\phi) = P(\phi') - \mu(\phi'),$$

where $P(\phi)$ denotes the total multiplicity of ϕ at all the basepoints.

Proof of Proposition 5.4. By Lemmas 5.2 and 5.3, the difference $D(\phi) - D(\phi')$ is a linear combination of components of $\Sigma \setminus \boldsymbol{\alpha}$, components of the complements $\Sigma \setminus \boldsymbol{\eta}(I)$, and domains of the forms $D_{c_j}^{\eta(I),\eta(I')}$ and $D_{p_i}^{\eta(I),\eta(I')}$. It is easy to verify that these domains all satisfy $P = \mu$, exactly as in the proof of [28, Lemma 11]. Proposition 5.4 then follows from the additivity of P and μ. □

Next, we describe the periodic domains of $\mathcal{H}_I$ that avoid the basepoints in $\mathbb{P}$. Let $S_I^1, \dots, S_I^{k_I}$ denote the components of L_I, labeled so that p_1 lies on $S_I^{k_I}$. Then $H_2(S^3 \setminus L_I; \mathbb{Z})$ is freely generated by the homology classes of tori $T_I^1, \dots, T_I^{k_I-1}$ obtained as the boundaries of regular neighborhoods of $S_I^1, \dots, S_I^{k_I-1}$. These tori correspond to positive periodic domains $\tilde{P}_I^1, \dots, \tilde{P}_I^{k_I}$ in $\Pi^0_{\tilde{\alpha},\eta(I)}$, where the boundary of $\tilde{P}_I^\ell$ consists of (1) the $\tilde{\boldsymbol{\alpha}}$ circles of the ladybugs associated to the points of $\mathbf{p}$ on $\mathcal{D}_I^\ell$, and (2) a copy of $\eta_{c_j}(I)$ for every crossing c_j such that S_I^ℓ enters and leaves a neighborhood of c_j exactly once. The torus T_I^ℓ can then be recovered by capping off the boundary components of $\tilde{P}_I^\ell$ with disks. Finally, let P_I^ℓ be the domain in $\Pi^0_{\alpha,\eta(I)}$ corresponding to $\tilde{P}_I^\ell$; although P_I^ℓ has both positive and negative multiplicities in general, its boundary multiplicities and its multiplicities at points of $\mathbb{A} \cup \mathbb{P}$ agree with those of $\tilde{P}_I^\ell$.

Let $[\omega_{\mathbf{r}}]_I$ denote the element of $H^2(S^3 \setminus L_I)$ associated to the marking $(\mathbb{A}, \omega_\Omega)$. The previous paragraph implies that the evaluation of $[\omega_{\mathbf{r}}]_I$ on $[T_I^\ell]$ is equal to the sum of the weights of the marked points on S_I^ℓ. The following proposition then follows from the genericity of $\mathbf{r}$ together with Lemma 2.3, Proposition 4.2, and Eq. (3.8).

Proposition 5.5. (1) *For any* $I \in \{0, 1\}$ *for which* $\mathcal{D}_I$ *is disconnected, the cohomology class* $[\omega_{\mathbf{r}}]_I$ *is nonzero, so the complex* $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$ *is acyclic.*

(2) *Let* $I^\infty = (\infty, \dots, \infty)$*, so that* $\mathcal{D}_{I^\infty} = \mathcal{D}$*. If* $\mathbf{r} = \mathbf{r}_\Omega$ *for some function* $\Omega\colon \{1, \dots, n\} \to \mathbb{Z}$*, then the cohomology class* $[\omega_{\mathbf{r}}]_{I^\infty}$ *is zero, so*

$$\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^\infty)) \cong \widetilde{\mathrm{CFK}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^\infty)) \otimes_{\mathbb{F}} \mathcal{M}.$$

5.3. Construction of the cube of resolutions

For distinct tuples $I, I' \in \{0, 1, \infty\}^n$, we write $I < I'$ if $I_j \le I'_j$ for $j = 1, \dots, n$. If I' is obtained from I by changing a single entry from 0 to 1, from 1 to ∞, or from ∞ to 0, we say that I' is a *cyclic successor* of I. In the first two cases, I' is called an *immediate successor* of I. A *successor sequence* (resp. *cyclic successor sequence*) is a sequence of tuples $I^0, \dots, I^k$ such that each I^j is an immediate (resp. cyclic) successor of I^{j-1}. For any cyclic successor sequence $I^0, \dots, I^k$, let

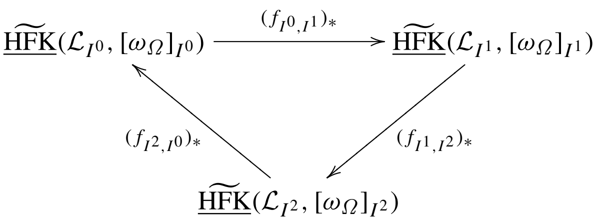
$$f_{I^0,\dots,I^k}\colon \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^0)) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^k))$$

be the map defined by

$$f_{I^0,\dots,I^k}(\mathbf{x}) = F_{\alpha,\eta(I^0),\dots,\eta(I^k)}\left(\mathbf{x} \otimes \left(\Theta_1^{I^0,I^1} + \Theta_2^{I^0,I^1}\right) \otimes \cdots \otimes \left(\Theta_1^{I^{k-1},I^k} + \Theta_2^{I^{k-1},I^k}\right)\right).$$

We shall eventually incorporate these maps into a cube of resolutions complex which is quasi-isomorphic to $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^\infty))$. First, we prove an analogue of Manolescu's unoriented skein exact triangle for coefficients in an arbitrary $\mathbb{F}[Z]$-module $\mathcal{M}$.

Theorem 5.6. *Suppose* I^0, I^1, I^2 *is a cyclic successor sequence of tuples in* $\{0, 1, \infty\}^n$ *which differ in only one coordinate. Then, the triangle*



is exact.

As in [27,39], Theorem 5.6 follows immediately from the proposition below.

Proposition 5.7. *Suppose* I^0, I^1, I^2 *is a cyclic successor sequence of tuples in* $\{0, 1, \infty\}^n$ *which differ in only one coordinate. Then,*

(1) *the composite* $f_{I^0,I^1} \circ f_{I^2,I^0}$ *is chain homotopic to zero,*

$$f_{I^0,I^1} \circ f_{I^2,I^0} = \partial \circ f_{I^2,I^0,I^1} + f_{I^2,I^0,I^1} \circ \partial;$$

(2) *the map*

$$f_{I^0,I^1,I^2} \circ f_{I^2,I^0} + f_{I^1,I^2} \circ f_{I^2,I^0,I^1}\colon \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^2)) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^2))$$

is a quasi-isomorphism.

Proof of Proposition 5.7. This is a straightforward adaptation of Manolescu's proofs of [27, Lemmas 6 and 7]. Simply note that the relevant polygons in Manolescu's proofs avoid the markings in $\mathbb{A}$ since every such marking lies in the same component of $\Sigma \setminus (\boldsymbol{\eta}(I^0) \cup \boldsymbol{\eta}(I^1) \cup \boldsymbol{\eta}(I^2))$ as a basepoint. □

For tuples $I < I'$ in $\{0, 1, \infty\}^n$, let

$$D_{I,I'} : \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I)) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I'))$$

denote the sum, over all successor sequences $I = I^0 < \cdots < I^k = I'$, of the maps $f_{I^0,\dots,I^k}$, and let $D_{I,I}$ denote the differential ∂ on $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$. For $S \subset \{0, 1, \infty\}^n$, let

$$X(S) = \bigoplus_{I \in S} \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I)),$$

and set $X = X(\{0, 1, \infty\}^n)$. We define a map $D : X \to X$ by

$$D = \bigoplus_{I \leq I'} D_{I,I'}.$$

Below, we show that D is a differential. As a warmup, we prove the following.

Lemma 5.8. *Suppose I^0, I^1, I^2 is a cyclic successor sequence of tuples in $\{0, 1, \infty\}^n$. If these tuples differ in only one coordinate, then*

$$F_{\eta(I^0),\eta(I^1),\eta(I^2)}\left(\left(\Theta_1^{I^0,I^1} + \Theta_2^{I^0,I^1}\right) \otimes \left(\Theta_1^{I^1,I^2} + \Theta_2^{I^1,I^2}\right)\right) = 0.$$

Otherwise,

$$F_{\eta(I^0),\eta(I^1),\eta(I^2)}\left(\left(\Theta_1^{I^0,I^1} + \Theta_2^{I^0,I^1}\right) \otimes \left(\Theta_1^{I^1,I^2} + \Theta_2^{I^1,I^2}\right)\right) = \Theta_1^{I^0,I^2} + \Theta_2^{I^0,I^2} + \Theta_3^{I^0,I^2} + \Theta_4^{I^0,I^2}.$$

Proof of Lemma 5.8. Let $\mathbf{z} \in \mathbb{T}_{\eta(I^0)} \cap \mathbb{T}_{\eta(I^2)}$. Each of the four tubular regions of $\Sigma \setminus (\boldsymbol{\eta}(I^0) \cup \boldsymbol{\eta}(I^1) \cup \boldsymbol{\eta}(I^2))$ in the neighborhood of a crossing contains a basepoint, as does the tubular region on each side of a ladybug. Therefore, the domain of any Whitney triangle $\psi \in \pi_2(\Theta_r^{I^0,I^1}, \Theta_s^{I^1,I^2}, \mathbf{z})$ which avoids $\mathbb{P}$ is a union of small triangles. Suppose I^0, I^1, and I^2 differ only in their jth coordinates. Near p_i and $c_{j'}$ for $j' \neq j$, these triangles look like those shaded in Fig. 10(a) and (b). Near c_j, the domain of ψ looks like one of the four triangles shaded in (d). Thus, $\mathbf{z}$ is of the form $\Theta_{\kappa(a,b)}^{I^0,I^2}$, for some $2:1$ map

$$\kappa : \{1, 2\} \times \{1, 2\} \to \{1, 2\}.$$

Moreover, $\mu(\psi) = 0$ and ψ has a unique holomorphic representative. The first statement of Lemma 5.8 follows immediately.

Now, suppose I^0, I^1 and I^1, I^2 differ in their j_1th and j_2th coordinates, respectively. Near p_i and $c_{j'}$ for $j' \neq j_1, j_2$, the domain of ψ looks like the shaded triangles in (a) and (b). Near c_{j_1}, the domain of ψ is a small triangle with vertices at intersection points between the curves $\eta_{c_{j_1}}(I^0)$, $\eta_{c_{j_1}}(I^1)$, and $\eta_{c_{j_1}}(I^2)$. Fig. 10(c) shows a picture of this triangle when $\eta_{c_{j_1}}(I^0)$ is isotopic to β_{j_1} and $\eta_{c_{j_1}}(I^1)$, $\eta_{c_{j_1}}(I^2)$ are isotopic to γ_{j_1}. The same reasoning applies near c_{j_2}. Therefore, $\mathbf{z}$ is of the form $\Theta_{\nu(a,b)}^{I^0,I^2}$ for some $1:1$ map

$$\nu : \{1, 2\} \times \{1, 2\} \to \{1, 2, 3, 4\}.$$

As above, $\mu(\psi) = 0$ and ψ has a unique holomorphic representative. The second statement of Lemma 5.8 follows immediately. □

Proposition 5.9. *For tuples $I < I'$ in $\{0, 1, \infty\}^n$,*

$$\sum_{\substack{I=I^0<\cdots<I^k=I' \\ \text{succ. seq.}}} F_{\eta(I^0),\ldots,\eta(I^k)}\Big(\Big(\Theta_1^{I^0,I^1} + \Theta_2^{I^0,I^1}\Big) \otimes \cdots \otimes \Big(\Theta_1^{I^{k-1},I^k} + \Theta_2^{I^{k-1},I^k}\Big)\Big) = 0. \quad (5.2)$$

It then follows from the $\mathcal{A}_\infty$ relation (3.11) that $D^2 = 0$.

Proof of Proposition 5.9. For $k = 1$, this is just the statement that $\Theta_1^{I^0,I^1} + \Theta_2^{I^0,I^1}$ is a cycle in $\widetilde{\underline{\mathrm{CFK}}}(\boldsymbol{\eta}(I^0), \boldsymbol{\eta}(I^1))$.

Suppose $k = 2$. If $I = I^0$ and $I' = I^2$ differ in only one coordinate, then the proposition follows from Lemma 5.8. Otherwise, there are exactly two tuples, I^1 and J^1, with $I^0 < I^1 < I^2$ and $I^0 < J^1 < I^2$. By Lemma 5.8, the contributions of these two successor sequences to the sum (5.2) cancel.

Now, suppose $k > 2$. For any $a_1, \ldots, a_k \in \{1, 2\}$ and $b \in \{1, \ldots, 2^{\epsilon(I^0,I^k)}\}$, there exists a class

$$\psi \in \pi_2(\Theta_{a_1}^{I^0,I^1}, \ldots, \Theta_{a_k}^{I^{k-1},I^k}, \Theta_b^{I^0,I^k})$$

with $P(\psi) - \mu(\psi) = 0$, gotten by concatenating the Whitney triangles described in the proof of Lemma 5.8 with the Whitney disks in $\pi_2(\Theta_p^{I^i,I^j}, \Theta_q^{I^i,I^j})$ described in the proof of Lemma 5.1. Suppose, for a contradiction, that the coefficient of some $\Theta \in \mathbb{T}_{\eta(I^0)} \cap \mathbb{T}_{\eta(I^k)}$ in the sum (5.2) is nonzero. Then there is a Whitney $(k + 1)$-gon

$$\psi' \in \pi_2(\Theta_{a_1}^{I^0,I^1}, \ldots, \Theta_{a_k}^{I^{k-1},I^k}, \Theta)$$

with $P(\psi') - \mu(\psi') = k - 2$. Since $\Theta_b^{I^0,I^k}$ has the minimal δ-grading among all generators of $\widetilde{\underline{\mathrm{CFK}}}(\boldsymbol{\eta}(I^0), \boldsymbol{\eta}(I^k))$, there is a class $\phi \in \pi_2(\Theta_b^{I^0,I^k}, \Theta)$ with $P(\phi) - \mu(\phi) \le 0$. Then $P(\psi * \phi) - \mu(\psi * \phi) \le 0$ as well. On the other hand, $D(\psi * \phi) - D(\psi')$ is a multi-periodic domain, so Proposition 5.4 implies that $P(\psi * \phi) - \mu(\psi * \phi) = P(\psi') - \mu(\psi')$, a contradiction. □

The main theorem of this section is as follows.

Theorem 5.10. *The complex $(X(\{0, 1\}^n), D)$ is quasi-isomorphic to $(\widetilde{\underline{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^\infty), \partial))$.*

This theorem follows rather quickly from the lemma below.

Lemma 5.11. *For $0 \le k \le n$, consider the complex $X(\{0, 1\}^{n-k-1} \times \{0, 1, \infty\} \times \{\infty\}^k)$, with its differential induced by D. Then,*

$$H_*(X(\{0, 1\}^{n-k-1} \times \{0, 1, \infty\} \times \{\infty\}^k), D) = 0. \quad (5.3)$$

Proof of Lemma 5.11. Consider the decreasing filtration of $X(\{0, 1\}^{n-k-1} \times \{0, 1, \infty\} \times \{\infty\}^k)$ induced by the grading which assigns to an element $\mathbf{x}$ in the summand $\widetilde{\underline{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$ the number $I_1 + \cdots + I_{n-k-1}$. The homology of the associated graded object is a direct sum

$$\bigoplus_{J \in \{0,1\}^{n-k-1}} H_*(X(J \times \{0, 1, \infty\} \times \{\infty\}^k), D). \quad (5.4)$$

Each complex in (5.4) is the mapping cone of a map

$$X(J \times \{0,1\} \times \{\infty\}^k) \to X(J \times \{\infty\} \times \{\infty\}^k). \tag{5.5}$$

Let

$$I^0 = J \times \{0\} \times \{\infty\}^k, \qquad I^1 = J \times \{1\} \times \{\infty\}^k, \qquad I^2 = J \times \{\infty\} \times \{\infty\}^k.$$

Then $X(J \times \{0,1\} \times \{\infty\}^k)$ is the mapping cone, $MC(f_{I^0,I^1})$, of

$$f_{I^0,I^1} : \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^0)) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^1)),$$

and the map in (5.5) is

$$f_{I^0,I^1,I^2} + f_{I^1,I^2} : MC(f_{I^0,I^1}) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^2)).$$

The quasi-isomorphism in part (2) of Proposition 5.7 factors through this map, which implies that $f_{I^0,I^1,I^2} + f_{I^1,I^2}$ is also a quasi-isomorphism. The terms in (5.4) are therefore zero, which implies (5.3). □

Proof of Theorem 5.10. For $0 \le k \le n$, the complex $X(\{0,1\}^{n-k-1} \times \{0,1,\infty\} \times \{\infty\}^k)$ is the mapping cone of

$$G_k : X(\{0,1\}^{n-k} \times \{\infty\}^k) \to X(\{0,1\}^{n-k-1} \times \{\infty\}^{k+1}),$$

where G_k is the sum, over all $I \in \{0,1\}^{n-k} \times \{\infty\}^k$ and $I' \in \{0,1\}^{n-k-1} \times \{\infty\}^{k+1}$, of the maps $D_{I,I'}$. By Lemma 5.11, G_k must be a quasi-isomorphism. The composition

$$G = G_0 \circ \ldots G_{n-1} : X(\{0,1\}^n) \to X(\{\infty\}^n)$$

is therefore a quasi-isomorphism, proving Theorem 5.10. □

Proposition 5.5 and Eqs. (3.8) and (3.9) immediately imply the following corollary.

Corollary 5.12. *For any system of weights* $\mathbf{r}$*,*

$$H_*(X(\{0,1\}^n), D) \cong \underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}, [\omega_{\mathbf{r}}]_{I^\infty}; \mathcal{M}) \cong \underline{\widehat{\mathrm{HFK}}}(L, [\omega_{\mathbf{r}}]_{I^\infty}; \mathcal{M}) \otimes_{\mathbb{F}} (V^{\otimes(m-|L|)}).$$

In particular, if $\mathbf{r} = \mathbf{r}_\Omega$ *for a function* $\Omega : \{1,\ldots,n\} \to \mathbb{Z}$*, then*

$$H_*(X(\{0,1\}^n), D) \cong \widehat{\mathrm{HFK}}(L) \otimes_{\mathbb{F}} (V^{\otimes(m-|L|)}) \otimes_{\mathbb{F}} \mathcal{M}. \quad \square$$

Note that $X(\{0,1\}^n)$ has a decreasing filtration induced by the grading Q which assigns to any element of the summand $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$ the number $|I|$. We shall refer to Q as the *filtration grading*. This filtration gives rise to a spectral sequence $\mathcal{S}^{\mathbf{r}}_{\mathcal{M}}$. (If $\mathbf{r} = \mathbf{r}_\Omega$, we may denote this spectral sequence $\mathcal{S}^{\Omega}_{\mathcal{M}}$ as in the Introduction.) The E_1 page of $\mathcal{S}^{\mathbf{r}}_{\mathcal{M}}$ is the direct sum

$$\bigoplus_{I \in \{0,1\}^n} \underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}_I, [\omega_{\mathbf{r}}]_I; \mathcal{M}),$$

and its d_1 differential is the sum of the maps $(f_{I,I'})_*$, over immediate successors I' of I. We shall be interested in the case that $\mathbf{r}$ is generic and $\mathcal{M} = \mathcal{F}$. In this case, the E_1 page of $\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}$ is a sum over connected resolutions,

$$\bigoplus_{I \in \mathcal{R}(D)} \widetilde{\mathrm{HFK}}(\mathcal{L}_I) \otimes_{\mathbb{F}} \mathcal{F}, \tag{5.6}$$

since $\underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}_I, [\omega_\Omega]_I; \mathcal{F})$ vanishes if $\mathcal{D}_I$ is disconnected, by Proposition 5.5, and is isomorphic to $\widetilde{\mathrm{HFK}}(\mathcal{L}_I) \otimes_{\mathbb{F}} \mathcal{F}$ if $\mathcal{D}_I$ is connected, by (3.6). Since no edge in the cube of resolutions of $\mathcal{D}$ can join two connected resolutions, the d_1 differential of $\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}$ is zero. Therefore, $E_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}) \cong E_1(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}})$. In Section 6, we prove that $\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}$ collapses at its E_3 page. Since $\mathcal{F}$ is a field, Corollary 5.12 implies that

$$H_*(E_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}), d_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}})) \cong \underline{\widehat{\mathrm{HFK}}}(L, [\omega_{\mathbf{r}}]) \otimes_{\mathbb{F}} V^{\otimes(m-|L|)};$$

if $\mathbf{r} = \mathbf{r}_\Omega$, then

$$H_*(E_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}), d_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}})) \cong \widehat{\mathrm{HFK}}(L) \otimes_{\mathbb{F}} V^{\otimes(m-|L|)} \otimes_{\mathbb{F}} \mathcal{F}.$$

In Section 7, we show that $(E_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}), d_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}))$ is isomorphic to the complex $(C^{\mathbf{r}}(\mathcal{D}), \partial^{\mathbf{r}})$ defined in Section 2. Combined with the grading calculations in Section 6, this proves Theorem 1.1.

We end this section with a brief discussion of orientations and gradings.

Recall that, for $I \in \{0,1\}^n$, the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\eta}(I), \mathbb{O}, \mathbb{X})$ determines L_I as an oriented link, where L_I is oriented as the boundary of the black regions in $\mathcal{D}_I$. Therefore, for $I, I' \in \{0,1\}^n$, the complexes $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$ and $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}(I), \boldsymbol{\eta}(I'))$ come equipped with Maslov and Alexander gradings.

Suppose I' is an immediate successor of I, differing in the jth entry. The Maslov and Alexander gradings of $\Theta_1^{I,I'}$ differ from those of $\Theta_2^{I,I'}$ by 1 (in the same direction); from now on, we shall assume that $\Theta_1^{I,I'}$ is the unique element of $\mathbb{T}_{\eta(I)} \cap \mathbb{T}_{\eta(I')}$ in the maximal Maslov grading. Furthermore, we may consider the chain maps

$$\Psi_i^{\alpha,\eta(I)}: \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I)) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$$
$$\Psi_i^{\eta(I),\eta(I')}: \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}(I), \boldsymbol{\eta}(I')) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}(I), \boldsymbol{\eta}(I'))$$

defined in Section 3.3, which count disks that go over the basepoints in $\mathbb{X}$. We shall use these maps to describe the differentials in the spectral sequence $\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}$. The following lemma will be useful.

Lemma 5.13. *Suppose I' is an immediate successor of I which differs from I in its jth coordinate. Let i_1 and i_3 be the special indices associated to the crossing c_j, as shown in Fig. 3. Then*

$$\Psi_{i_1}^{\eta(I),\eta(I')}(\Theta_1^{I,I'}) = \Psi_{i_3}^{\eta(I),\eta(I')}(\Theta_1^{I,I'}) = \Theta_2^{I,I'},$$

while $\Psi_i^{\eta(I),\eta(I')}(\Theta_1^{I,I'}) = 0$ for $i \neq i_1, i_3$.

Proof of Lemma 5.13. It is not hard to see that there is a unique $\mathbf{z} \in \mathbb{T}_{\eta(I)} \cap \mathbb{T}_{\eta(I')}$ such that there exists a Whitney disk $\phi \in \pi_2(\Theta_1^{I,I'}, \mathbf{z})$ with $X_{i_1}(\phi) = 1$, which avoids $\mathbb{O}$ and all other X_i. Namely, $\mathbf{z}$ is the point $\Theta_2^{I,I'}$ and the domain of $D(\phi)$ is an annulus, as in the proof of Lemma 5.1. There are actually two such disks in $\pi_2(\Theta_1^{I,I'}, \Theta_2^{I,I'})$ with $\mu = 1$, exactly one of which admits a holomorphic representative. (Compare [37, proof of Lemma 9.4].) This proves that $\Psi_{i_1}^{\eta(I),\eta(I')}(\Theta_1^{I,I'}) = \Theta_2^{I,I'}$. The other statements follow similarly. □

6. On δ-gradings

The summands $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$ of $X(\{0,1\}^n)$ are endowed with canonical absolute δ-gradings, by Lemma 4.1, and the complex $X(\{\infty\}^n) = \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^\infty))$ has an absolute δ-grading

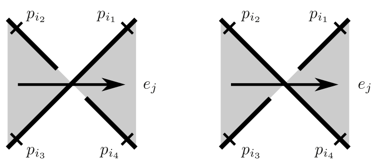


Fig. 3. Two possibilities for the neighborhood of c_j.

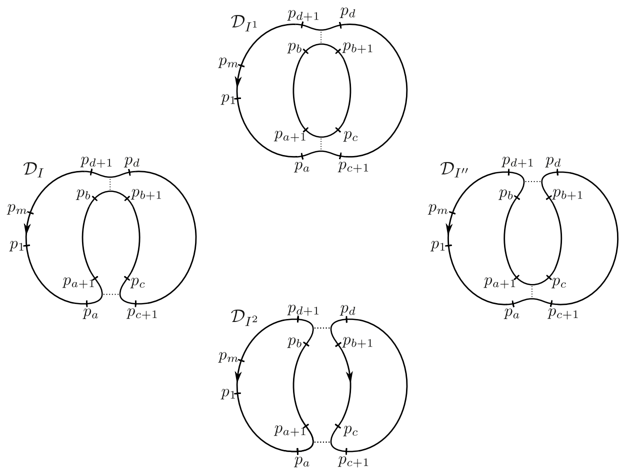


Fig. 4. The resolutions $\mathcal{D}_I$, $\mathcal{D}_{I^1}$, $\mathcal{D}_{I^2}$ and $\mathcal{D}_{I''}$ in the case that $\gamma_{I^1} = \gamma_I \cup e_{j_1}$, along with the marked points p_i. The dotted lines indicate the traces of the crossings c_{j_1} (bottom) and c_{j_2} (top). (If $m = d$, then p_1 plays the role of p_{d+1}.)

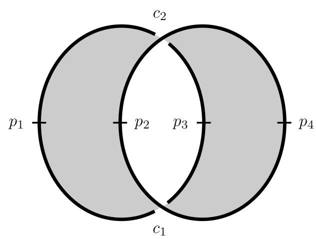


Fig. 5. Diagram for a two-component unlink whose cube of resolutions is Fig. 4.

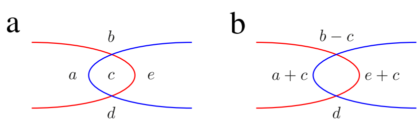


Fig. 6. We have labeled markings by the values that ω takes on them. In (a), the isotopy gets stuck at the point labeled c. In (b), we have removed this point and adjusted the values of ω on the four nearby points.

determined by the orientation of the original link L. Let Δ denote the grading on $X(\{0,1\}^n)$ obtained by shifting the δ-grading on each summand $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$ by $(|I| - n_-(\mathcal{D}))/2$. The two main results of this section are as follows.

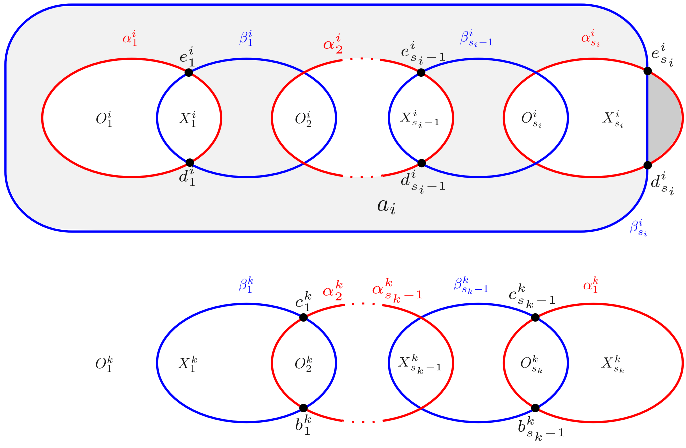


Fig. 7. A Heegaard diagram for $\mathcal{L}$. There is a copy of the upper portion for each $i = 1, \dots, k-1$.

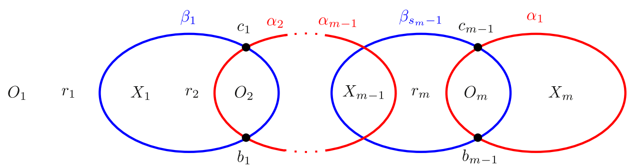


Fig. 8. Heegaard diagram $\mathcal{H}$ for the unknot, with twisting as prescribed in Proposition 4.3. Points of $\mathbb{A}$ are labeled with their values of ω.

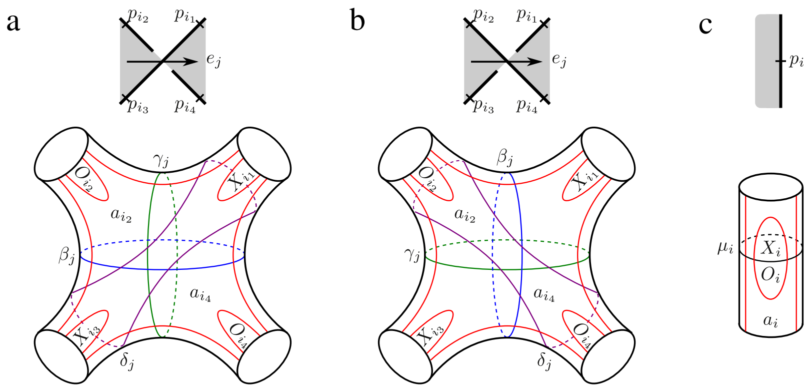


Fig. 9. (The portions of Σ near a crossing c_j (a–b) or a marked point p_i (c). The labeling conventions in (a–b) are the same as in Fig. 3.

Theorem 6.1. *With respect to Δ,*

(1) *the differential D on $X(\{0,1\}^n)$ is homogeneous of degree* 1, *and*
(2) *the quasi-isomorphism*

$$G : X(\{0,1\}^n) \to X(\{\infty\}^n)$$

coming from Theorem 5.10 *is grading preserving.*

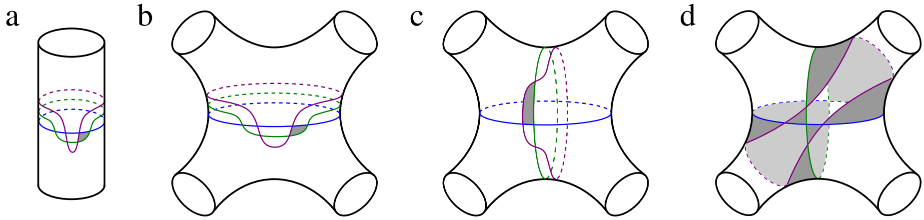


Fig. 10. Some of the possible $(\boldsymbol{\eta}(I^0), \boldsymbol{\eta}(I^1), \boldsymbol{\eta}(I^2))$-triangles.

Theorem 6.2. *If* $\mathbf{r}$ *is generic, then the differential* $d_k(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}})$ *vanishes for* $k > 2$. *Therefore,*

$$H_*(E_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}), d_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}})) \cong \underline{\widehat{\mathrm{HFK}}}(L, [\omega]_{\mathbf{r}}) \otimes_{\mathbb{F}} V^{\otimes(m-|L|)};$$

as graded vector spaces over $\mathcal{F}$, *with respect to the* δ-*grading on* $\widehat{\mathrm{HFK}}(L)$.

Proof of Theorem 6.2. By definition, $d_k(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}})$ is homogeneous of degree k with respect to the filtration grading Q, defined in Section 5.3. By Theorem 6.1, Δ descends to a grading on the pages of $\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}$. Recall, from the previous section, that $E_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}})$ consists of a copy of the group $\widetilde{\mathrm{HFK}}(\mathcal{L}_I) \otimes_{\mathbb{F}} \mathcal{F}$ for each $I \in \mathcal{R}(\mathcal{D})$. Since $\mathcal{L}_I$ is a pointed unknot, this group is supported in the Δ-grading $(|I| - n_-(\mathcal{D}))/2$; that is, the gradings Δ and Q on $E_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}})$ are related by

$$\Delta = (Q - n_-(\mathcal{D}))/2.$$

This relationship therefore holds for all $k \geq 2$. Suppose that x is a nonzero, homogeneous element of $E_k(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}})$. If $d_k(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}})(x) = y \neq 0$, then

$$\begin{aligned} 0 &= (2\Delta(y) - Q(y)) - (2\Delta(x) - Q(y)) \\ &= 2(\Delta(y) - \Delta(x)) - (Q(y) - Q(x)) \\ &= 2 - k. \end{aligned}$$

Thus, $d_k(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}})$ vanishes for $k > 2$. The second statement follows immediately from Theorem 6.1 and Corollary 5.12. □

The rest of this section is devoted to proving Theorem 6.1.

6.1. The relative δ-grading

First, we show that the maps $f_{I^0,\dots,I^0}$ are homogeneous with respect to the relative δ-grading. For a Whitney polygon ψ, let $\delta(\psi)$ denote the difference $P(\psi) - \mu(\psi)$. Note that this quantity is additive under concatenation of polygons.

Proposition 6.3. *Suppose* $I^0 < \cdots < I^k$ *is a successor sequence of tuples in* $\{0, 1, \infty\}^n$. *For* $i = 0, \dots, k$, *let* δ_{I^i} *be an arbitrary absolute lift of the relative* δ-*grading on* $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^i))$. *Then, for each* i, j *with* $0 \leq i < j \leq k$, *there are constants* $\delta(f_{I^i,\dots,I^j})$ *such that* $f_{I^i,\dots,I^j}$ *is homogeneous of degree* $\delta(f_{I^i,\dots,I^j})$ *with respect to these absolute lifts, and*

$$\delta(f_{I^i,\dots,I^j}) = \delta(f_{I^i,\dots,I^l}) + \delta(f_{I^l,\dots,I^j}) - 1, \tag{6.1}$$

for any $i < l < j$.

Proof of Proposition 6.3. Choose some $\mathbf{x}^s \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta(I^i)}$ and $\mathbf{y}^s \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta(I^j)}$ for $s = 1, 2$, and suppose ψ_s is a Whitney $(j - i + 2)$-gon in $\pi_2(\mathbf{x}^s, \Theta^{I^i,I^{i+1}}_{e^s_i}, \dots, \Theta^{I^{j-1},I^j}_{e^s_{j-1}}, \mathbf{y}^s)$, where

$e_i^s, \dots, e_{j-1}^s \in \{1, 2\}$. Such polygons always exist since the pairs $(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^l))$ span $H_1(\Sigma; \mathbb{Z})$ for $l = i, \dots, j$ (see, e.g., [37, Proposition 8.3]). We claim that

$$\delta_{I^j}(\mathbf{y}^1) - \delta_{I^i}(\mathbf{x}^1) + \delta(\psi_1) = \delta_{I^j}(\mathbf{y}^2) - \delta_{I^i}(\mathbf{x}^2) + \delta(\psi_2), \tag{6.2}$$

which enables us to define the quantity

$$\delta(f_{I^i,\dots,I^j}) = \delta_{I^j}(\mathbf{y}^1) - \delta_{I^i}(\mathbf{x}^1) + \delta(\psi_1) + i - j + 1 \tag{6.3}$$

independently of $\mathbf{x}^s$, $\mathbf{y}^s$, and ψ_s. To prove (6.2), let ψ_s' be the Whitney $(j - i + 1)$-gon in $\pi_2(\mathbf{x}^s, \Theta_1^{I^i,I^{i+1}}, \dots, \Theta_1^{I^{j-1},I^j}, \mathbf{y}^s)$ obtained by concatenating ψ_s with Whitney disks $\phi_i \in \pi_2(\Theta_1^{I^l,I^{l+1}}, \Theta_2^{I^l,I^{l+1}})$ where necessary. Since each ϕ_i satisfies $P(\phi_i) = \mu(\phi_i) = 1$, we have that $\delta(\psi_s') = \delta(\psi_s)$. Choose some Whitney disks $\phi_x \in \pi_2(\mathbf{x}^1, \mathbf{x}^2)$ and $\phi_y \in \pi_2(\mathbf{y}^1, \mathbf{y}^2)$, and consider the concatenation

$$\psi_2'' = \phi_x * \psi_2' * \overline{\phi}_y \in \pi_2(\mathbf{x}^1, \Theta_1^{I^i,I^{i+1}}, \dots, \Theta_1^{I^{j-1},I^j}, \mathbf{y}^1).$$

The difference $D(\psi_1') - D(\psi_2'')$ is a multi-periodic domain, so

$$\delta(\psi_1') - \delta(\psi_2'') = 0,$$

by Proposition 5.4. Thus,

$$\begin{aligned} \delta(\psi_1) = \delta(\psi_1') &= \delta(\psi_2'') \\ &= \delta(\psi_2') + \delta(\phi_x) - \delta(\phi_y) \\ &= \delta(\psi_2) + (\delta_{I^i}(\mathbf{x}^1) - \delta_{I^i}(\mathbf{x}^2)) - (\delta_{I^j}(\mathbf{y}^1) - \delta_{I^j}(\mathbf{y}^2)), \end{aligned}$$

from which (6.2) follows. Now, if $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta(I^j)}$ appears with nonzero coefficient in $f_{I^i,\dots,I^j}(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta(I^i)}$, then there exists a $(j-i+2)$-gon $\psi \in \pi_2(\mathbf{x}, \Theta_{e_i}^{I^i,I^{i+1}}, \dots, \Theta_{e_{j-1}}^{I^{j-1},I^j}, \mathbf{y})$ with

$$\delta(\psi) = P(\psi) - \mu(\psi) = 0 - (i - j + 1) = j - i - 1.$$

By (6.3), $\delta_{I^j}(\mathbf{y}) - \delta_{I^i}(\mathbf{x}) = \delta(f_{I^i,\dots,I^j})$. It follows that $f_{I^i,\dots,I^j}$ is homogeneous of degree $\delta(f_{I^i,\dots,I^j})$.

For the second part, let $\mathbf{x}$ and $\mathbf{y}$ be as above, and let $\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta(I^l)}$. Choose Whitney polygons

$$\psi_1 \in \pi_2(\mathbf{x}, \Theta_{e_i}^{I^i,I^{i+1}}, \dots, \Theta_{e_{l-1}}^{I^{l-1},I^l}, \mathbf{z}) \quad \text{and} \quad \psi_2 \in \pi_2(\mathbf{z}, \Theta_{e_i}^{I^l,I^{l+1}}, \dots, \Theta_{e_{j-1}}^{I^{j-1},I^j}, \mathbf{y}),$$

and let $\psi = \psi_1 * \psi_2$. Then,

$$\begin{aligned} \delta(f_{I^i,\dots,I^l}) + \delta(f_{I^l,\dots,I^j}) &= (\delta_{I^l}(\mathbf{z}) - \delta_{I^i}(\mathbf{x}) + \delta(\psi_1) + i - l + 1) \\ &\quad + (\delta_{I^j}(\mathbf{y}) - \delta_{I^l}(\mathbf{z}) + \delta(\psi_2) + l - j + 1) \\ &= \delta_{I^j}(\mathbf{y}) - \delta_{I^i}(\mathbf{x}) + \delta(\psi) + i - j + 2 \\ &= \delta(f_{I^i,\dots,I^j}) + 1, \end{aligned}$$

completing the proof of Proposition 6.3. □

Remark 6.4. Proposition 6.3 shows that the grading shifts $\delta(f_{I^i,\dots,I^j})$ are well defined and satisfy additivity properties under composition even if some of the maps $f_{I^i,\dots,I^j}$ are zero.

The next result shows that the maps $D_{I,I'}$ are homogeneous with respect to the relative δ-grading.

Proposition 6.5. *Suppose $\bar{I} = I^0 < \cdots < I^k$ and $\bar{J} = J^0 < \cdots < J^k$ are successor sequences of tuples in $\{0, 1, \infty\}^n$ with $I^0 = J^0$ and $I^k = J^k$. For any absolute lifts δ_{I^0} and δ_{I^k} of the relative δ-gradings on $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^0))$ and $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^k))$, the grading shifts $\delta(f_{I^0,\dots,I^k})$ and $\delta(f_{J^0,\dots,J^k})$ are equal. In particular, the map*

$$D_{I^0,I^k} : \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^0)) \to (\boldsymbol{\alpha}, \boldsymbol{\eta}(I^k))$$

is homogeneous of degree $\delta(D_{I^0,I^k}) = \delta(f_{I^0,\dots,I^k})$ with respect to these absolute lifts.

Proof of Proposition 6.5. The sequences $\bar{I}$ and $\bar{J}$ can be connected by an ordered list of sequences in which one sequence in the list differs from the next in a single place. It is therefore enough to prove Proposition 6.5 for $\bar{I}$ and $\bar{J}$, where

$$\bar{J} = I^0 < \cdots < I^{i-1} < J^i < I^{i+1} < \cdots < I^k.$$

For $i = 0, \dots, k$, let δ_{I^i} and δ_{J^i} denote arbitrary absolute lifts of the relative δ-gradings on the complexes $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^i))$ and $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(J^i))$. By (6.1), we need only show that

$$\delta(f_{I^{i-1},J^i}) + \delta(f_{J^i,I^{i+1}}) = \delta(f_{I^{i-1},I^i}) + \delta(f_{I^i,I^{i+1}}).$$

It is helpful to have in mind the following diagram, which commutes up to homotopy.

$$\begin{array}{ccc}
\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^{i-1})) & \xrightarrow{f_{I^{i-1},I^i}} & \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^i)) \\
\downarrow {\scriptstyle f_{I^{i-1},J^i}} & & \downarrow {\scriptstyle f_{I^i,I^{i+1}}} \\
\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(J^i)) & \xrightarrow[f_{J^i,I^{i+1}}]{} & \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^{i+1})).
\end{array}$$

Choose generators

$$\mathbf{x}^1 \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta(I^{i-1})}, \qquad \mathbf{y}^1 \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta(I^i)}, \qquad \mathbf{x}^2 \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta(J^i)}, \qquad \mathbf{y}^2 \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta(I^{i+1})},$$

and Whitney triangles

$$\psi_{I^{i-1},I^i} \in \pi_2(\mathbf{x}^1, \Theta_1^{I^{i-1},I^i}, \mathbf{y}^1), \qquad \psi_{I^i,I^{i+1}} \in \pi_2(\mathbf{y}^1, \Theta_1^{I^i,I^{i+1}}, \mathbf{y}^2),$$
$$\psi_{I^{i-1},J^i} \in \pi_2(\mathbf{x}^1, \Theta_1^{I^{i-1},J^i}, \mathbf{x}^2), \qquad \psi_{J^i,I^{i+1}} \in \pi_2(\mathbf{x}^2, \Theta_1^{J^i,I^{i+1}}, \mathbf{y}^2).$$

Let $\psi_1 \in \pi_2(\mathbf{x}^1, \Theta_1^{I^{i-1},I^i}, \Theta_1^{I^i,I^{i+1}}, \mathbf{y}^2)$ and $\psi_2 \in \pi_2(\mathbf{x}^1, \Theta_1^{I^{i-1},J^i}, \Theta_1^{J^i,I^{i+1}}, \mathbf{y}^2)$ denote the Whitney rectangles obtained by concatenation,

$$\psi_1 = \psi_{I^{i-1},I^i} * \psi_{I^i,I^{i+1}}, \qquad \psi_2 = \psi_{I^{i-1},J^i} * \psi_{J^i,I^{i+1}}.$$

As in the proof of Lemma 5.8, there exists some generator $\Theta_s^{J^i,I^i} \in \mathbb{T}_{\eta(J^i)} \cap \mathbb{T}_{\eta(I^i)}$ (one of the four generators with minimal δ-grading) such that there are Whitney triangles

$$\tau_1 \in \pi_2(\Theta_1^{I^{i-1},J^i}, \Theta_s^{J^i,I^i}, \Theta_1^{I^{i-1},I^i}) \quad \text{and} \quad \tau_2 \in \pi_2(\Theta_s^{J^i,I^i}, \Theta_1^{I^i,I^{i+1}}, \Theta_1^{J^i,I^{i+1}}),$$

whose domains are disjoint unions of small triangles, so that

$$P(\tau_1) = \mu(\tau_1) = P(\tau_2) = \mu(\tau_2) = 0.$$

Let ϕ_1 and ϕ_2 denote the Whitney pentagons in $\pi_2(\mathbf{x}^1, \Theta_1^{I^{i-1},J^i}, \Theta_s^{J^i,I^i}, \Theta_1^{I^i,I^{i+1}}, \mathbf{y}^2)$ obtained by concatenating τ_1 with ψ_1 at $\Theta_1^{I^{i-1},I^i}$ and τ_2 with ψ_2 at $\Theta_1^{J^i,I^{i+1}}$, respectively. The difference $D(\phi_1) - D(\phi_2)$ is a multi-periodic domain. Therefore,

$$\begin{aligned}
0 &= \delta(\phi_1) - \delta(\phi_2) \\
&= \delta(\psi_1) - \delta(\psi_2) \\
&= (\delta(\psi_{I^{i-1},I^i}) + \delta(\psi_{I^i,I^{i+1}})) - (\delta(\psi_{I^{i-1},J^i}) + \delta(\psi_{J^i,I^{i+1}})) \\
&= (\delta(f_{I^{i-1},I^i}) + \delta_{I^{i-1}}(\mathbf{x}^1) - \delta_{I^i}(\mathbf{y}^1) + \delta(f_{I^i,I^{i+1}}) + \delta_{I^i}(\mathbf{y}^1) - \delta_{I^{i+1}}(\mathbf{y}^2)) \\
&\quad - (\delta(f_{I^{i-1},J^i}) + \delta_{I^{i-1}}(\mathbf{x}^1) - \delta_{J^i}(\mathbf{x}^2) + \delta(f_{J^i,I^{i+1}}) + \delta_{J^i}(\mathbf{x}^2) - \delta_{I^{i+1}}(\mathbf{y}^2)) \\
&= (\delta(f_{I^{i-1},I^i}) + \delta(f_{I^i,I^{i+1}})) - (\delta(f_{I^{i-1},J^i}) + \delta(f_{J^i,I^{i+1}})),
\end{aligned}$$

completing the proof of Proposition 6.5. □

Before proceeding further, we pause to record a fact about Alexander and Maslov gradings that will be useful in Section 7. Recall that, for $I \in \{0,1\}^n$, we orient the diagrams $\mathcal{D}_I$ as boundaries of the black regions. These orientations determine absolute Maslov and Alexander gradings on the complexes $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$, per the discussion in Section 5.3.

Proposition 6.6. *Suppose $I^0 < \cdots < I^k$ is a successor sequence of tuples in $\{0,1\}^n$. For any $e_1, \ldots, e_k \in \{1,2\}$, the map*

$$F_{\alpha,\eta(I^0),\ldots,\eta(I^k)}(\cdot \otimes \Theta_{e_1}^{I^0,I^1} \otimes \cdots \otimes \Theta_{e_k}^{I^{k-1},I^k}) : \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^0)) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^k))$$

is homogeneous with respect to both the Alexander and Maslov gradings. Moreover, the Alexander and Maslov grading shifts of

$$F_{\alpha,\eta(I^0),\ldots,\eta(I^k)}(\cdot \otimes \Theta_{e_1}^{I^0,I^1} \otimes \cdots \otimes \Theta_1^{I^{i-1},I^i} \otimes \cdots \otimes \Theta_{e_k}^{I^{k-1},I^k})$$

are 1 greater than those of

$$F_{\alpha,\eta(I^0),\ldots,\eta(I^k)}(\cdot \otimes \Theta_{e_1}^{I^0,I^1} \otimes \cdots \otimes \Theta_2^{I^{i-1},I^i} \otimes \cdots \otimes \Theta_{e_k}^{I^{k-1},I^k}).$$

Proof of Proposition 6.6. This follows from the same reasoning as was used in the proof of Proposition 6.3. The key element in the latter was Proposition 5.4, which, in turn, follows from the fact that any doubly periodic domain D in the multi-diagram $\mathcal{H}$ satisfies $\mu(D) = P(D)$. To prove that

$$F_{\alpha,\eta(I^0),\ldots,\eta(I^k)}(\cdot \otimes \Theta_{e_0}^{I^0,I^1} \otimes \cdots \otimes \Theta_{e_{k-1}}^{I^{k-1},I^k})$$

is homogeneous with respect to the Alexander grading, we simply need the modification that $O(D) = X(D)$, which is clearly true. These two facts also imply that $\mu(D) = 2O(D)$, which is the modification we need for the homogeneity statement about Maslov gradings. The second statement in Proposition 6.6 follows from the fact that the Maslov and Alexander gradings of $\Theta_1^{I^{i-1},I^i}$ are each 1 greater than those of $\Theta_2^{I^{i-1},I^i}$. □

6.2. The absolute δ-grading

In this subsection, we compute certain absolute δ-grading shifts. These calculations, in conjunction with Propositions 6.3 and 6.5, complete the proof of Theorem 6.1.

Suppose $I^0 < I^1 < I^2$ is a successor sequence of tuples in $\{0, 1, \infty\}^n$ which differ only in their jth coordinates, and consider the maps

$$\begin{tikzcd} \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^2)) \arrow[r, "f_0"] \arrow[rr, bend right, "H_1"'] & \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^0)) \arrow[r, "f_1"] \arrow[rr, bend right, "H_2"'] & \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^1)) \arrow[r, "f_2"] & \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^2)), \end{tikzcd} \tag{6.4}$$

where

$$f_0 = f_{I^2,I^0}, \qquad f_1 = f_{I^0,I^1}, \qquad f_2 = f_{I^1,I^2},$$

and

$$H_1 = f_{I^2,I^0,I^1}, \qquad H_2 = f_{I^0,I^1,I^2}.$$

According to Proposition 5.7, the sum $\Phi = f_2 \circ H_1 + H_2 \circ f_0$ is a grading-preserving quasi-isomorphism. Now, fix an orientation on $\mathcal{D}_{I^2}$. If the crossing c_j is positive, then $\mathcal{D}_{I^1}$ naturally inherits an orientation from $\mathcal{D}_{I^2}$. We choose an orientation of $\mathcal{D}_{I^0}$ that agrees with the orientation of $\mathcal{D}_{I^2}$ on every component of $\mathcal{D}_{I^0}$ that does not pass through a neighborhood of c_j. Likewise, if c_j is negative, then $\mathcal{D}_{I^0}$ inherits an orientation from $\mathcal{D}_{I^2}$, and we choose an orientation of $\mathcal{D}_{I^1}$ that agrees with that of $\mathcal{D}_{I^2}$ on every component of $\mathcal{D}_{I^1}$ away from c_j. For $i = 0, 1, 2$, let $n_\pm(\mathcal{D}_{I^i})$ denote the number of $\pm$ crossings in $\mathcal{D}_{I^i}$ with respect to these orientations.

Proposition 6.7. *If c_j is positive, then $\delta(f_0) = n_-(\mathcal{D}_{I^2}) - n_-(\mathcal{D}_{I^0})$ and $\delta(f_2) = \frac{1}{2}$. If c_j is negative, then $\delta(f_0) = \frac{1}{2}$ and $\delta(f_2) = n_+(\mathcal{D}_{I^2}) - n_+(\mathcal{D}_{I^1})$. In either case, $\delta(H_1) = -\delta(f_2)$ and $\delta(H_2) = -\delta(f_0)$.*

Before proving Proposition 6.7, we illustrate how it is used to prove Theorem 6.1, starting with the corollary below.

Proposition 6.8. *Suppose $I^0 < \cdots < I^k$ is a successor sequence of tuples in $\{0, 1\}^n$. Then $\delta(f_{I^0 \ldots I^k}) = (2 - k)/2$.*

Proof of Proposition 6.8. Suppose I^i and I^{i+1} differ in their jth entries, and let J be the tuple obtained by replacing this entry with ∞. We may identify $f_{I^i,I^{i+1}}$ with the map f_1 in (6.4). Note that $\mathcal{D}_J$ is a diagram for an unlink with only one crossing. By Proposition 6.3, $\delta(f_0)+\delta(f_1) = \delta(H_1)+1$. If c_j is positive, then $\delta(f_0) = 0$ and $\delta(H_1) = -\frac{1}{2}$, by Proposition 6.7; otherwise, $\delta(f_0) = \frac{1}{2}$ and $\delta(H_1) = 0$. In either case, $\delta(f_1) = \frac{1}{2}$. According to Proposition 6.3,

$$\delta(f_{I^0,\ldots,I^k}) = \delta(f_{I^0,I^1}) + \cdots + \delta(f_{I^{k-1},I^k}) - (k - 1) = (2 - k)/2,$$

as claimed. □

Proof of Theorem 6.1. Suppose $I^0 \leq I^k$ are tuples in $\{0, 1\}^n$ which differ in k entries. The grading shift of D_{I^0,I^k} with respect to the grading Δ is

$$(2 - k)/2 - (|I^0| - n_-(\mathcal{D}))/2 + (|I^k| - n_-(\mathcal{D}))/2 = 1,$$

by Propositions 6.5 and 6.8. This proves the first statement of Theorem 6.1.

Now, let G_I denote the restriction of G to the summand $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$. Recall that G_I is the sum, over all sequences $I = I^0 < \cdots < I^n = I^\infty$ with $I^k \in \{0, 1\}^{n-k} \times \{\infty\}^k$, of the compositions $D_{I^{n-1},I^n} \circ \cdots \circ D_{I^0,I^1}$. It follows easily from Propositions 6.3 and 6.5

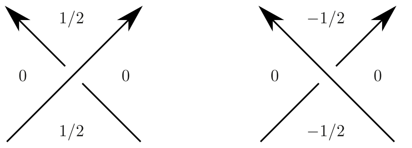


Fig. 11. The local contributions to $\delta(\mathbf{x})$ near a crossing.

that G_I is homogeneous. Choose a sequence $I = I^0 < \cdots < I^n = I^\infty$ as above, and let $J = (0, \ldots, 0)$. Choose absolute lifts of the relative δ-gradings on the complexes $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^i))$. By Propositions 6.3, 6.5 and 6.8,

$$\begin{aligned}\delta(D_{J,I^1}) = \delta(D_{J,I^0}) + \delta(D_{I^0,I^1}) - 1 &= (2 - |I|)/2 + \delta(D_{I^0,I^1}) - 1 \\ &= \delta(D_{I^0,I^1}) - |I|/2.\end{aligned}$$

Adding $\delta(D_{I^1,I^2}) + \cdots + \delta(D_{I^{n-1},I^n})$ to both sides, we have $\delta(G_J) = \delta(G_I) - |I|/2$; that is, $\delta(G_I) = |I|/2 + C$ for some constant C. For the second statement of Theorem 6.1, it suffices to show that $C = -n_-(\mathcal{D})$.

Define a tuple $I^0 \in \{0, 1\}^n$ according to the following rule: if c_j is a positive crossing, let $(I^0)_j = 1$; otherwise, let $(I^0)_j = 0$. Note that $\mathcal{D}_{I^0}$ is the oriented (Seifert) resolution of $\mathcal{D}$. For $i = 1, \ldots, n$, let I^i be the tuple obtained by changing the last i entries of I^0 to ∞. One of the terms appearing in G_{I^0} is $D_{I^{n-1},I^n} \circ \cdots \circ D_{I^0,I^1}$. In this composition, $n_+(\mathcal{D})$ of the maps are of the form f_2, as in (6.4), while $n_-(\mathcal{D})$ are of the form H_2. Therefore,

$$\delta(G_{I^0}) = n_+(\mathcal{D})/2 - n_-(\mathcal{D})/2 = |I^0|/2 - n_-(\mathcal{D})/2,$$

which implies that $C = -n_-(\mathcal{D})$. $\square$

The rest of this section is devoted to proving Proposition 6.7. For this, it helps to know the δ-gradings of certain generators. Let $A_1, \ldots, A_k$ denote the regions in the diagram $\mathcal{D}_I$ that are not adjacent to the marking p_m. Recall that a *Kauffman state* is a bijection which assigns to each crossing c of $\mathcal{D}_I$ one of the regions A_i incident to c.

A generator $\mathbf{x}$ of $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$ is said to be *Kauffman* if $\mathbf{x}$ does not contain any intersections points between ladybug and non-ladybug curves. A Kauffman generator $\mathbf{x}$ determines a Kauffman state $s_{\mathbf{x}}$ as follows: for each crossing c, let $s_{\mathbf{x}}(c)$ be the region whose corresponding $\boldsymbol{\alpha}$ curve intersects $\eta(I)_c$ in a point of $\mathbf{x}$. (This correspondence is 2^{m-1}-to-1.) Let $\delta(\mathbf{x}, c) \in \{0, \pm 1/2\}$ be the quantity defined in Fig. 11, according to which region is assigned to c in $s_{\mathbf{x}}$. Ozsváth and Szabó [33] prove that

$$\delta(\mathbf{x}) = \sum_c \delta(\mathbf{x}, c). \tag{6.5}$$

Proof of Proposition 6.7. We only consider the case in which c_j is a positive crossing; the proof for a negative crossing is extremely similar.

First, suppose that the projections $\mathcal{D}_{I^0}$, $\mathcal{D}_{I^1}$, and $\mathcal{D}_{I^2}$ are connected. In this case, we use an argument due to Manolescu and Ozsváth [28]. Choose some Kauffman generators $\mathbf{x}^0 \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta(I^0)}$ and $\mathbf{x}^1 \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta(I^1)}$. As in [28, Section 3.5], we may find corresponding Kauffman generators $\mathbf{y}^0, \mathbf{y}^1 \in \mathbb{T}_\alpha \cap \mathbb{T}_{\eta(I^2)}$ such that (1) $s_{\mathbf{x}_i}(c) = s_{\mathbf{y}_i}(c)$ for each $c \neq c_j$, and (2) there exist homotopy classes $\psi_0 \in \pi_2(\mathbf{y}_0, \Theta_{e_0}^{I^2,I^0}, \mathbf{x}_0)$ and $\psi_1 \in \pi_2(\mathbf{x}_1, \Theta_{e_1}^{I^1,I^2}, \mathbf{y}_1)$ with $\delta(\psi_i) = 0$ (for some $e_i \in \{1, 2\}$).

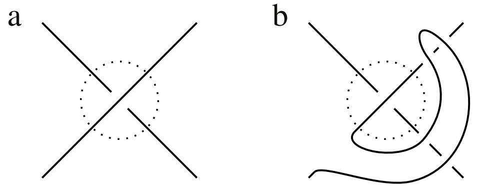


Fig. 12. (a) A crossing of $\mathcal{D}$. (b) The modified crossing in $\mathcal{D}'$.

Since $\mathcal{D}_{I^1}$ is the oriented resolution of $\mathcal{D}_{I^2}$, $\delta(\mathbf{x}_1, c) = \delta(\mathbf{y}_1, c)$ for every crossing $c \neq c_j$, while $\delta(\mathbf{y}_1, c_j) = \frac{1}{2}$. Therefore, $\delta(f_2) = \delta(\mathbf{y}_1) - \delta(\mathbf{x}_1) = \frac{1}{2}$, as claimed. On the other hand, the sign of a crossing in $\mathcal{D}_{I^0}$ need not be the same as its sign in $\mathcal{D}_{I^2}$. For any crossing $c \neq c_j$ that is negative in $\mathcal{D}_{I^2}$ and positive in $\mathcal{D}_{I^0}$, we have $\delta(\mathbf{x}_0, c) = \delta(\mathbf{y}_0, c) + \frac{1}{2}$; the number of such crossings is $n_-(\mathcal{D}_{I^2}) - n_-(\mathcal{D}_{I^0})$. Likewise, if $c \neq c_j$ is positive in $\mathcal{D}_{I^2}$ and negative in $\mathcal{D}_{I^0}$, then $\delta(\mathbf{x}_0, c) = \delta(\mathbf{y}_0, c) - \frac{1}{2}$; the number of such crossings is $n_+(\mathcal{D}_{I^2}) - 1 - n_+(\mathcal{D}_{I^0})$. Since $n_+(\mathcal{D}_{I^2}) + n_-(\mathcal{D}_{I^2}) = n$ and $n_+(\mathcal{D}_{I^0}) + n_-(\mathcal{D}_{I^0}) = n - 1$,

$$\begin{aligned}\delta(f_0) &= \delta(\mathbf{x}_0) - \delta(\mathbf{y}_0)\\ &= \frac{1}{2}\big((n_-(\mathcal{D}_{I^2}) - n_-(\mathcal{D}_{I^0}))\big) - \frac{1}{2}\big((n_+(\mathcal{D}_{I^2}) - n_+(\mathcal{D}_{I^0}) - 1)\big)\\ &= n_-(\mathcal{D}_{I^2}) - n_-(\mathcal{D}_{I^0}),\end{aligned}$$

as claimed.

If either $\mathcal{D}_{I^0}$ or $\mathcal{D}_{I^1}$ is disconnected, then the corresponding complex has no Kauffman generators, so the argument above does not apply. We remedy this situation as follows. Let $\mathcal{D}'$ be the planar diagram obtained from $\mathcal{D}$ by performing a finger move just outside of each crossing as in Fig. 12, and let $\mathcal{D}'_{I^0}$, $\mathcal{D}'_{I^1}$ and $\mathcal{D}'_{I^2}$ be the corresponding resolutions of $\mathcal{D}'$, leaving the newly introduced crossings unresolved. Notice that all three of these diagrams are connected, so the argument above applies.

Let $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\eta}'(I^0), \boldsymbol{\eta}'(I^1), \boldsymbol{\eta}'(I^2), \mathbb{O}, \mathbb{X})$ denote the Heegaard multi-diagram encoding $\mathcal{D}'_{I^0}$, $\mathcal{D}'_{I^1}$, and $\mathcal{D}'_{I^2}$ that is obtained from $\mathcal{D}'$ using the procedure in Section 5.1, except that we do not place ladybugs on the tubes corresponding to the edges that are contained entirely in Fig. 12(b). This diagram is related to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\eta}(I^0), \boldsymbol{\eta}(I^1), \boldsymbol{\eta}(I^2), \mathbb{O}, \mathbb{X})$ by a sequence of handleslides, isotopies, and index one/two stabilizations avoiding $\mathbb{O} \cup \mathbb{X}$. (Essentially, these Heegaard moves account for the Reidemeister II moves introduced by the operation in Fig. 12.) We therefore have diagrams,

$$\begin{array}{ccccccc}\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^2)) & \xrightarrow{f_0} & \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^0)) & \quad & \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^1)) & \xrightarrow{f_2} & \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^2))\\ \downarrow{\scriptstyle \Phi_2} & & \downarrow{\scriptstyle \Phi_0} & & \downarrow{\scriptstyle \Phi_1} & & \downarrow{\scriptstyle \Phi_2}\\ \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}'(I^2)) & \xrightarrow[f_0']{} & \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}'(I^0)) & & \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}'(I^1)) & \xrightarrow[f_2']{} & \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}'(I^2)),\end{array}$$

which commute up to homotopy, where Φ_0, Φ_1, and Φ_2 are the grading-preserving chain homotopy equivalences associated to these Heegaard moves. An argument very similar to that in the proof of Proposition 6.5 shows that $\delta(f_0) = \delta(f_0') = n_-(\mathcal{D}'_{I^2}) - n_-(\mathcal{D}'_{I^0}) = n_-(\mathcal{D}_{I^2}) - n_-(\mathcal{D}_{I^0})$ and $\delta(f_2) = \delta(f_2') = \frac{1}{2}$, as required.

Finally, note that $H_2 \circ f_0 + f_2 \circ H_1$ is a grading-preserving quasi-isomorphism, so at least one of these terms is nonzero. Therefore, $\delta(H_2) + \delta(f_0) = \delta(f_2) + \delta(H_1) = 0$, by Proposition 6.3, completing the proof of Proposition 6.7. □

7. The d_2 differential

From now on, we shall assume that $\mathbf{r}$ is generic. Recall from Section 5.3 that the E_2 term of $\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}$ is the direct sum

$$\bigoplus_{I \in \mathcal{R}(\mathcal{D})} \underline{\widetilde{\mathrm{HFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I); \mathcal{F}).$$

With respect to this direct sum decomposition, the differential $d_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}})$ is a sum of maps

$$d_{I,I''} : \underline{\widetilde{\mathrm{HFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I); \mathcal{F}) \to \underline{\widetilde{\mathrm{HFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I''); \mathcal{F})$$

over all pairs I, I'' for which I'' is a double successor of I. The purpose of this section is to compute these $d_{I,I''}$.

Suppose that $I, I'' \in \mathcal{R}(\mathcal{D})$ and that I'' is a double successor of I which differs from I in its j_1th and j_2th entries. Let J be the tuple obtained from I by changing its j_1th and j_2th entries from 0s to ∞s. Then $\mathcal{D}_J$ is a 2-crossing diagram for the 2-component unlink L_J. The four complete resolutions $\mathcal{D}_I$, $\mathcal{D}_{I^1}$, $\mathcal{D}_{I^2}$, and $\mathcal{D}_{I''}$ described in Section 2 are obtained from $\mathcal{D}_J$ by resolving these two crossings. Recall that $d_{I,I''}$ is defined in terms of maps that count pseudo-holomorphic polygons in the multi-diagram

$$\mathcal{H}_{I,I''} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\eta}(I), \boldsymbol{\eta}(I^1), \boldsymbol{\eta}(I^2), \boldsymbol{\eta}(I''), \mathbb{O}, \mathbb{X}).$$

Our strategy for computing $d_{I,I''}$ is as follows. First, we describe a sequence of Heegaard moves from $\mathcal{H}_{I,I''}$ to a "standard" genus-3 multi-diagram which encodes the four resolutions above. We then determine the relevant polygon-counting maps for this genus-3 diagram. Fortunately, it suffices to explicitly compute only a handful of these maps; the rest are determined via the Ψ_i maps defined in Section 3.3. Next, we argue that this model computation determines $d_{I,I''}$ to the extent that we can recover the isomorphism type of the complex $(E_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}), d_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}))$. Finally, we show that this complex is isomorphic to $(C^{\mathbf{r}}(\mathcal{D}), \partial^{\mathbf{r}})$.

As in Section 2, we assume that the marked points are ordered $p_1, \dots, p_m$ according to the orientation of $\mathcal{D}_I$. For $i = 1, \dots, m$, the value of $\omega_{\mathbf{r}}$ on the unique point of $\mathbb{A}$ that is contained in the same component of $\Sigma \setminus \boldsymbol{\eta}(I)$ as O_i equals r_i. As a notational convenience, we define

$$R(i, j) = \begin{cases} r_i + \cdots + r_j & i \le j \\ 0 & i > j, \end{cases}$$

so that $A = R(1, a)$, $B = R(a+1, b)$, $C = R(b+1, c)$, and $D = R(c+1, d)$.

7.1. A model computation

We may reduce $\mathcal{H}_{I,I''}$ to an admissible genus-3 multi-diagram via a sequence of handleslides and isotopies in the complement of $\mathbb{O} \cup \mathbb{X} \cup \mathbb{A}$, followed by index one/two destabilizations, as follows. Consider a crossing c_j, where $j \ne j_1, j_2$. The curves $\eta_{c_j}(I)$, $\eta_{c_j}(I^1)$, $\eta_{c_j}(I^2)$ and $\eta_{c_j}(I'')$ are pairwise isotopic and each intersects either one or two of the $\boldsymbol{\alpha}$ curves corresponding

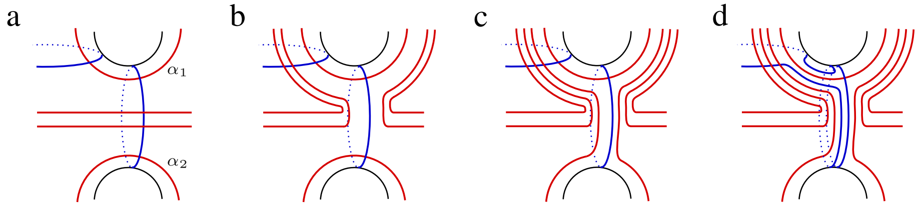


Fig. 13. A sequence of handleslides; in this example, there is one ladybug $\boldsymbol{\alpha}$ curve passing through the vertical curves $\eta_{c_j}(I)$, $\eta_{c_j}(I^1)$, $\eta_{c_j}(I^2)$, and $\eta_{c_j}(I'')$. For convenience, we have not shown the $\boldsymbol{\eta}(I^1)$, $\boldsymbol{\eta}(I^2)$, or $\boldsymbol{\eta}(I'')$ curves here; they are simply small translates of the blue $\boldsymbol{\eta}(I)$ curves in this picture.

to regions of $\mathbb{R}^2 \setminus \mathcal{D}$ in exactly one point. Call these curves α_1 and, if necessary, α_2. First, we handleslide all of the ladybug $\boldsymbol{\alpha}$ circles which pass through $\eta_{c_j}(I)$ over α_1, as in Fig. 13(b). (This takes two handleslides for each such $\boldsymbol{\alpha}$ curve.) Second, we handleslide α_2 over α_1 (if applicable), as in (c). Third, we handleslide all other $\boldsymbol{\eta}(I)$ (resp. $\boldsymbol{\eta}(I^1)$, $\boldsymbol{\eta}(I^2)$, and $\boldsymbol{\eta}(I'')$) curves which intersect α_1 over $\eta_{c_j}(I)$ (resp. $\eta_{c_j}(I^1)$, $\eta_{c_j}(I^2)$, and $\eta_{c_j}(I'')$), as in (d). The resulting multi-diagram is the connected sum of a multi-diagram of smaller genus with a standard torus piece. Handlesliding further, we can "move" this torus piece until it is adjacent to the region containing O_1. We perform these operations for each $j \neq j_1, j_2$, and then destabilize $n-2$ times.[5]

The genus-3 multi-diagram so obtained is the one we would associate to the planar diagram $\mathcal{D}_J$, following Section 5.1. Let us refer to this multi-diagram as $\hat{\mathcal{H}}^3_{I,I''}$. There are two cases to consider. If the smoothing of c_{j_1} in $\mathcal{D}_I$ connects the white regions — i.e., $\gamma_{I^1} = \gamma_I \cup e_{j_1}$ — then $\hat{\mathcal{H}}^3_{I,I''}$ is isotopic to the multi-diagram

$$\mathcal{H}^3_{I,I''} = (\Sigma_3, \boldsymbol{a}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\epsilon}, \mathbb{O}, \mathbb{X}),$$

depicted in Fig. 14, where $\boldsymbol{a}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, and $\boldsymbol{\epsilon}$ are the images of the tuples $\boldsymbol{\alpha}$, $\boldsymbol{\eta}(I)$, $\boldsymbol{\eta}(I^1)$, $\boldsymbol{\eta}(I^2)$, and $\boldsymbol{\eta}(I'')$, respectively, after these Heegaard moves. On the other hand, if the smoothing of c_{j_2} connects the black regions — i.e., $\gamma_{I^1} = \gamma_I \setminus e_{j_1}$ — then $\hat{\mathcal{H}}^3_{I,I''}$ is isotopic to the multi-diagram in Fig. 15, also denoted by $\mathcal{H}^3_{I,I''}$. (Note that, in either case, the ladybug curves in $\mathcal{H}^3_{I,I''}$ are stretched just enough to achieve admissibility, rather than all the way to the region containing X_1 as in the definition of $\mathcal{H}$.) We shall distinguish these two cases using the number $\nu = \nu_{I,I''}$, defined to be 1 in the first case and 0 in the second, as in Section 2.

In Figs. 14 and 15, we have indicated, by circles and squares, some intersection points between the $\boldsymbol{a}$ curves and the $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, and $\boldsymbol{\epsilon}$ curves. For each $i = 2, \dots, m$, let w_i (resp. x_i, y_i, and z_i) be the circular intersection point between a_{p_i} and some $\boldsymbol{\beta}$ (resp. $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, and $\boldsymbol{\epsilon}$) curve, and let w_i' (resp. x_i', y_i', and z_i') be the square intersection point between a_{p_i} and the same $\boldsymbol{\beta}$ (resp. $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, and $\boldsymbol{\epsilon}$) curve. Note that every point of $\mathbb{T}_a \cap \mathbb{T}_\beta$ (resp. $\mathbb{T}_a \cap \mathbb{T}_\gamma$, $\mathbb{T}_a \cap \mathbb{T}_\delta$, and $\mathbb{T}_a \cap \mathbb{T}_\epsilon$) contains either w_i or w_i' (resp. x_i or x_i', y_i or y_i', and z_i or z_i') for $i = 2, \dots, m$, and

$$\left|\mathbb{T}_a \cap \mathbb{T}_\beta\right| = \left|\mathbb{T}_a \cap \mathbb{T}_\gamma\right| = |\mathbb{T}_a \cap \mathbb{T}_\delta| = |\mathbb{T}_a \cap \mathbb{T}_\epsilon| = 2^{m-1}.$$

By construction, the unique point $\mathbf{w}_0$ (resp. $\mathbf{x}_0$, $\mathbf{y}_0$, and $\mathbf{z}_0$) of $\mathbb{T}_a \cap \mathbb{T}_\beta$ (resp. $\mathbb{T}_a \cap \mathbb{T}_\gamma$, $\mathbb{T}_a \cap \mathbb{T}_\delta$, and $\mathbb{T}_a \cap \mathbb{T}_\epsilon$) in the top Maslov grading contains all of the w_i (resp. x_i, y_i, and z_i). For $2, \dots, m$, let $\mathbf{w}_i$ be the generator obtained from $\mathbf{w}_0$ by replacing w_i by w_i', and define $\mathbf{x}_i$, $\mathbf{y}_i$, and $\mathbf{z}_i$ similarly.

[5] This is destabilization in the sense of multi-diagrams; see [40,47].

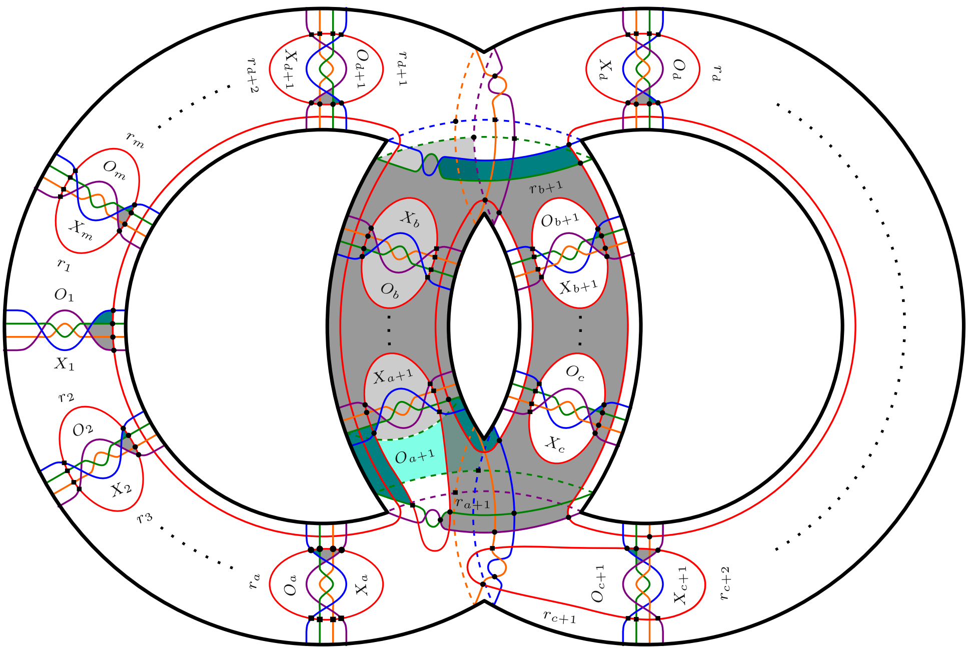


Fig. 14. $\mathcal{H}^3_{I,I''}$ in the case that $\gamma_{I^1} = \gamma_I \cup e_{j_1}$. The tuples $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, and $\boldsymbol{\epsilon}$ are drawn in red, blue, green, orange, and purple, respectively. The crossing c_{j_1} is on the top and c_{j_2} is on the bottom. The shaded regions represent the domains of the triangle classes considered in the proof of Proposition 7.5. Points in $\mathbb{A}$ are labeled with their corresponding values of $\omega_{\mathbf{r}}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

These constitute all of the generators in the second-to-top Maslov gradings of their respective complexes.

The two lemmas below are easy exercises in counting holomorphic disks; compare with Proposition 4.2 and Lemma 5.13.

Lemma 7.1. *The differentials on* $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha},\boldsymbol{\beta})$, $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha},\boldsymbol{\epsilon})$, $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\beta},\boldsymbol{\gamma})$, $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\beta},\boldsymbol{\delta})$, $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\gamma},\boldsymbol{\epsilon})$, *and* $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\delta},\boldsymbol{\epsilon})$ *are all zero. The differentials on* $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha},\boldsymbol{\gamma})$ *and* $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha},\boldsymbol{\delta})$ *are given by*

$$\partial_{\alpha\gamma}(\mathbf{x}) = \begin{cases} (1+T^{B+C})(\mathbf{x}\setminus\{x_{a+1}\}\cup\{x'_{a+1}\}) & x_{a+1}\in\mathbf{x} \\ 0 & x_{a+1}\notin\mathbf{x} \end{cases}$$

$$\partial_{a\delta}(\mathbf{y}) = \begin{cases} (1+T^{C+D})(\mathbf{y}\setminus\{y_{c+1}\}\cup\{y'_{c+1}\}) & y_{c+1}\in\mathbf{y} \\ 0 & y_{c+1}\notin\mathbf{y}. \end{cases}$$

Lemma 7.2. *We have*

$$\Psi_i^{\beta\gamma}(\Theta_1^{\beta\gamma}) = \begin{cases} \Theta_2^{\beta\gamma} & i\in\{a,c\} \\ 0 & \textit{otherwise} \end{cases} \qquad \Psi_i^{\gamma\epsilon}(\Theta_1^{\gamma\epsilon}) = \begin{cases} \Theta_2^{\gamma\epsilon} & i\in\{b,d\} \\ 0 & \textit{otherwise} \end{cases}$$

$$\Psi_i^{\beta\delta}(\Theta_1^{\beta\delta}) = \begin{cases} \Theta_2^{\beta\delta} & i\in\{b,d\} \\ 0 & \textit{otherwise} \end{cases} \qquad \Psi_i^{\delta\epsilon}(\Theta_1^{\delta\epsilon}) = \begin{cases} \Theta_2^{\delta\epsilon} & i\in\{a,c\} \\ 0 & \textit{otherwise}, \end{cases}$$

while $\Psi_i^{\beta\gamma}(\Theta_2^{\beta\gamma}) = \Psi_i^{\gamma\epsilon}(\Theta_2^{\gamma\epsilon}) = \Psi_i^{\beta\delta}(\Theta_2^{\beta\delta}) = \Psi_i^{\delta\epsilon}(\Theta_2^{\delta\epsilon}) = 0$ *for* $i = 1,\ldots,m$.

The genericity of $\mathbf{r}$ implies that $B + C$ and $C + D$ are nonzero, so

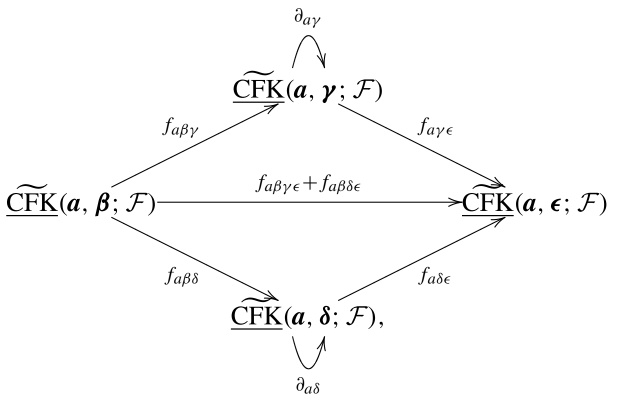
$$H_*(\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\gamma}; \mathcal{F}), \partial_{a\gamma}) = H_*(\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\delta}; \mathcal{F}), \partial_{a\delta}) = 0; \tag{7.1}$$

compare with Lemma 2.3. In other words, every cycle in $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\gamma}; \mathcal{F})$ or $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\delta}; \mathcal{F})$ is a boundary, so we may define maps $\partial_{a\gamma}^{-1}$ and $\partial_{a\delta}^{-1}$ up to addition of cycles. Indeed, for any $\mathbf{x} \in \mathbb{T}_a \cap \mathbb{T}_\gamma$ containing x'_{a+1}, we may take $\partial_{a\gamma}^{-1}(\mathbf{x}) = (1 + T^{B+C})^{-1}(\mathbf{x} \setminus \{x'_{a+1}\} \cup \{x_{a+1}\})$, and define $\partial_{a\delta}^{-1}$ similarly.

Let $f_{a\beta\gamma}$, $f_{a\beta\delta}$, $f_{a\gamma\epsilon}$, $f_{a\delta\epsilon}$, $f_{a\beta\gamma\epsilon}$, and $f_{a\beta\delta\epsilon}$ be the maps defined by

$$\begin{aligned} f_{a\beta\gamma}(\mathbf{x}) &= F_{a\beta\gamma}(\mathbf{x} \otimes (\Theta_1^{\beta\gamma} + \Theta_2^{\beta\gamma})), \\ f_{a\beta\gamma\epsilon}(\mathbf{x}) &= F_{a\beta\gamma\epsilon}(\mathbf{x} \otimes (\Theta_1^{\beta\gamma} + \Theta_2^{\beta\gamma}) \otimes (\Theta_1^{\gamma\epsilon} + \Theta_2^{\gamma\epsilon})), \end{aligned}$$

and so on. According to the discussion in Section 5.3, these maps fit into a commutative diagram,



(7.2)

which may be viewed as a filtered complex, where the filtration is by horizontal position. Let $\mathcal{S}_{I,I''}$ denote the spectral sequence associated to this filtered complex. Its d_1 differential vanishes due to (7.1). Since the differentials $\partial_{a\beta}$ and $\partial_{a\epsilon}$ are also zero, we may identify the complexes $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\beta}; \mathcal{F})$ and $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\epsilon}; \mathcal{F})$ with their homologies. With respect to this identification, $E_2(\mathcal{S}_{I,I''})$ is the mapping cone

$$\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\beta}; \mathcal{F}) \xrightarrow{\ g\ } \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\epsilon}; \mathcal{F}), \tag{7.3}$$

where, for any generator $\mathbf{w} \in \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\beta}; \mathcal{F})$, we have

$$\begin{aligned} g(\mathbf{w}) &= (f_{a\beta\gamma\epsilon} + f_{a\beta\delta\epsilon} + f_{a\gamma\epsilon} \circ \partial_{a\gamma}^{-1} \circ f_{a\beta\gamma} + f_{a\delta\epsilon} \circ \partial_{a\delta}^{-1} \circ f_{a\beta\delta})(\mathbf{w}) \\ &= F_{a\beta\gamma\epsilon}(\mathbf{w} \otimes (\Theta_1^{\beta\gamma} + \Theta_2^{\beta\gamma}) \otimes (\Theta_1^{\gamma\epsilon} + \Theta_2^{\gamma\epsilon})) \\ &\quad + F_{a\beta\delta\epsilon}(\mathbf{w} \otimes (\Theta_1^{\beta\delta} + \Theta_2^{\beta\delta}) \otimes (\Theta_1^{\delta\epsilon} + \Theta_2^{\delta\epsilon})) \\ &\quad + F_{a\gamma\epsilon}(\partial_{a\gamma}^{-1} F_{a\beta\gamma}(\mathbf{w} \otimes (\Theta_1^{\beta\gamma} + \Theta_2^{\beta\gamma})) \otimes (\Theta_1^{\gamma\epsilon} + \Theta_2^{\gamma\epsilon})) \\ &\quad + F_{a\delta\epsilon}(\partial_{a\delta}^{-1} F_{a\beta\delta}(\mathbf{w} \otimes (\Theta_1^{\beta\delta} + \Theta_2^{\beta\delta})) \otimes (\Theta_1^{\delta\epsilon} + \Theta_2^{\delta\epsilon})). \end{aligned}$$

Our goal, then, will be to understand the map g.

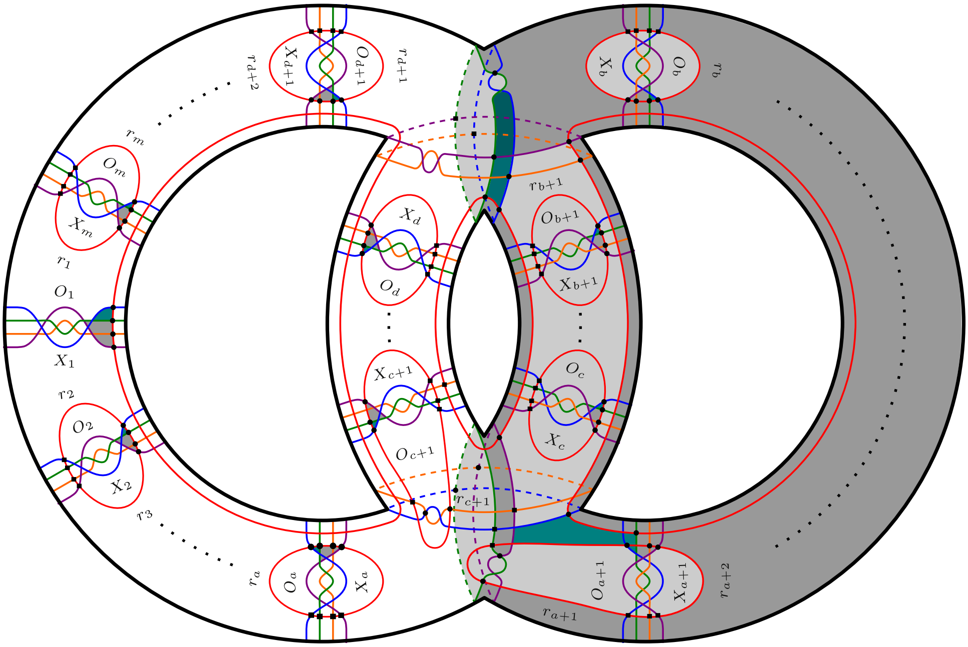


Fig. 15. $\mathcal{H}^3_{I,I''}$ in the case that $\gamma_{I^1} = \gamma_I \setminus e_{j_1}$.

7.2. *Commutation with the basepoint action*

For $k, l \in \{1, 2\}$, define

$$\begin{aligned} g^{k,l}(\mathbf{w}) &= F_{\alpha\beta\gamma\epsilon}(\mathbf{w} \otimes \Theta_k^{\beta\gamma} \otimes \Theta_l^{\gamma\epsilon}) + F_{\alpha\beta\delta\epsilon}(\mathbf{w} \otimes \Theta_l^{\beta\delta} \otimes \Theta_k^{\delta\epsilon}) \\ &\quad + F_{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}^{-1} F_{\alpha\beta\gamma}(\mathbf{w} \otimes \Theta_k^{\beta\gamma}) \otimes \Theta_l^{\gamma\epsilon}) + F_{\alpha\delta\epsilon}(\partial_{\alpha\delta}^{-1} F_{\alpha\beta\delta}(\mathbf{w} \otimes \Theta_l^{\beta\delta}) \otimes \Theta_k^{\delta\epsilon}). \end{aligned} \tag{7.4}$$

Clearly, $g = g^{1,1} + g^{1,2} + g^{2,1} + g^{2,2}$. The lemma below follows directly from Proposition 6.6.

Lemma 7.3. *The maps $g^{k,l}$ are each homogeneous with respect to the Maslov grading. Moreover, the Maslov grading shifts of these maps are related by*

$$M(g^{1,1}) = M(g^{1,2}) + 1 = M(g^{2,1}) + 1 = M(g^{2,2}) + 2.$$

This decomposition enables us to understand how g interacts with the maps $\Psi_i^{\alpha\beta}$ and $\Psi_i^{\alpha\epsilon}$.

Proposition 7.4. *For any $i = 1, \ldots, m$ and $k, l \in \{1, 2\}$, we have $g^{k,l} \circ \Psi_i^{\alpha\beta} = \Psi_i^{\alpha\epsilon} \circ g^{k,l}$, with the following exceptions.*

(1) *If $i = a$ or $i = c$, then $g^{1,l} \circ \Psi_i^{\alpha\beta} = \Psi_i^{\alpha\epsilon} \circ g^{1,l} + g^{2,l}$.*
(2) *If $i = b$ or $i = d$, then $g^{k,1} \circ \Psi_i^{\alpha\beta} = \Psi_i^{\alpha\epsilon} \circ g^{k,1} + g^{k,2}$.*

Proof of Proposition 7.4. From the $\mathcal{A}_\infty$ relation (3.13) and the fact that $\partial_{\alpha\beta} = \partial_{\alpha\epsilon} = 0$, we have, for each $\mathbf{w} \in \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\beta}; \mathcal{F})$ and $\mathbf{x} \in \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\gamma}; \mathcal{F})$, that

$$F_{\alpha\beta\gamma}(\Psi_i^{\alpha\beta}(\mathbf{w}),\Theta_k^{\beta\gamma}) = F_{\alpha\beta\gamma}(\mathbf{w},\Psi_i^{\beta\gamma}(\Theta_k^{\beta\gamma})) + \Psi_i^{\alpha\gamma}(F_{\alpha\beta\gamma}(\mathbf{w},\Theta_k^{\beta\gamma})) + \partial_{\alpha\gamma}(\Psi_i^{\alpha\beta\gamma}(\mathbf{w},\Theta_k^{\beta\gamma})), \tag{7.5}$$

$$F_{\alpha\gamma\epsilon}(\Psi_i^{\alpha\gamma}(\mathbf{x}),\Theta_l^{\gamma\epsilon}) = F_{\alpha\gamma\epsilon}(\mathbf{x},\Psi_i^{\gamma\epsilon}(\Theta_l^{\gamma\epsilon})) + \Psi_i^{\alpha\epsilon}(F_{\alpha\gamma\epsilon}(\mathbf{x},\Theta_l^{\gamma\epsilon})) + \Psi_i^{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}(\mathbf{x}),\Theta_k^{\gamma\epsilon}). \tag{7.6}$$

Applying $F_{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}^{-1}(\cdot),\Theta_l^{\gamma\epsilon})$ to both sides of (7.5), we have that

$$\begin{aligned}F_{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}^{-1}(F_{\alpha\beta\gamma}(\Psi_i^{\alpha\beta}(\mathbf{w}),\Theta_k^{\beta\gamma})),\Theta_l^{\gamma\epsilon}) &= F_{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}^{-1}(F_{\alpha\beta\gamma}(\mathbf{w},\Psi_i^{\beta\gamma}(\Theta_k^{\beta\gamma}))),\Theta_l^{\gamma\epsilon})\\ &+ F_{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}^{-1}(\Psi_i^{\alpha\gamma}(F_{\alpha\beta\gamma}(\mathbf{w},\Theta_k^{\beta\gamma}))),\Theta_l^{\gamma\epsilon}) + F_{\alpha\gamma\epsilon}(\Psi_i^{\alpha\beta\gamma}(\mathbf{w},\Theta_k^{\beta\gamma}),\Theta_l^{\gamma\epsilon}).\end{aligned} \tag{7.7}$$

In the second term on the right of (7.7), we may commute $\partial_{\alpha\gamma}^{-1}$ past $\Psi_i^{\alpha\gamma}$. We substitute $\mathbf{x} = \partial_{\alpha\gamma}^{-1}(F_{\alpha\beta\gamma}(\mathbf{w},\Theta_k^{\beta\gamma}))$ into (7.6) to obtain

$$\begin{aligned}F_{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}^{-1}(F_{\alpha\beta\gamma}(\Psi_i^{\alpha\beta}(\mathbf{w}),\Theta_k^{\beta\gamma})),\Theta_l^{\gamma\epsilon}) &= F_{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}^{-1}(F_{\alpha\beta\gamma}(\mathbf{w},\Psi_i^{\beta\gamma}(\Theta_k^{\beta\gamma}))),\Theta_l^{\gamma\epsilon})\\ &+ F_{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}^{-1}(F_{\alpha\beta\gamma}(\mathbf{w},\Theta_k^{\beta\gamma})),\Psi_i^{\gamma\epsilon}(\Theta_l^{\gamma\epsilon})) + \Psi_i^{\alpha\epsilon}(F_{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}^{-1}(F_{\alpha\beta\gamma}(\mathbf{w},\Theta_k^{\beta\gamma})),\Theta_l^{\gamma\epsilon}))\\ &+ \Psi_i^{\alpha\gamma\epsilon}(F_{\alpha\beta\gamma}(\mathbf{w},\Theta_k^{\beta\gamma}),\Theta_k^{\gamma\epsilon}) + F_{\alpha\gamma\epsilon}(\Psi_i^{\alpha\beta\gamma}(\mathbf{w},\Theta_k^{\beta\gamma}),\Theta_l^{\gamma\epsilon}).\end{aligned} \tag{7.8}$$

Similarly, the $\mathcal{A}_\infty$ relation for the quadrilateral-counting maps yields

$$\begin{aligned}F_{\alpha\beta\gamma\epsilon}(\Psi^{\alpha\beta}(\mathbf{w}),\Theta_k^{\beta\gamma},\Theta_l^{\gamma\epsilon}) &= F_{\alpha\beta\gamma\epsilon}(\mathbf{w},\Psi_i^{\beta\gamma}(\Theta_k^{\beta\gamma}),\Theta_l^{\gamma\epsilon})\\ &+ F_{\alpha\beta\gamma\epsilon}(\mathbf{w},\Theta_k^{\beta\gamma},\Psi_i^{\gamma\epsilon}(\Theta_l^{\gamma\epsilon})) + \Psi_i^{\alpha\epsilon}(F_{\alpha\beta\gamma\epsilon}(\mathbf{w},\Theta_k^{\beta\gamma},\Theta_l^{\gamma\epsilon}))\\ &+ F_{\alpha\gamma\epsilon}(\Psi_i^{\alpha\beta\gamma}(\mathbf{w},\Theta_k^{\beta\gamma}),\Theta_l^{\gamma\epsilon}) + F_{\alpha\beta\epsilon}(\mathbf{w},\Psi_i^{\beta\gamma\epsilon}(\Theta_k^{\beta\gamma},\Theta_l^{\gamma\epsilon}))\\ &+ \Psi_i^{\alpha\gamma\epsilon}(F_{\alpha\beta\gamma}(\mathbf{w},\Theta_k^{\beta\gamma}),\Theta_l^{\gamma\epsilon}) + \Psi_i^{\alpha\beta\epsilon}(\mathbf{w},F_{\beta\gamma\epsilon}(\Theta_k^{\beta\gamma},\Theta_l^{\gamma\epsilon})).\end{aligned} \tag{7.9}$$

Adding (7.8) and (7.9) and canceling terms, we have

$$\begin{aligned}&F_{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}^{-1}(F_{\alpha\beta\gamma}(\Psi_i^{\alpha\beta}(\mathbf{w}),\Theta_k^{\beta\gamma})),\Theta_l^{\gamma\epsilon}) + F_{\alpha\beta\gamma\epsilon}(\Psi^{\alpha\beta}(\mathbf{w}),\Theta_k^{\beta\gamma},\Theta_l^{\gamma\epsilon})\\ &= \Psi_i^{\alpha\epsilon}(F_{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}^{-1}(F_{\alpha\beta\gamma}(\mathbf{w},\Theta_k^{\beta\gamma})),\Theta_l^{\gamma\epsilon}) + F_{\alpha\beta\gamma\epsilon}(\mathbf{w},\Theta_k^{\beta\gamma},\Theta_l^{\gamma\epsilon}))\\ &\quad + F_{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}^{-1}(F_{\alpha\beta\gamma}(\mathbf{w},\Psi_i^{\beta\gamma}(\Theta_k^{\beta\gamma}))),\Theta_l^{\gamma\epsilon}) + F_{\alpha\beta\gamma\epsilon}(\mathbf{w},\Psi_i^{\beta\gamma}(\Theta_k^{\beta\gamma}),\Theta_l^{\gamma\epsilon})\\ &\quad + F_{\alpha\gamma\epsilon}(\partial_{\alpha\gamma}^{-1}(F_{\alpha\beta\gamma}(\mathbf{w},\Theta_k^{\beta\gamma})),\Psi_i^{\gamma\epsilon}(\Theta_l^{\gamma\epsilon})) + F_{\alpha\beta\gamma\epsilon}(\mathbf{w},\Theta_k^{\beta\gamma},\Psi_i^{\gamma\epsilon}(\Theta_l^{\gamma\epsilon}))\\ &\quad + F_{\alpha\beta\epsilon}(\mathbf{w},\Psi_i^{\beta\gamma\epsilon}(\Theta_k^{\beta\gamma},\Theta_l^{\gamma\epsilon})) + \Psi_i^{\alpha\beta\epsilon}(\mathbf{w},F_{\beta\gamma\epsilon}(\Theta_k^{\beta\gamma},\Theta_l^{\gamma\epsilon})).\end{aligned} \tag{7.10}$$

Similarly,

$$\begin{aligned}&F_{\alpha\delta\epsilon}(\partial_{\alpha\delta}^{-1}(F_{\alpha\beta\delta}(\Psi_i^{\alpha\beta}(\mathbf{w}),\Theta_l^{\beta\delta})),\Theta_k^{\delta\epsilon}) + F_{\alpha\beta\delta\epsilon}(\Psi^{\alpha\beta}(\mathbf{w}),\Theta_l^{\beta\delta},\Theta_k^{\delta\epsilon})\\ &= \Psi_i^{\alpha\epsilon}(F_{\alpha\delta\epsilon}(\partial_{\alpha\delta}^{-1}(F_{\alpha\beta\delta}(\mathbf{w},\Theta_l^{\beta\delta})),\Theta_k^{\delta\epsilon}) + F_{\alpha\beta\delta\epsilon}(\mathbf{w},\Theta_l^{\beta\delta},\Theta_k^{\delta\epsilon}))\\ &\quad + F_{\alpha\delta\epsilon}(\partial_{\alpha\delta}^{-1}(F_{\alpha\beta\delta}(\mathbf{w},\Theta_l^{\beta\delta})),\Psi_i^{\delta\epsilon}(\Theta_k^{\delta\epsilon})) + F_{\alpha\beta\delta\epsilon}(\mathbf{w},\Theta_l^{\beta\delta},\Psi_i^{\delta\epsilon}(\Theta_k^{\delta\epsilon}))\\ &\quad + F_{\alpha\delta\epsilon}(\partial_{\alpha\delta}^{-1}(F_{\alpha\beta\delta}(\mathbf{w},\Psi_i^{\beta\delta}(\Theta_l^{\beta\delta}))),\Theta_k^{\delta\epsilon}) + F_{\alpha\beta\delta\epsilon}(\mathbf{w},\Psi_i^{\beta\delta}(\Theta_l^{\beta\delta}),\Theta_k^{\delta\epsilon})\\ &\quad + F_{\alpha\beta\epsilon}(\mathbf{w},\Psi_i^{\beta\delta\epsilon}(\Theta_l^{\beta\delta},\Theta_k^{\delta\epsilon})) + \Psi_i^{\alpha\beta\epsilon}(\mathbf{w},F_{\beta\delta\epsilon}(\Theta_l^{\beta\delta},\Theta_k^{\delta\epsilon})).\end{aligned} \tag{7.11}$$

The first lines in (7.10) and (7.11) sum to $g^{k,l}(\Psi_i^{a\beta}(\mathbf{w}))$, and the sum of the second lines equals $\Psi_i^{a\epsilon}(g^{k,l}(\mathbf{w}))$. By Lemma 7.2, if $k = 1$ and $i \in \{b,d\}$, then the sum of the third lines of (7.10) and (7.11) equals $g^{2,l}(\mathbf{w})$; otherwise, it equals zero. Similarly, the sum of the fourth lines equals $g^{k,2}(\mathbf{w})$ if $l = 1$ and $i \in \{a,c\}$, and zero otherwise. Thus, to finish the proof of Proposition 7.4, we only need to show that the sum of the fifth lines is zero; that is,

$$\begin{aligned} &F_{a\beta\epsilon}(w, \Psi_i^{\beta\gamma\epsilon}(\Theta_k^{\beta\gamma}, \Theta_l^{\gamma\epsilon}) + \Psi_i^{\beta\delta\epsilon}(\Theta_l^{\beta\delta}, \Theta_k^{\delta\epsilon})) \\ &\quad + \Psi_i^{a\beta\epsilon}(w, F_{\beta\gamma\epsilon}(\Theta_k^{\beta\gamma}, \Theta_l^{\gamma\epsilon}) + F_{\beta\delta\epsilon}(\Theta_l^{\beta\delta}, \Theta_k^{\delta\epsilon})) = 0. \end{aligned} \tag{7.12}$$

This follows from an argument nearly identical to those in the proofs of Lemma 5.8 and Proposition 5.9. Let $\Theta_{k,l}^{\beta\epsilon} \in \mathbb{T}_\beta \cap \mathbb{T}_\epsilon$ denote the generator consisting of the point of $\beta_{c_{j_2}} \cap \epsilon_{c_{j_2}}$ nearest the point of $\beta_{c_{j_2}} \cap \gamma_{c_{j_2}}$ in $\Theta_k^{\beta\gamma}$; the point of $\beta_{c_{j_1}} \cap \epsilon_{c_{j_1}}$ nearest the point of $\gamma_{c_{j_1}} \cap \epsilon_{c_{j_1}}$ in $\Theta_l^{\gamma\epsilon}$; and, for each $i = 1, \dots, m$, the intersection point $\theta_{p_i}^{\beta\epsilon}$ of $\beta_{p_i} \cap \epsilon_{p_i}$ with smallest δ-grading contribution, as in Section 5.1. As in Lemma 5.1, $\Theta_{1,1}^{\beta\epsilon}$, $\Theta_{2,1}^{\beta\epsilon}$, $\Theta_{1,2}^{\beta\epsilon}$, and $\Theta_{2,2}^{\beta\epsilon}$ are the generators in $\mathbb{T}_\beta \cap \mathbb{T}_\epsilon$ with minimal δ-grading. Moreover, it is not hard to see that

$$F_{\beta\gamma\epsilon}(\Theta_k^{\beta\gamma}, \Theta_l^{\gamma\epsilon}) = F_{\beta\delta\epsilon}(\Theta_l^{\beta\delta}, \Theta_k^{\delta\epsilon}) = \Theta_{k,l}^{\beta\epsilon};$$

the domains that contribute to these maps are simply disjoint unions of small triangles. Furthermore, the δ-grading shifts of $\Psi_i^{\beta\gamma\epsilon}$ and $\Psi_i^{\beta\gamma\epsilon}$ are 1 less than those of $F_{\beta\gamma\epsilon}$ and $F_{\beta\delta\epsilon}$, respectively. Thus,

$$\Psi_i^{\beta\gamma\epsilon}(\Theta_k^{\beta\gamma}, \Theta_l^{\gamma\epsilon}) = \Psi_i^{\beta\delta\epsilon}(\Theta_l^{\beta\delta}, \Theta_k^{\delta\epsilon}) = 0,$$

and both terms on the left side of (7.12) vanish. □

Next, we describe the actions of the maps $\Psi_i^{a\beta}$ and $\Psi_i^{a\epsilon}$. The diagrams $(\Sigma_3, \boldsymbol{a}, \boldsymbol{\beta}, \mathbb{O}, \mathbb{X})$ and $(\Sigma_3, \boldsymbol{a}, \boldsymbol{\epsilon}, \mathbb{O}, \mathbb{X})$ both satisfy the hypotheses of Proposition 4.3 with respect to the marking $(\mathbb{A}, \omega_{\mathbf{r}})$. Therefore, without any direct computation, we know that

$$\sum_{i=1}^{m} T^{R(1,i)}\, \Psi_i^{a\beta} = 0, \tag{7.13}$$

$$\begin{aligned} &\sum_{i\in\{1,\dots,a,d+1,\dots,m\}} T^{R(1,i)}\, \Psi_i^{a\epsilon} + \sum_{i=c+1}^{d} T^{A+R(c+1,i)}\, \Psi_i^{a\epsilon} \\ &\quad + \sum_{i=b+1}^{c} T^{A+D+R(b+1,i)}\, \Psi_i^{a\epsilon} + \sum_{i=a+1}^{b} T^{A+D+C+R(a+1,i)}\, \Psi_i^{a\epsilon} = 0. \end{aligned} \tag{7.14}$$

Any element of $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\beta}; \mathcal{F})$ may be obtained from $\mathbf{w}_0$ through a sum of compositions of the $\Psi_i^{a\beta}$ maps, by Proposition 4.3. Therefore, by Proposition 7.4, the values $g^{1,1}(\mathbf{w}_0)$, $g^{1,2}(\mathbf{w}_0)$, $g^{2,1}(\mathbf{w}_0)$, and $g^{2,2}(\mathbf{w}_0)$ determine the entire function g. We shall see in a moment that $g^{2,1}(\mathbf{w}_0)$ is a nonzero multiple of $\mathbf{z}_0$, the unique generator of $\underline{\mathrm{CFK}}(\boldsymbol{a}, \boldsymbol{\epsilon}; \mathcal{F})$ in the top Maslov grading. It follows that the other values $g^{k,l}(\mathbf{w}_0)$ are completely determined by $g^{2,1}(\mathbf{w}_0)$. Indeed, it must be the case that $g^{1,1}(\mathbf{w}_0) = 0$ by Lemma 7.3. Next, by (7.13) and Proposition 7.4, we have

$$\begin{aligned} 0 &= g\left(\sum_{i=1}^{m} T^{R(1,i)}\, \Psi_i^{\alpha\beta}(\mathbf{w}_0)\right) \\ &= \sum_{i=1}^{m} T^{R(1,i)}(g^{1,1} + g^{1,2} + g^{2,1} + g^{2,2})(\Psi_i^{\alpha\beta}(\mathbf{w}_0)) \\ &= \sum_{i=1}^{m} T^{R(1,i)}\, \Psi_i^{\alpha\epsilon}((g^{1,1} + g^{1,2} + g^{2,1} + g^{2,2})(\mathbf{w}_0)) \\ &\quad + (T^{A} + T^{A+B+C})(g^{2,1} + g^{2,2})(\mathbf{w}_0) \\ &\quad + (T^{A+B} + T^{A+B+C+D})(g^{1,2} + g^{2,2})(\mathbf{w}_0). \end{aligned} \tag{7.15}$$

The sum of the terms in the top Maslov grading must equal zero, so

$$(T^{A} + T^{A+B+C})g^{2,1}(\mathbf{w}_0) + (T^{A+B} + T^{A+B+C+D})g^{1,2}(\mathbf{w}_0) = 0,$$

which determines $g^{1,2}(\mathbf{w}_0)$. Likewise, the sum of the terms in the second-to-top Maslov grading must equal zero, so

$$\begin{aligned} &\sum_{i=1}^{m} T^{R(1,i)}\, \Psi_i^{\alpha\epsilon}((g^{1,2} + g^{2,1})(\mathbf{w}_0)) \\ &\quad + (T^{A} + T^{A+B} + T^{A+B+C} + T^{A+B+C+D})g^{2,2}(\mathbf{w}_0) = 0, \end{aligned} \tag{7.16}$$

which determines $g^{2,2}(\mathbf{w}_0)$. Thus, g is determined by the following.

Proposition 7.5. *The map $g^{2,1}$ satisfies*

$$g^{2,1}(\mathbf{w}_0) = \frac{T^{B+\nu C}}{1 + T^{B+C}}\mathbf{z}_0.$$

Proof of Proposition 7.5. In each of Figs. 14 and 15, the turquoise regions represent the domain of a Whitney triangle $\psi_1 \in \pi_2(\mathbf{w}_0, \Theta_2^{\beta\gamma}, \mathbf{x}_{a+1})$, and the (partially overlapping) gray regions represent the domain of a triangle $\psi_2 \in \pi_2(\mathbf{x}_0, \Theta_1^{\gamma\epsilon}, \mathbf{z}_0)$. Both of these domains avoid the basepoints, and their weights are $\langle \omega_{\mathbf{r}}, \psi_1 \rangle = 0$ and $\langle \omega_{\mathbf{r}}, \psi_2 \rangle = B + \nu C$. Moreover, one can verify using Sarkar's formula for the Maslov index of polygons [48] that $\mu(\psi_1) = \mu(\psi_2) = 0$. Since the map $F_{\alpha\beta\gamma}(\cdot \otimes \Theta_2^{\beta\gamma})$ is homogeneous, it follows that

$$F_{\alpha\beta\gamma}(\mathbf{w}_0 \otimes \Theta_2^{\beta\gamma}) = \sum_{i=1}^{m-1} s_i \mathbf{w}_i$$

for some coefficients $s_i \in \mathcal{F}$. Since $\mathbf{w}_0$ is a cycle, the right side of (7.17) must be as well, which implies that $s_i = 0$ for $i \neq a + 1$. Furthermore, it is easy to verify that ψ_1 is the only positive class in $\pi_2(\mathbf{w}_0, \Theta_2^{\beta\gamma}, \mathbf{x}_{a+1})$ Thus, we may conclude that

$$F_{\alpha\beta\gamma}(\mathbf{w}_0 \otimes \Theta_2^{\beta\gamma}) = s \cdot \mathbf{x}_{a+1} \tag{7.17}$$

for some $s \in \mathbb{F}$. A similar argument shows that

$$F_{\alpha\gamma\epsilon}(\mathbf{x}_0 \otimes \Theta_1^{\gamma\epsilon}) = t \cdot T^{B+\nu C}\mathbf{z}_0 \tag{7.18}$$

for some $t \in \mathbb{F}$. Therefore,

$$F_{a\gamma\epsilon}(\partial_{a\gamma}^{-1}(F_{a\beta\gamma}(\mathbf{w}_0 \otimes \Theta_2^{\beta\gamma}) \otimes \Theta_1^{\gamma\epsilon})) = s \cdot t \cdot \frac{T^{B+\nu C}}{1+T^{B+C}}\mathbf{z}_0. \tag{7.19}$$

We shall see that s and t are both equal to 1. Remarkably, we will not need any direct analysis of moduli spaces to prove this fact.

Note that the Maslov grading shift of $F_{a\beta\gamma}(\cdot, \Theta_k^{\beta\gamma})$ is equal to that of $F_{a\beta\delta}(\cdot, \Theta_k^{\beta\delta})$, so $F_{a\beta\delta}(\mathbf{w}_0, \Theta_2)$ must be in the second-to-top Maslov grading in $\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\delta}; \mathcal{F})$. This implies that $F_{a\beta\delta}(\mathbf{w}_0, \Theta_1)$ is in the top Maslov grading and, hence, is a multiple of $\mathbf{y}_0$. However, this multiple must be zero, since $\mathbf{w}_0$ is a cycle while $\partial_{a\delta}(\mathbf{y}_0) \neq 0$. Thus,

$$F_{a\delta\epsilon}(\partial_{a\delta}^{-1}(F_{a\beta\delta}(\mathbf{w}_0 \otimes \Theta_1^{\beta\delta}) \otimes \Theta_2^{\delta\epsilon})) = 0. \tag{7.20}$$

Next, we claim that the two terms in $g^{2,1}(\mathbf{w}_0)$ which count holomorphic rectangles both vanish; that is,

$$F_{a\beta\gamma\epsilon}(\mathbf{w}_0 \otimes \Theta_2^{\beta\gamma} \otimes \Theta_1^{\gamma\epsilon}) = F_{a\beta\delta\epsilon}(\mathbf{w}_0 \otimes \Theta_1^{\beta\delta} \otimes \Theta_2^{\delta\epsilon}) = 0. \tag{7.21}$$

It follows from (7.17), (7.18), and the fact that $g^{2,1}$ is homogeneous that both terms in (7.21) are multiples of $\mathbf{z}_0$. To prove (7.21), we show that the domain of any Whitney rectangle ψ in $\pi_2(\mathbf{w}_0, \Theta_2^{\beta\gamma}, \Theta_1^{\gamma\epsilon}, \mathbf{z}_0)$ or $\pi_2(\mathbf{w}_0, \Theta_1^{\beta\delta}, \Theta_2^{\delta\epsilon}, \mathbf{z}_0)$ in Fig. 14 which avoids $\mathbb{O} \cup \mathbb{X}$ has some negative multiplicities (the same argument works for the diagram in Fig. 15). For $i = 1, \ldots, a$, the local multiplicities of $D(\psi)$ near p_i are as shown in Fig. 16(a), for some integers p, q. To avoid negative multiplicities, we are forced to have $p = q = 0$; it follows that the multiplicity of the top region equals that of the bottom region. For $i = 1$, this top region has multiplicity 0 since it contains X_m. Inductively, the region directly to the right of X_a in Fig. 14 has multiplicity 0. The multiplicities of $D(\psi)$ in the regions near O_{a+1} and O_{c+1} are therefore as shown in Fig. 16(b) and (c), for some integers r, s, t, and we are forced to have $r = s = t = 0$. Since neither $\Theta_1^{\beta\gamma}$ nor $\Theta_1^{\delta\epsilon}$ is a corner of $D(\psi)$, the multiplicity on the underside of the upper-right tube must be -1. As a result, ψ has no holomorphic representative.

Therefore, the only potentially nonzero contribution to $g^{2,1}(\mathbf{w}_0)$ (of the four terms in (7.4)) is that in (7.19). If $s \cdot t = 0$, then $g(\mathbf{w}_0)$ is also zero, by (7.15) and (7.16). This implies that g is identically zero, by Proposition 7.4. On the other hand, Theorem 5.10 tells us that the homology of the filtered complex in (7.2) is $\underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}_J, [\omega_{\mathbf{r}}]_J; \mathcal{F})$. Since the spectral sequence $\mathcal{S}_{I,I''}$ converges no later than the E_3 page, $\underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}_J, [\omega_{\mathbf{r}}]_J; \mathcal{F})$ is isomorphic to the homology of the mapping cone of g, by (7.3). This means that, if $g \equiv 0$, then $\underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}_J, [\omega_{\mathbf{r}}]_J; \mathcal{F})$ has rank 2^m over $\mathcal{F}$. But this is plainly impossible: if $[\omega_{\mathbf{r}}]_J = 0$, then, by (3.6),

$$\underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}_J, [\omega_{\mathbf{r}}]_J; \mathcal{F}) \cong \widetilde{\mathrm{HFK}}(\mathcal{L}_J) \otimes_{\mathbb{F}} \mathcal{F},$$

which has rank 2^{m-1} over $\mathcal{F}$; otherwise, if $[\omega_{\mathbf{r}}]_J \neq 0$, then $\underline{\widetilde{\mathrm{HFK}}}(\mathcal{L}_J, [\omega_{\mathbf{r}}]_J; \mathcal{F}) = 0$, by Proposition 4.2. Therefore, it must be the case that $s \cdot t = 1$, completing the proof of Proposition 7.5. □

7.3. Sufficiency of the model computation

In this subsection, we show that the model computation above suffices to describe the complex $(E_2(\mathcal{S}_{\mathcal{F}}^{\mathbf{r}}), d_2(\mathcal{S}_{\mathcal{F}}^{\mathbf{r}}))$ up to isomorphism.

The sequence of Heegaard moves from $\mathcal{H}_{I,I''}$ to $\mathcal{H}_{I,I''}^3$ described at the beginning of Section 7.1 induces chain homotopy equivalences,

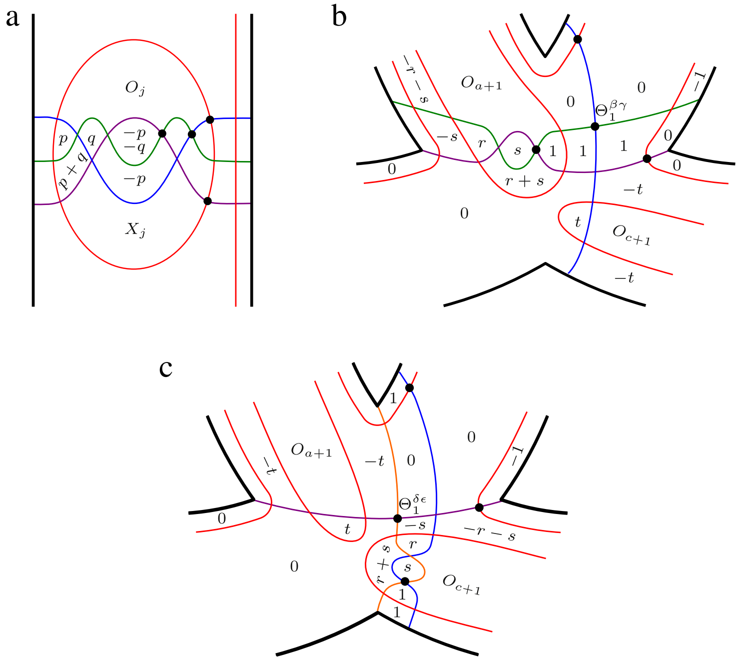


Fig. 16. Local multiplicities of the Whitney rectangle ψ.

$$\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\beta}) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I)), \qquad \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\gamma}) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^1)),$$
$$\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\delta}) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I^2)), \qquad \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\epsilon}) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I'')),$$

which preserve both Maslov and Alexander gradings. These are compositions of the maps corresponding to stabilizations with triangle-counting maps corresponding to isotopies and handleslides. We shall denote these chain homotopy equivalences by $\Phi_{I,I''}$. There are also maps $\overline{\Phi}_{I,I''}$ in the reverse direction which are homotopy inverses of the $\Phi_{I,I''}$. These are compositions of triangle-counting maps with the maps corresponding to destabilizations.

These Heegaard moves also induce homogeneous maps (denoted by $\Phi_{I,I''}$ as well),

$$\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\beta}, \boldsymbol{\gamma}) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}(I), \boldsymbol{\eta}(I^1)), \qquad \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\gamma}, \boldsymbol{\epsilon}) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}(I^1), \boldsymbol{\eta}(I'')),$$
$$\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\beta}, \boldsymbol{\delta}) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}(I), \boldsymbol{\eta}(I^2)), \qquad \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\delta}, \boldsymbol{\epsilon}) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\eta}(I^2), \boldsymbol{\eta}(I'')),$$

which give rise to injections on homology taking the part of $\underline{\widetilde{\mathrm{HFK}}}(\boldsymbol{\beta}, \boldsymbol{\gamma})$ in the top Maslov grading to that of $\underline{\widetilde{\mathrm{HFK}}}(\boldsymbol{\eta}(I), \boldsymbol{\eta}(I^1))$, etc. (See [40].) Hence, $\Phi_{I,I''}(\Theta_1^{\beta\gamma}) = T^e \Theta_1^{I,I^1}$ for some $e \in \mathbb{Z}$. Every point of $\mathbb{A}$ is contained in the same region as a basepoint in the triple-diagram associated to each pair of consecutive Heegaard diagrams in the sequence from $(\Sigma, \boldsymbol{\eta}(I), \boldsymbol{\eta}(I^1))$ to $(\Sigma_3, \boldsymbol{\beta}, \boldsymbol{\gamma})$. It follows that $\Phi_{I,I''}$ does not pick up any nontrivial powers of T; that is, $e = 1$. By the same token,

$$\Phi_{I,I''}(\Theta_1^{\beta\gamma}) = \Theta_1^{I,I^1}, \qquad \Phi_{I,I''}(\Theta_1^{\beta\delta}) = \Theta_1^{I,I^2},$$
$$\Phi_{I,I''}(\Theta_1^{\gamma\epsilon}) = \Theta_1^{I^1,I''}, \qquad \Phi_{I,I''}(\Theta_1^{\delta\epsilon}) = \Theta_1^{I^2,I''}.$$

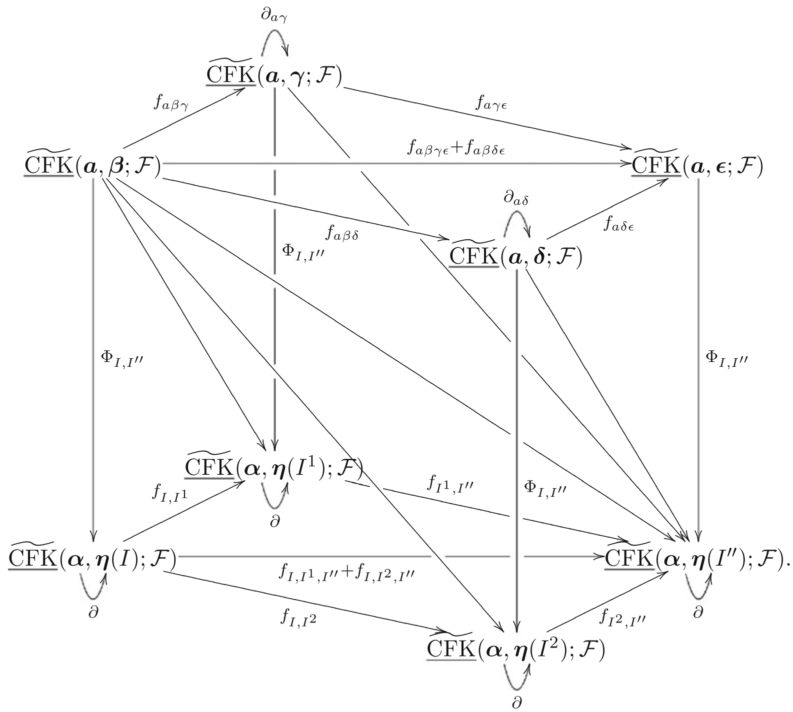


Fig. 17. The complex induced by the Heegaard moves from $\mathcal{H}^3_{I,I''}$ to $\mathcal{H}_{I,I''}$.

Furthermore, since $\Phi_{I,I''}$ is equivariant with respect to the maps Ψ_i, we have

$$\Phi_{I,I''}(\Theta_2^{\beta\gamma}) = \Theta_2^{I,I^1}, \qquad \Phi_{I,I''}(\Theta_2^{\beta\delta}) = \Theta_2^{I,I^2},$$
$$\Phi_{I,I''}(\Theta_2^{\gamma\epsilon}) = \Theta_2^{I^1,I''}, \qquad \Phi_{I,I''}(\Theta_2^{\delta\epsilon}) = \Theta_2^{I^2,I''},$$

by Lemmas 5.13 and 7.2.

The $\mathcal{A}_\infty$ relations (3.11), applied to the large multi-diagram which includes all of the multi-diagrams in the sequence from $\mathcal{H}_{I,I''}$ to $\mathcal{H}^3_{I,I''}$, show that these maps fit into a complex as shown in Fig. 17. We may view this complex as a filtered map between two filtered complexes, which induces a map of spectral sequences. On the E_2 page, we have a commutative square.

$$\begin{array}{ccc} \widetilde{\underline{\mathrm{HFK}}}(\boldsymbol{\alpha},\boldsymbol{\beta};\mathcal{F}) & \xrightarrow{\ g\ } & \widetilde{\underline{\mathrm{HFK}}}(\boldsymbol{\alpha},\boldsymbol{\epsilon};\mathcal{F}) \\ \cong \downarrow (\Phi_{I,I''})_* & & \cong \downarrow (\Phi_{I,I''})_* \\ \widetilde{\underline{\mathrm{HFK}}}(\boldsymbol{\alpha},\boldsymbol{\eta}(I);\mathcal{F}) & \xrightarrow{\ d_{I,I''}\ } & \widetilde{\underline{\mathrm{HFK}}}(\boldsymbol{\alpha},\boldsymbol{\eta}(I'');\mathcal{F}). \end{array} \tag{7.22}$$

The maps $(\Phi_{I,I''})_*$ parameterize $\widetilde{\underline{\mathrm{HFK}}}(\boldsymbol{\alpha},\boldsymbol{\eta}(I);\mathcal{F})$ and $\widetilde{\underline{\mathrm{HFK}}}(\boldsymbol{\alpha},\boldsymbol{\eta}(I'');\mathcal{F})$ by groups that we understand concretely. (7.22) then says that, with respect to these parameterizations, the map $d_{I,I''}$ is described by g. To show that this determines the *global* structure of $(E_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}), d_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}))$, we must verify that any two of these parameterizations agree where they overlap. Specifically, consider another double successor pair $J, J'' \in \mathcal{R}(\mathcal{D})$ for which either $I = J$, $I = J''$, $I'' = J$

or $I'' = J''$.[6] Without loss of generality, let us assume that $I = J$; the other three cases are treated identically. Let

$$\mathcal{H}^3_{J,J''} = (\Sigma_3, \boldsymbol{a}', \boldsymbol{\beta}', \boldsymbol{\gamma}', \boldsymbol{\delta}', \boldsymbol{\epsilon}', \mathbb{O}, \mathbb{X})$$

be the genus-3 Heegaard diagram obtained from $\mathcal{H}_{J,J''}$, as described in Section 7.1, and let $\mathbf{w}'_0$ denote the point in $\mathbb{T}_{a'} \cap \mathbb{T}_{\beta'}$ of maximal Maslov grading. Since $\underline{\widetilde{\mathrm{HFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$ has rank 1 in this grading, we know that $(\Phi_{J,J''})_*([\mathbf{w}'_0]) = \lambda(\Phi_{I,I''})_*([\mathbf{w}'_0])$ for some nonzero $\lambda \in \mathcal{F}$. Since any element of $\underline{\widetilde{\mathrm{HFK}}}(\boldsymbol{a}, \boldsymbol{\beta})$ (resp. $\underline{\widetilde{\mathrm{HFK}}}(\boldsymbol{a}', \boldsymbol{\beta}')$) can be obtained from $[\mathbf{w}_0]$ (resp. $[\mathbf{w}'_0]$) via the action of the maps $\psi_i^{a\beta}$ (resp. $\psi_i^{a'\beta'}$), and $(\Phi_{I,I''})_*$ and $(\Phi_{J,J''})_*$ are equivariant with respect to these actions, the constant λ completely determines the relationship between the two parameterizations. In fact, the following proposition implies that $\lambda = 1$.

Proposition 7.6. *The elements $\Phi_{I,I''}(\mathbf{w}_0)$ and $\Phi_{J,J''}(\mathbf{w}'_0)$ represent the same homology class in $\underline{\widetilde{\mathrm{HFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I))$.*

Proof of Proposition 7.6. It suffices to show that the composition

$$\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\beta}; \mathcal{F}) \xrightarrow{\Phi_{I,I''}} \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I); \mathcal{F}) \xrightarrow{\overline{\Phi}_{J,J''}} \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}', \boldsymbol{\beta}'; \mathcal{F}) \tag{7.23}$$

sends $\mathbf{w}_0$ to $\mathbf{w}'_0$. Since the maps induced by (de)stabilizations commute with those induced by isotopies and handleslides, $\Phi_{I,I''}$ and $\Phi_{J,J''}$ can be factored into the compositions

$$\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}, \boldsymbol{\beta}; \mathcal{F}) \xrightarrow{\Phi} \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}^1, \boldsymbol{\beta}^1; \mathcal{F}) \xrightarrow{\Phi'_{I,I''}} \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I); \mathcal{F}),$$
$$\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I); \mathcal{F}) \xrightarrow{\overline{\Phi}'_{J,J''}} \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}^k, \boldsymbol{\beta}^k; \mathcal{F}) \xrightarrow{\overline{\Phi}} \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}', \boldsymbol{\beta}'; \mathcal{F}),$$

where $(\Sigma, \boldsymbol{a}^1, \boldsymbol{\beta}^1, \mathbb{O}, \mathbb{X})$ and $(\Sigma, \boldsymbol{a}^k, \boldsymbol{\beta}^k, \mathbb{O}, \mathbb{X})$ are obtained from $H^3_{I,I''}$ and $H^3_{J,J''}$ by stabilizing $n-2$ times, and Φ and $\overline{\Phi}$ are the maps induced by stabilization and destabilization, respectively.

By definition, the map Φ sends $\mathbf{w}_0$ to the unique generator $\mathbf{w}^1_0$ in $\mathbb{T}_{a^1} \cap \mathbb{T}_{\beta^1}$ of maximal Maslov grading. Likewise, $\overline{\Phi}$ sends the unique generator $\mathbf{w}^k_0$ in $\mathbb{T}_{a^k} \cap \mathbb{T}_{\beta^k}$ of maximal Maslov grading to $\mathbf{w}'_0$. To prove Proposition 7.6, it then suffices to show that the composition

$$\underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}^1, \boldsymbol{\beta}^1; \mathcal{F}) \xrightarrow{\Phi'_{I,I''}} \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I); \mathcal{F}) \xrightarrow{\overline{\Phi}'_{J,J''}} \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}^k, \boldsymbol{\beta}^k; \mathcal{F})$$

sends $\mathbf{w}^1_0$ to $\mathbf{w}^k_0$. The map $\overline{\Phi}'_{J,J''} \circ \Phi_{I,I''}$ is a composition $\Phi_{k-1} \circ \cdots \circ \Phi_1$, where

$$\Phi_i : \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}^i, \boldsymbol{\beta}^i; \mathcal{F}) \to \underline{\widetilde{\mathrm{CFK}}}(\boldsymbol{a}^{i+1}, \boldsymbol{\beta}^{i+1}; \mathcal{F})$$

is the triangle-counting map induced by the handleslide or isotopy taking $(\Sigma, \mathbf{a}^i, \boldsymbol{\beta}^i, \mathbb{O}, \mathbb{X})$ to $(\Sigma, \boldsymbol{a}^{i+1}, \boldsymbol{\beta}^{i+1}, \mathbb{O}, \mathbb{X})$, where either $\boldsymbol{a}^i = \boldsymbol{a}^{i+1}$ or $\boldsymbol{\beta}^i = \boldsymbol{\beta}^{i+1}$. Note that, for some intermediate j, we have $\boldsymbol{a}^j = \boldsymbol{\alpha}$ and $\boldsymbol{\beta}^j = \boldsymbol{\eta}(I)$.

Recall from the previous section that a Kauffman generator is one which does not contain any intersection point between a ladybug curve and a non-ladybug curve. Let $\mathbf{w}^i_0$ denote the unique Kauffman generator in $\mathbb{T}_{a^i} \cap \mathbb{T}_{\beta^i}$ of maximal Maslov grading. It is not hard to see that, for each $i = 1, \ldots, k-1$, there is a Whitney triangle ψ_i in either $\pi_2(\Theta^{a^{i+1}a^i}, \mathbf{w}^i_0, \mathbf{w}^{i+1}_0)$ or

[6] This J is not related to the J used earlier in this section.

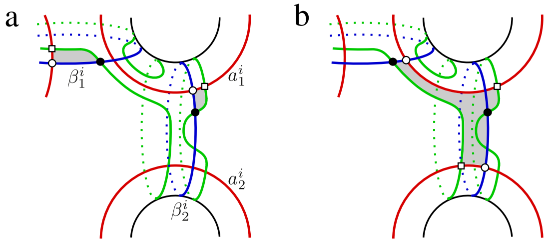


Fig. 18. The white circles and squares represent $\mathbf{w}_0^i$ and $\mathbf{w}_0^{i+1}$, respectively, and the $\boldsymbol{\beta}^{i+1}$ curves are in green. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$\pi_2(\mathbf{w}_0^i, \Theta^{\beta^i \beta^{i+1}}, \mathbf{w}_0^{i+1})$ (depending on whether $\boldsymbol{a}^i = \boldsymbol{a}^{i+1}$ or $\boldsymbol{\beta}^i = \boldsymbol{\beta}^{i+1}$, respectively) which avoids $\mathbb{O} \cup \mathbb{X} \cup \mathbb{A}$. If $(\Sigma, \boldsymbol{a}^{i+1}, \boldsymbol{\beta}^{i+1}, \mathbb{O}, \mathbb{X})$ is obtained from $(\Sigma, \boldsymbol{a}^i, \boldsymbol{\beta}^i, \mathbb{O}, \mathbb{X})$ by an isotopy or handleslide of a ladybug curve, then $\mathbf{w}_0^{i+1}$ is "very close" to $\mathbf{w}_0^i$, and the domain of ψ_i is just a disjoint union of small triangles. Otherwise, if $(\Sigma, \boldsymbol{a}^{i+1}, \boldsymbol{\beta}^{i+1}, \mathbb{O}, \mathbb{X})$ is obtained from $(\Sigma, \boldsymbol{a}^i, \boldsymbol{\beta}^i, \mathbb{O}, \mathbb{X})$ by a handleslide of a non-ladybug curve, then there are two possibilities.

Without loss of generality, assume $\boldsymbol{a}^i = \boldsymbol{a}^{i+1}$ and that $\boldsymbol{\beta}^{i+1}$ is obtained from $\boldsymbol{\beta}^i$ by handlesliding β_1^i over β_2^i, as shown in Fig. 18. If there is no point of $\mathbf{w}_0^i$ on $a_1^i \cap \beta_1^i$, then $\mathbf{w}_0^{i+1}$ is "very close" to $\mathbf{w}_0^i$, and the domain of ψ_i is a disjoint union of small triangles; see Fig. 18(a). Otherwise, $\mathbf{w}_0^{i+1}$ is "very close" to $\mathbf{w}_0^i$ away from the portion of the diagram shown in Fig. 18(b). In these distant regions, the domain of ψ_i is a disjoint union of small triangles; near a_1^i and β_2^i, the domain of ψ_i consists of the hexagon shown in the figure.

The concatenation $\psi = \psi_1 * \cdots * \psi_k$ is therefore a Whitney $(k+2)$-gon connecting $\mathbf{w}_0^1$ to $\mathbf{w}_0^k$, with evaluation $\langle \omega_{\mathbf{r}}, \psi \rangle = 0$. Suppose that ψ' is another concatenation of triangles connecting these two generators and missing $\mathbb{O} \cup \mathbb{X}$. Then $D(\psi') - D(\psi)$ is a multi-periodic domain P on the large multi-diagram that encodes all intermediate diagrams between $(\Sigma, \boldsymbol{a}^1, \boldsymbol{\beta}^1, \mathbb{O}, \mathbb{X})$ and $(\Sigma, \mathbf{a}^k, \boldsymbol{\beta}^k, \mathbb{O}, \mathbb{X})$. One can show, exactly as in the proof of Lemma 5.3, that any such periodic domain is the sum of doubly periodic domains in $\Pi^0_{a^i, a^{i+1}}$ or $\Pi^0_{\beta^i, \beta^{i+1}}$, for $i = 1, \ldots, k-1$, with a periodic domain P' in $\Pi^0_{a^1, \beta^1}$. The former domains must miss $\mathbb{A}$ since the handleslides and isotopies all avoid $\mathbb{A}$, and, since $(\Sigma, \boldsymbol{a}^1, \boldsymbol{\beta}^1, \mathbb{O}, \mathbb{X})$ is a diagram for the unknot in S^3, we have $P' = 0$. Thus, P misses $\mathbb{A}$, so $\langle \omega_{\mathbf{r}}, \psi' \rangle = \langle \omega_{\mathbf{r}}, \psi \rangle = 0$. This implies that the coefficient of $\mathbf{w}_0^k$ in $\overline{\Phi}'_{J,J''} \circ \Phi_{I,I''}(\mathbf{w}_0^1)$ is 1. $\square$

Proof of Theorems 1.1 and 2.7. For each $I \in \mathcal{R}(\mathcal{D})$ such that either (1) there is a double successor I'' of I with $I'' \in \mathcal{R}(\mathcal{D})$ or (2) I is a double successor of some $J \in \mathcal{R}(\mathcal{D})$, Proposition 7.6 gives us a canonical class $w^I \in \underline{\widetilde{\mathrm{HFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I); \mathcal{F})$. For all other $I \in \mathcal{R}(\mathcal{D})$, we may take w^I to be any generator of $\underline{\widetilde{\mathrm{HFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I); \mathcal{F})$ in the top Maslov grading.

Recall that $\mathcal{Y}_I$ is the vector space over $\mathcal{F}$ generated by $y_1, \ldots, y_m$, modulo the relation

$$\sum_{i=1}^{m} T^{r_{\sigma_I}(1)+\cdots+r_{\sigma_I}(i)} y_{\sigma_I(i)} = 0.$$

By Proposition 4.3, there are isomorphisms

$$\rho_I : \Lambda^*(\mathcal{Y}_I) \to \underline{\widetilde{\mathrm{HFK}}}(\boldsymbol{\alpha}, \boldsymbol{\eta}(I); \mathcal{F})$$

such that $\rho_I(1) = w_I$ and $\rho_I(y_i x) = \psi_i^{\alpha\eta(I)}(\rho_I(x))$ for all $x \in \Lambda^*(\mathcal{Y}_I)$. By expressing Propositions 7.4 and 7.5, (7.15) and (7.16) in terms of these identifications, we see that, if I''

is a double successor of I, then $d_{I,I''}$ is as described in Section 2. Thus, the maps ρ_I induce an isomorphism of chain complexes from $(C(\mathcal{D}), \partial^{\mathbf{r}})$ to $(E_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}), d_2(\mathcal{S}^{\mathbf{r}}_{\mathcal{F}}))$, and the grading Δ agrees with the grading on $(C(\mathcal{D}), \partial^{\mathbf{r}})$ defined in Section 2. This identification, combined with Theorem 6.2, completes the proof. □

Remark 7.7. One can also use the computations in this section to determine the d_1 differential of the *untwisted* spectral sequence $\mathcal{S}_{\mathbb{F}}$ (which does not depend on $\mathbf{r}$). Unfortunately, the rank of its E_2 page, after dividing by $2^{m-|L|}$ to adjust for the number of marked points, is not an invariant of L. For instance, the complex associated to a 0-crossing diagram of the unknot with m marked points consists of a single copy of $\Lambda_*(\mathcal{Y}_I)$ (where I is the empty tuple), with rank 2^{m-1}. On the other hand, for a 3-crossing diagram for the unknot obtained by changing one crossing of a diagram for the trefoil, with one marked point on each of the six edges, a *Mathematica* computation shows that the E_2 page has rank 48 rather than $32 = 2^5$.

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