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Continuity points of quasi-continuous mappings ☆

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Abstract

It is known that the fragmentability of a topological space X by a metric whose topology contains the topology of X is equivalent to the existence of a winning strategy for one of the players in a special two players "fragmenting game". In this paper we show that the absence of a winning strategy for the other player is equivalent to each of the following two properties of the space X:

for every quasi-continuous mapping $f:Z\to X$, where Z is a complete metric space, there exists a point $z_0\in Z$ at which f is continuous;

for every quasi-continuous mapping $f:Z\to X$, where Z is an α-favorable space, there exists a dense subset of Z at the points of which f is continuous.

In fact, we show that the set of points of continuity of f is of the second Baire category in every non-empty open subset of Z. Using this we derive some results concerning joint continuity of separately continuous functions.

Keywords: Fragmentability; Quasi-continuity

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1. Introduction.

In the paper [21] Kempisty introduced a notion similar to continuity for real-valued functions defined in $\mathbb{R}$. For general topological spaces this notion can be given the following equivalent formulation.

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Definition 1. The mapping $g : Z \to X$ between the topological spaces Z and X is said to be *quasi-continuous* at $z_0 \in Z$ if for every open subset $U \subset X$, $g(z_0) \in U$, there exists some open set $V \subset Z$ such that

(a) $z_0 \in \overline{V}$ (the closure of V in Z), and

(b) $g(V) \subset U$.

The mapping g is called quasi-continuous if it is quasi-continuous at each point of Z.

The roots of this notion can be traced back to Volterra (see [2, p. 95]). Since then quasi-continuity penetrated a variety of mathematical problems. The properties of quasi-continuous mappings have been studied intensively (see, for instance, [3,29,33,34]). Quasi-continuity of real-valued separately continuous functions of two variables was studied very frequently in connection with the existence of points of joint continuity for such functions (see [30,28,36–39,46]). The notion of quasi-continuity recently turned out to be instrumental in the proof that some semitopological groups are actually topological ones (see [5,6]) and in the proof of some generalizations of Michael's selection theorem (see [12]).

There are simple examples of quasi-continuous mappings which are nowhere continuous. Take $Z = [0, 1)$ with the usual topology, $X = [0, 1)$ with the Sorgenfrey topology and the identity mapping $g : Z \to X$. The map g is quasi-continuous but nowhere continuous. Nevertheless, under some mild requirements imposed on the spaces Z and X, each quasi-continuous map becomes continuous at many points of the space Z. Levine [27] has shown that, if X has countable base, then every quasi-continuous map $g : Z \to X$ could be discontinuous only at the points of some first Baire category subset of Z. Bledsoe [4] proved similar result for the case when X is a metric space. Results of this kind could be found in many articles (see, for instance, the survey papers [37,38] of Piotrowski). In [40, p. 114] Piotrowski asked for which large classes of spaces X every quasi-continuous mapping $f : Y \to X$ defined in a Baire space Y has at least one point of continuity. In this paper we characterize the spaces X for which every quasi-continuous mapping $f : Z \to X$, defined in a complete metric space Z, has a point of continuity. Very good approximation to this characterization (and to the answer of the question of Piotrowski) provides the notion fragmentability of a given topological space X. We recall here the definition of this notion (see [19]).

Definition 2. Let X be a topological space and ρ some metric defined on $X \times X$. The space X is said to be *fragmented by the metric* ρ, if for every $\varepsilon > 0$ and every subset $A \subset X$ there exists a non-empty relatively open subset $B \subset A$ with ρ-diameter$(B) \leqslant \varepsilon$. In such a case the space X is called *fragmentable*.

The proof of the next simple result shows some of the techniques associated with quasi-continuity of mappings and fragmentability of spaces.

Theorem 1. *Let Z be a Baire space and $f : Z \to X$ a quasi-continuous map from Z into the topological space X which is fragmented by some metric ρ. Then there exists a dense*

G_δ-subset $C \subset Z$ at the points of which $f : Z \to (X, \rho)$ is continuous. In particular, if the topology generated by the metric ρ contains the topology of the space X, then $f : Z \to X$ is continuous at every point of the set C.

Proof. Consider, for every $n = 1, 2, \ldots$, the set $V_n := \bigcup\{V\colon\ V$ open in Z and ρ-diameter$(f(V)) \leqslant n^{-1}\}$. The set V_n is open in Z. It is also dense in Z. Indeed, suppose W is a non-empty open subset of Z. Consider the set $A := f(W) \subset X$. By fragmentability of X there is some relatively open subset $B = A \cap U = f(W) \cap U$, where U is open in X, such that ρ-diameter$(B) \leqslant n^{-1}$. Quasi-continuity of f implies that there is some non-empty open $V \subset W$ with $f(V) \subset U \cap f(W) = B$. This shows that $\emptyset \neq V \subset V_n \cap W$. Hence, V_n is dense in Z. Obviously, at each point of $C := \bigcap_{n\geqslant 1} V_n$ the map f is ρ-continuous. □

Note that, according to a result of Ribarska [41,42], if the space X is compact and fragmentable, then it is also fragmentable by some metric that *majorizes the topology of* X. I.e., the metric topology generated by the new fragmenting metric contains the topology of the compact space X. Therefore, the following result has place for compact spaces X.

Corollary 1. *Let Z be a Baire space and $f : Z \to X$ a quasi-continuous map from Z into the fragmentable compact space X. Then there exists a dense G_δ-subset $C \subset Z$ at the points of which $f : Z \to X$ is continuous.*

Later in this paper (Section 3) we use a topological game to describe a large class GD of "game determined" spaces X for which the existence of a fragmenting metric implies fragmentability by a metric that majorizes the topology of X. All p-spaces and all Moore spaces belong to the class GD. It contains as a subclass the class of "spaces with countable separation" which was introduced in [25]. The latter subclass of GD contains all Borel subsets of any compact space. Moreover, any set that can be obtained from Borel subsets of a given compact space by means of the Souslin operation has countable separation. Therefore any Čech-analytic space X is also in GD. For any space X from GD fragmentability implies that the set $C(f)$ of points of continuity of a given quasi-continuous mapping $f : Z \to X$ is residual in Z (its complement is of the first Baire category).

For nonfragmentable spaces X one could not expect that the set $C(f)$ is always residual in Z. However density of this set (and slightly more than density!) can have place even without fragmentability of X. In Section 2 of this paper we use a topological game to characterize the spaces X such that, for every quasi-continuous map $f : Z \to X$ from a complete metric space (or an α-favorable space) Z, the set $C(f)$ is dense in Z. As a matter of fact, the set $C(f)$ turns out to be of the second Baire category in every non-empty open subset of Z. Similar results are formulated for minimal set-valued mappings as well.

In Section 4 we study the enlargement of a minimal set-valued mappings $F : Z \to X$ obtained by taking the closure $\overline{Gr(F)}$ of the graph $Gr(F)$ of F in $Z \times bX$ where bX is some compactifications of X. This closure determines a new set-valued mapping $\widetilde{F} : Z \to bX$ for which $F(z) \subset \widetilde{F}(z) \subset bX$ whenever $z \in Z$. We characterize the situations when the set

$C(F) := \{z \in Z\colon\ \widetilde{F}(z) \subset X\}$ is dense in Z. The class of spaces GD plays an important role in this characterization. For instance, if $X \in \mathrm{GD}$, then $C(F)$ is residual in Z. As immediate corollaries from these results we get in Section 5 conditions for dense or residual subcontinuity of quasi-continuous mappings.

The last Section 6 is devoted to some examples which outline the validity of the main statements as well as to some applications concerning the existence of points of (joint) continuity of separately continuous mappings.

If not stated otherwise, all topological spaces appearing in this paper are assumed to have enough separation properties. For instance, points are assumed to be closed sets and whenever a point x does not belong to some closed set H there exist disjoint open sets U and V such that $x \in U$ and $H \subset V$.

2. Dense continuity of quasi-continuous mappings

To formulate our main results we need to recast fragmentability of X in terms of a topological *fragmenting game* $G(X)$ in the space X (see [23–25]). This game involves two players Σ and Ω. The players select, one after the other, *non-empty* subsets of X. Ω starts the game by selecting the whole space X. Σ answers by choosing any subset A_1 of X and Ω goes on by taking a subset $B_1 \subset A_1$ which is *relatively open* in A_1. After that, on the nth stage of development of the game, Σ takes any subset A_n of the last move B_{n-1} of Ω and the latter answers by taking again a *relatively open* subset B_n of the set A_n just chosen by Σ. Acting this way, the players produce a sequence of non-empty sets $A_1 \supset B_1 \supset A_2 \supset \cdots \supset A_n \supset B_n \supset \cdots$, which is called a *play* and will be denoted by $p = (A_i, B_i)_{i \geqslant 1}$ (there is no need to include in this notation the space X which is the first (and obligatory) move of Ω). The player Ω is said to *have won* this play if the set $\bigcap_{n \geqslant 1} A_n$ contains at most one point. Otherwise the player Σ is said to have won the play.

A *partial play* is a finite sequence which consists of the first several moves $A_1 \supset B_1 \supset A_2 \supset \cdots \supset A_n$ (or $A_1 \supset B_1 \supset A_2 \supset \cdots \supset B_n$) of a play. A *strategy* ω for the player Ω is a mapping which assigns to each partial play $A_1 \supset B_1 \supset A_2 \supset \cdots \supset A_n$ some set B_n such that $A_1 \supset B_1 \supset A_2 \supset \cdots \supset A_n \supset B_n$ is again a partial play. A strategy σ for Σ is defined in a symmetric way. Sometimes we will denote the first choice A_1 under a strategy σ by $\sigma(X)$. A σ*-play* (ω*-play*) is a play in which Σ (Ω) selects his/her moves according to σ (ω). The strategy ω (σ) is said to be a *winning* one if every ω-play (σ-play) is won by Ω (Σ). The game $G(X)$ or the space X is called Ω*-favorable* (Σ*-favorable*), if there is a winning strategy for the player Ω (Σ). The game $G(X)$ (or the space X) is called Σ*-unfavorable*, if there does not exist winning strategy for the player Σ. Examples show (see the last section, Example 1) that there are compact spaces X which are unfavorable for both players.

It was proved in [23] that the fragmentability of a given topological space X is equivalent to the existence of a winning strategy for the player Ω in the game $G(X)$. I.e., X is fragmentable if, and only if, the game $G(X)$ is Ω-favorable. By a change of the rule for winning a play in the game $G(X)$ (but keeping intact the rules for the moves of the players) one can express in a similar way the existence of a fragmenting metric which majorizes the topology of the space X. We will denote by $G'(X)$ the game in which the plays are the same as in $G(X)$ but the rule for winning a play is the following one. The player Ω is said to have won the play $p=(A_i,B_i)_{i\geqslant 1}$ in the game $G'(X)$, if the set $\bigcap_{n\geqslant 1} A_n$ is either empty or consists of exactly one point x such that for every open neighborhood U of x there is some positive integer n with $A_n\subset U$. Otherwise the player Σ is said to have won the play $(A_i,B_i)_{i\geqslant 1}$. As shown in [24,25], the topological space X is fragmentable by a metric which majorizes its topology if, and only if, the player Ω has a winning strategy in the game $G'(X)$.

The next result shows what one could expect from spaces X in which the other player, Σ, does not have a winning strategy in $G'(X)$. As already mentioned the absence of a winning strategy for Σ does not necessarily imply that Ω has a winning strategy in $G'(X)$. I.e., the condition "the game $G'(X)$ is Σ-unfavorable (or the space X is Σ-unfavorable)" is weaker than the condition "X is fragmentable by a metric which majorizes its topology". Correspondingly, the conclusion is also weaker. The set of points of continuity $C(f)$ is not necessarily residual in Z. It is however of the second Baire category in every non-empty open subset of Z. I.e., for every non-empty open subset $V\subset Z$ the set $C(f)\cap V$ is not of the first Baire category (equivalently, the set $C(f)\cap V$ cannot be covered by a countable union of subsets whose closures in Z have no interior points).

Theorem 2. *For the topological space X the following conditions are equivalent*:

(i) *$G'(X)$ is Σ-unfavorable*;

(ii) *every quasi-continuous mapping $f:Z\to X$ from the complete metric space Z into X is continuous at at least one point of Z*;

(iii) *every quasi-continuous mapping $f:Z\to X$ from the complete metric space Z into X is continuous at the points of some subset which is of the second Baire category in every non-empty open subset of Z*;

(iv) *every quasi-continuous mapping $f:Z\to X$ from an α-favorable space Z into X is continuous at the points of some subset which is of the second Baire category in every non-empty open subset of Z.*

To recall the concept of α-favorability we need the well known Banach–Mazur game.

Let Z be a topological space. The Banach–Mazur game $BM(Z)$ is played by two players α and β, who select alternatively non-empty open subsets of Z. α starts the game by selecting $W_0=Z$. β answers by taking some non-empty open subset V_0 of Z. On the nth move, $n\geqslant 1$, the player α takes a non-empty open subset $W_n\subset V_{n-1}$ and β answers by taking a non-empty open subset V_n of W_n. Using this way of selection, the players get a sequence $(W_n,V_n)_{n=0}^{\infty}$ which is called a *play*. The player β is said to *have won* this play if $\bigcap_{n\geqslant 1} W_n=\emptyset$; otherwise this play is won by α. A *partial play* is a finite sequence which consists of the first several consecutive moves in the game. A *strategy* ζ for the player α is a mapping which assigns to each partial play $(V_0,W_1,V_1,W_2,V_2,\dots,W_{n-1},V_{n-1})$ some non-empty open subset W_n of V_{n-1}. A ζ*-play* is a play in which α selects his/her moves according to ζ. The strategy ζ is said to be a *winning* one if every ζ-play is won by α. The space Z is called *α-favorable* if there exists a winning strategy for α in $BM(Z)$.

Let us remind that the space Z is called *Čech complete*, if it is a G_δ-subset of some compact space. Z is said to be *almost Čech complete*, if it contains dense Čech complete subset. It is known that complete metric spaces are Čech complete and that every almost Čech complete space is α-favorable. Below we will also use the simple observation that, for any α-favorable space Z and any subset H which is of the first Baire category in Z, there exists a strategy ζ for player α such that $\bigcap_{i\geqslant 0} W_i \neq \emptyset$ and $H \cap (\bigcap_{i\geqslant 0} W_i) = \emptyset$ whenever $(V_i, W_i)_{i\geqslant 0}$ is a ζ-play.

Proof of Theorem 2. We show that (i) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (i). The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are obvious.

(i) $\Rightarrow$ (iv) Suppose X is Σ-unfavorable for $G'(X)$ and $f: Z \to X$ is a quasi-continuous mapping from the α-favorable space Z. Let H be a first Baire category subset of Z. There is some winning strategy ζ for the player α in $BM(Z)$ which "avoids" the set H. I.e., $\bigcap_{i\geqslant 0} W_i \neq \emptyset$ and $H \cap (\bigcap_{i\geqslant 0} W_i) = \emptyset$ whenever $(V_i, W_i)_{i\geqslant 0}$ is a ζ-play. Take an open $V_0 \neq \emptyset$, $V_0 \subset Z$. We will show that f is continuous at some point of $V_0 \setminus H$. To do this we first construct a strategy σ for the player Σ in $G'(X)$ and then use the fact that Σ does not win some σ-play. Put the first move of β in $BM(Z)$ to be V_0 and let $W_1 = \zeta(V_0)$ be the answer of α. Assign $A_1 := f(W_1)$ to be the first move in the strategy σ. Suppose that the answer of Ω in $G'(X)$ is B_1, a non-empty relatively open subset of A_1. Quasi-continuity of f implies there exists some non-empty open subset V_1 of W_1, such that $f(V_1) \subset B_1$. Suppose the set V_1 is the next move of the player β in the game $BM(Z)$. The player α, of course, uses the strategy ζ to answer this move and selects the set $W_2 = \zeta(V_0, W_1, V_1)$. Then we define the second move of Σ in $G'(X)$ to be $A_2 = \sigma(A_1, B_1) := f(W_2)$. Proceeding like this, we construct inductively the strategy σ. Together with each σ-play $(A_i, B_i)_{i\geqslant 1}$ in $G'(X)$ we construct also a ζ-play $(W_i, V_i)_{i\geqslant 1}$ in $BM(Z)$ with $A_n = f(W_n)$ and $W_n = \zeta(V_0, W_1, V_1, \ldots, W_{n-1}, V_{n-1})$ for $n = 1, 2, \ldots$.

As ζ is a winning strategy for α, we have $\bigcap_{i\geqslant 1} W_i \neq \emptyset$. Therefore

$$\emptyset \neq f\left(\bigcap_{i\geqslant 1} W_i\right) \subset \bigcap_{i\geqslant 1} f(W_i) = \bigcap_{i\geqslant 1} A_i.$$

Since X is Σ-unfavorable, there is some σ-play $(A_i, B_i)_{i\geqslant 1}$ that is won by Ω; hence the non-empty set $\bigcap_{i\geqslant 1} A_i$ has just one point x and, for every open set $U \ni x$, there is some n with $A_n = f(\overline{W_n}) \subset U$. All this means that $f(z) = x$ for every $z \in \bigcap_{i\geqslant 1} W_i \subset V_0 \setminus H$ and that f is continuous at such z.

(ii) $\Rightarrow$ (i) Let σ be an arbitrary strategy for the player Σ in $G'(X)$. We will show that it is not a winning one. Consider the space P of all σ-plays $p = (A_i, B_i)_{i\geqslant 1}$ endowed with the Baire metric d; that is, if $p = (A_i, B_i)_{i\geqslant 1} \in P$ and $p' = (A'_i, B'_i)_{i\geqslant 1} \in P$, then $d(p, p') = 0$ if $p = p'$ and otherwise $d(p, p') = 1/n$ where $n = \min\{k\colon\ B_k \neq B'_k\}$. Note that all the plays in P start with the same set $A_1 = \sigma(X)$, the first choice of the strategy σ. Also, if $A_i = A'_i$ and $B_i = B'_i$ for all $i \leqslant n$, then

$$A_{n+1} = \sigma(A_1, B_1, \ldots, A_n, B_n) = \sigma(A'_1, B'_1, \ldots, A'_n, B'_n) = A'_{n+1}.$$

In other words, if $p \neq p'$, then there is some n, such that $B_n \neq B'_n$, $A_i = A'_i$ for $i \leqslant n$ and $B_i = B'_i$ for $i < n$. It is easy to verify that (P, d) is a complete metric space.

Consider the (set-valued) mapping $F : P \to X$ defined by $F((A_i, B_i)_{i \geqslant 1}) = \bigcap_{i \geqslant 1} A_i$. If, for some σ-play p we have $F(p) = \emptyset$, then the play p is won by Ω and there is nothing to prove. Therefore, without loss of generality, we may (and do) assume that F is non-empty-valued at every point of P. Let $f : P \to X$ be an arbitrary selection of the non-empty-valued map $F : P \to X$ (i.e., $f(p) \in F(p)$ for every $p \in P$). Next we will show that f is quasi-continuous (see Corollary 2 below). Then, by property (ii), f will turn out to be continuous at some point $p_0 \in P$. Finally we will show (see Proposition 1 below) that the play p_0 is won by Ω. This will show that σ is not a winning strategy and will complete the proof. □

Having in mind this and our needs in the next sections we recall here a notion of minimality (see Definition 1.1 in [26]) for set-valued mappings.

Definition 3. The set-valued map $G : Z \to X$ between the topological spaces Z and X is said to be minimal at $z_0 \in Z$ if for every open $U \subset X$ with $U \cap G(z_0) \neq \emptyset$ there exists some open V in Z such that
(a) $z_0 \in \overline{V}$, and
(b) $G(V) = \bigcup\{G(z)\colon\ z \in V\} \subset U$.
The mapping G is said to be minimal, if it is minimal at each point of Z.

This definition is a direct generalization of quasi-continuity. It is shaped after the characterizing property of minimal upper semicontinuous compact-valued mappings (see [7]) which are, of course, minimal in the above sense. If X is a completely regular space and $C(X)$ is the space of all bounded continuous functions in X with the *sup*-norm, then the mapping $M : C(X) \to X$ which puts into correspondence to each function $h \in C(X)$ the (possibly empty) set $M(h)$ of all maximizers of h in X, is also minimal (see [8,9]). Below we will show that the above mapping $F : P \to X$ is minimal as well.

Corollary 2. *Every single-valued selection of a non-empty-valued minimal mapping is quasi-continuous. Every quasi-continuous mapping is minimal.*

The next simple lemma which is similar to Proposition 2.3 of [26] is important for our considerations.

Lemma 1. *Let the play $p_0 = (A_i, B_i)_{i \geqslant 1}$ be an element of the space P and U an open subset of X with $U \cap A_n \neq \emptyset$ for every $n = 1, 2, 3, \ldots$. Then there exists an open subset V in P such that*
(a) $p_0 \in \overline{V}$, *and*
(b) $F(V) = \bigcup\{F(p)\colon\ p \in V\} \subset U$.

Proof. Let $p_0 = (A_i, B_i)_{i \geqslant 1}$ and U be as required in the formulation of the lemma. Given a positive integer n, consider the non-empty set $B'_n := A_n \cap U$ (which is relatively open in

A_n and is a possible move of the player Ω). Denote by A'_{n+1} the set $\sigma(A_1, \ldots, B'_n)$ which is the answer of player Σ by means of the strategy σ. Let $p' \in P$ be some play in $G'(X)$ which starts with the partial play $(A_1, \ldots, A'_{n+1})$. Clearly, $d(p_0, p') \leqslant n^{-1}$. Moreover, the closed d-ball $D(p_0, n^{-1}) := \{p\colon\ d(p_0, p) \leqslant n^{-1}\}$ contains the ball $D(p', (n+1)^{-1})$ and for every play p'' in the latter ball we have $F(p'') \subset B'_n \subset U$. Put V_n to be the interior of $D(p', (n+1)^{-1})$. Thus, for every integer $n > 0$, we found an open subset $V_n \subset D(p_0, n^{-1})$ such that $F(V_n) \subset U$. The set $V := \bigcup_{n \geqslant 1} V_n$ satisfies the requirements of (a) and (b). □

This lemma immediately yields:

Corollary 3. *The (set-valued) mapping $F : P \to X$ defined above is minimal.*

To formulate the next result we need one more definition.

Definition 4. A set-valued mapping $G : Z \to X$ is said to be upper semicontinuous at $z_0 \in Z$ if for every open $U \supset G(z_0)$ there exists an open $V \ni z_0$ such that

$$G(V) := \bigcup \bigl\{G(z)\colon\ z \in V\bigr\} \subset U.$$

G is said to be upper semicontinuous (usc), if it is usc at every $z \in Z$.

We will use the abbreviation *usco* for mappings G which are usc and, in addition, $G(z)$ is compact for every $z \in Z$.

Proposition 1. *Let f be an arbitrary selection of the minimal mapping $F : P \to X$. If f is continuous at some point $p_0 \in P$, then the play $p_0 = (A_i, B_i)_{i \geqslant 1}$ is won by the player Ω in the game $G'(X)$ and the mapping $F : P \to X$ is single-valued and upper semicontinuous at p_0.*

Proof. Let W be an open subset of X with $f(p_0) \in W$. Since f is continuous at $p_0 = (A_i, B_i)_{i \geqslant 1}$ there exists some open V', $p_0 \in V'$, with $f(V') \subset W$. We will show that there is some integer $n > 0$ for which $A_n \subset \overline{W}$. Suppose this is not the case. Then the open set $U := X \setminus \overline{W}$ intersects all sets A_n, $n = 1, 2, \ldots$. By the above lemma, there is some open set $V \subset P$ such that $p_0 \in \overline{V}$ and $f(V) \subset F(V) \subset U$. Then there is a point $p' \in V \cap V' \neq \emptyset$. For p' we have the contradiction: $f(p') \in U \cap W = \emptyset$. This shows that, for some $n > 0$, $A_n \subset \overline{W}$. In other words, $F(D(p_0, n^{-1})) \subset \overline{W}$. Since W was an arbitrary open neighborhood of $f(p_0)$, we derive that $F(p_0) = f(p_0)$, that F is upper semicontinuous at p_0 and that the play p_0 is won by the player Ω in the game $G'(X)$. This completes the proofs of both Proposition 1 and Theorem 2. □

It is easy to provide the set-valued versions of Theorems 1 and 2.

The proof of the next statement is almost identical with the proof of Theorem 1 and is omitted.

Theorem 3. *Let Z be a Baire space and $F : Z \to X$ a minimal non-empty-valued map from Z into the topological space X which is fragmented by some metric ρ. Then there exists a dense G_δ-subset $C \subset Z$ at the points of which F is single-valued and usc with respect to the metric ρ. In particular, if the topology generated by the metric ρ contains the topology of the space X, then $F : Z \to X$ is single-valued and usc at every point of the set C.*

The game approach we used above answers also the following question. What are the properties of the space X which ensure that every minimal (non-empty-valued) mapping $F : Z \to X$, where Z is a complete metric space (or, more generally, an α-favorable space) is single-valued at the points of some dense subset of Z? Or when is such a mapping F single-valued and upper semicontinuous at the points of a dense subset of Z? The answers are given by the following statements. The proofs are very similar to the proof of Theorem 2 and are omitted.

Theorem 4. *For the topological space X the following conditions are equivalent*:

(i) *The game $G(X)$ is Σ-unfavorable*;

(ii) *For every minimal non-empty-valued mapping $F : Z \to X$ where Z is a complete metric space there exists a point $z_0 \in Z$ such that $F(z_0)$ is a singleton*;

(iii) *For every minimal non-empty-valued mapping $F : Z \to X$ where Z is an α-favorable space, the set of points of Z at which F is single-valued is of the second Baire category in every open subset of Z.*

Theorem 5. *For the topological space X the following conditions are equivalent*:

(i) *The game $G'(X)$ is Σ-unfavorable*;

(ii) *For every minimal non-empty-valued mapping $F : Z \to X$ where Z is a complete metric space there exists a point $z_0 \in Z$ such that $F(z_0)$ is a singleton and F is upper semicontinuous at z_0*;

(iii) *For every minimal non-empty-valued mapping $F : Z \to X$ where Z is an α-favorable space, the set of points of Z at which F is single-valued and upper semicontinuous is of the second Baire category in every non-empty open subset of Z.*

There is a large class of spaces (containing all compact spaces X) for which condition (i) in the above Theorems 2 and 5 is equivalent to the (formally less restrictive) requirement that the game $G(X)$ is Σ-unfavorable (condition (i) from Theorem 4). This class will be introduced and investigated in the next section.

3. Game determined spaces

We consider in X one more game which we call "*Determination game*" and denote by $DG(X)$. The reason for this terminology will become clear later. The game $DG(X)$ is a generalization of of the game $G'(X)$. The same players Ω and Σ are involved in $DG(X)$

and the plays $p = (A_i, B_i)_{i \geqslant 1}$ are the same as in $G'(X)$ and $G(X)$. The only difference is with the winning rule. The player Ω is said to have won the play $p = (A_i, B_i)_{i \geqslant 1}$, if the set $K(p) := \bigcap \overline{A_i}$ is either empty or is such a compact set in X that for every open $U \supset K(p)$ there exists some integer $n > 0$ with $A_i \subset U$. Otherwise the player Σ wins the play p. We will call the space X *game determined* if Ω has a winning strategy in $DG(X)$. The class of all game determined spaces will be denoted by GD.

Note first that, if ω is a winning strategy for Ω in $G'(X)$, then it is winning in $DG(X)$ too. Therefore every space X that is fragmentable by a metric majorizing its topology belongs to GD. In particular, all metric spaces are in this class.

The p-spaces which were introduced by Arhangel'skii in [1] are game determined as are all Moore spaces. Every space X which is "$(p - \sigma)$-fragmentable" (Bouziad [5]) is game determined.

There is another large class of spaces which are game determined.

Definition 5. Let X be a subset of some space Y. We say that the set X *has countable separation in* Y (see [24,25]), if in Y there exists a countable family of open sets $(U_i)_{i \geqslant 1}$ such that for every pair of points $x \in X$ and $y \in Y \setminus X$ some U_i from the family contains exactly one of the two points x and y.

Note that in the above definition it is not specified which of the two points x or y is in U_i. Further, if X has countable separation in Y then the set $Y \setminus X$ also has countable separation in Y. Every open subset of Y as well as every closed subset of Y has countable separation in Y (the separating family consists of only one element in this case). It is easy to see that, for a given Y, the family of subsets with countable separation is closed under taking countable unions and countable intersections. This implies that all Borel subsets of the space Y have a countable separation in it. Moreover, every set obtained by applying the Souslin operation to a family of sets with countable separation in some Y has countable separation in Y as well.

Also, X has countable separation in Y if, and only if, it has countable separation in $\overline{X}^Y$, the closure of X in Y.

It was shown in [25] that, if X has countable separation in some compact space Y, then it has countable separation in any other compactification of X. This is why we will say that the completely regular space X *has countable separation*, if it has countable separation in some (and then in all) of its compactifications.

Proposition 2. *Every space X with countable separation is game determined* (*belongs to* GD).

Proof. Denote by Y some compact space in which X has countable separation and let $(U_i)_{i \geqslant 1}$ be a family of open subsets of Y which "separates" the points of X from the points of $Y \setminus X$. We will define a strategy ω for the player Ω which is winning in $DG(X)$. Suppose $A_1 \neq \emptyset$ is a first choice of Σ. There are two possibilities: $A_1 \cap U_1 = \emptyset$ or $A_1 \cap U_1 \neq \emptyset$. In the first case we put $B_1 = \omega(A_1) := A_1$. In the second case we take as $B_1 = \omega(A_1)$

some subset of A_1 which is relatively open in A_1 and $\overline{B_1}^Y \subset U_1$. In both cases the set $\overline{B_1}^Y$ is defined in such a way that it either does not intersect U_1 or is entirely contained in it. Proceeding inductively (on the length of the partial plays) we construct the strategy ω in such a way that, for every ω-play $p = (A_i, B_i)_{i\geqslant 1}$ and every $i \geqslant 1$ just one of the two options hold: $\overline{B_i}^Y \cap U_i = \emptyset$ or $\overline{B_i}^Y \subset U_i$. The countable separation of X implies that the compact set $\emptyset \neq K(p) = \bigcap_{i\geqslant 1} \overline{A_i}^Y$ is entirely contained either in X or in $Y \setminus X$. If $K(p) \subset Y \setminus X$, then $\bigcap_{i\geqslant 1} \overline{A_i}^X \subset K(p) \cap X = \emptyset$. If $K(p) \subset X$, then $K(p) = \bigcap_{i\geqslant 1} \overline{A_i}^X$ and, by the compactness of Y, we have that for every open $U \supset K(p)$ there is some integer $n > 0$ with $\overline{A_n}^X \subset U$. □

By the above mentioned result of Ribarska [41,42] the games $G'(X)$ and $G(X)$ are simultaneously favorable (and, therefore, simultaneously unfavorable) for the player Ω provided X is a compact space. In [25] this result was generalized and shown to have place for spaces X with countable separation. The next result goes in the same direction and establishes that, for game determined spaces X, the games $G(X)$ and $G'(X)$ are simultaneously favorable (or unfavorable) for any of the two players. In such cases we will say that *the two games are equivalent*.

Proposition 3. *If X is a game determined space, then $G(X)$ and $G'(X)$ are equivalent games.*

Proof. Let ω^* be a strategy for Ω which is winning in $DG(X)$.

($G(X)$ is Ω-favorable) $\Leftrightarrow$ ($G'(X)$ is Ω-favorable). It suffices to show that, if ω is a winning strategy for Ω in $G(X)$, then there is a winning strategy ω' for the same player in the game $G'(X)$. This will be done by "blending" (or "merging") the strategies ω and ω^*. Let $A'_1 \neq \emptyset$ be an arbitrary first move of Σ in $G'(X)$. Put $A_1 := \overline{A'_1}$ and $B_1 := \omega(A_1) \neq \emptyset$. Since A_1 is closed and B_1 is relatively open in A_1, there exists some set $H_1^* \neq \emptyset$ which is relatively open in B_1 (and hence in A_1) with $\overline{H_1^*} \subset B_1$. The set H_1^* can be considered as a first move of Σ in the game $DG(X)$. The set $B_1^* := \omega^*(H_1^*) \neq \emptyset$ is relatively open in H_1^* and, therefore, in A_1. Then the set $B_1^* \cap A'_1$ is non-empty and relatively open in A'_1. Define $B'_1 = \omega'(A'_1) := B_1^* \cap A'_1$. Proceeding inductively we define the strategy ω' in such a way that every ω'-play $p' = (A'_i, B'_i)_{i\geqslant 1}$ is accompanied by some ω-play $p = (A_i, B_i)_{i\geqslant 1}$ and some ω^*-play $p^* = (H_i^*, B_i^*)_{i\geqslant 1}$ so that, for every $i \geqslant 1$,

(a) $A_i = \overline{A'_i}$;
(b) H_i^* is relatively open in B_i and $\overline{H_i^*} \subset B_i$;
(c) $B'_i = B_i^* \cap A'_i$.

The set B'_i is non-empty because B_i^* is non-empty and relatively open in $A_i = \overline{A'_i}$. We will see now that ω' is a winning strategy in the game $G'(X)$. Note that $\bigcap A'_i \subset \bigcap A_i = \bigcap \overline{H_i^*}$. Suppose that $\bigcap A'_i \neq \emptyset$. Since $p = (A_i, B_i)_{i\geqslant 1}$ is an ω-play, the set $\bigcap A_i$ contains just one point. Hence $\bigcap A'_i = \bigcap A_i = \bigcap \overline{H_i^*}$. Since $p^* = (H_i^*, B_i^*)_{i\geqslant 1}$ is an ω^*-play, the play p' is won by Ω (in the game $G'(X)$).

($G(X)$ is Σ-favorable) $\Leftrightarrow$ ($G'(X)$ is Σ-favorable). It suffices to establish that the existence of a winning strategy σ' for Σ in $G'(X)$ implies the existence of a winning strategy for the same player in $G(X)$. In order to construct σ we will "merge" the strategies σ' and ω^*. This will be done in such a way that every σ-play will be accompanied by some ω^*-play and some σ'-play which will help establish the claim.

Let $A'_1 = \sigma(X)$ be the first choice of Σ under the strategy σ'. Put $A^*_1 := A'_1$, $B^*_1 := \omega^*(A^*_1)$ and $A_1 = \sigma(X) := \overline{B^*_1}$. This is the first choice of Σ in the strategy σ. Suppose all partial σ-plays of length n have already been defined in such a way that every partial σ-play $(A_1 \supset B_1 \supset \cdots \supset B_{n-1} \supset A_n)$ is accompanied by some partial ω^*-play $(A^*_1 \supset B^*_1 \supset \cdots \supset A_n \supset B_n)$ and some partial σ'-play $(A'_1 \supset B'_1 \supset \cdots \supset B'_{n-1} \supset A'_n)$ with the following properties fulfilled for every $i = 1, 2, \ldots, n$:

(d) $A^*_i = A'_i$;

(e) $A_i = \overline{B^*_i}$.

To make the next step in the definition of σ let the nth move of Ω in $G(X)$ be the non-empty relatively open subset B_n of A_n. Clearly, $B_n = A_n \cap U$ where U is an open subset of X. Find some open V with $\overline{V} \subset U$ and $V \cap A_n \neq \emptyset$. By property (e), $B'_n := V \cap B^*_n \neq \emptyset$. The set B'_n is relatively open in B^*_n which is relatively open in $A^*_n = A'_n$. Therefore the set

$$A'_{n+1} = \sigma'(A'_1, \ldots, A'_n, B'_n)$$

is well defined. Note that $A'_{n+1} \subset B'_n \subset B^*_n$. Put $A^*_{n+1} = A'_{n+1}$, $B^*_{n+1} = \omega^*(A^*_1, \ldots, B^*_n, A^*_{n+1})$ and

$$A_{n+1} = \omega(A_1, \ldots, A_n, B_n) := \overline{B^*_{n+1}}.$$

This completes the induction step in the definition of the strategy σ.

Let $p = (A_i, B_i)$ be a σ-play accompanied by the ω^*-play $p^* = (A^*_i, B^*_i)$ and the σ'-play $p' = (A'_i, B'_i)$ so that the properties (d) and (e) have place. Since σ' is a winning strategy in $G'(X)$, $\bigcap A'_i \neq \emptyset$. By (d) we have $\bigcap A^*_i = \bigcap B^*_i = \bigcap A'_i \neq \emptyset$. Then (e) implies that the set $K(p^*) := \bigcap A_i = \bigcap \overline{B^*_i} \supset \bigcap B^*_i \neq \emptyset$. To prove that σ is a winning strategy in the game $G(X)$ it suffices to prove that the set $K(p^*)$ has more than one point. Suppose that $K(p^*)$ is a singleton. Then $K(p^*) = \bigcap A'_i$. Since p^* is an ω^*-play and ω^* is winning for Ω in $DG(X)$, for every open $U \supset K(p^*)$ there is some integer $n > 0$ with $A^*_n = A'_n \subset U$. This means that Ω wins the σ'-play $p' = (A'_i, B'_i)$ in the game $G'(X)$ which contradicts the assumption that σ' is a winning strategy in $G'(X)$. $\square$

Corollary 4. *If X is game determined, then the chain of equivalent conditions* (i)–(iv) *in Theorems* 2 *and* 5 *can be extended by one more equivalent condition*:

(v) *$G(X)$ is Σ-unfavorable.*

Corollary 5. *The game determined space X is fragmentable by a metric that majorizes its topology if, and only if it is fragmentable.*

Corollary 6. *The space $X = [0, 1)$ with the Sorgenfrey topology is not game determined.*

Proof. Denote by Z the set $[0, 1)$ with the usual metric in $\mathbb{R}$. X is fragmented by the metric of Z. Let g be the identity mapping of Z onto X. As mentioned in the Introduction,

g is quasi-continuous but nowhere continuous. The previous corollary says that, if X were game determined, it would be fragmentable by a metric which majorizes the topology of X. Then, by Theorem 1, there would exist points of continuity of g which is not the case. □

Remark 1. For the Banach space E the following statements are equivalent.

(i) The space (E, weak) is game determined;

(ii) The space (E, weak) is fragmented by a metric which majorizes the weak topology. I.e., the game $G'((E, \mathit{weak}))$ is Ω-favorable;

(iii) The space (E, weak) is fragmented by a metric which majorizes the norm topology;

(iv) The space (E, weak) is sigma-fragmented (see [13–18] for the definition) by the norm.

This follows from Theorems 1.3 and 2.1 of [25]. Similar statements hold for the space $(C(T), \tau_p)$ of all continuous functions in the compact space T with the pointwise convergence topology.

The characterization of Banach spaces E for which the game $G'((E, \mathit{weak}))$ is Σ-unfavorable is given in [22]. It turns out that this is the case if, and only if, the game $DG((E, \mathit{weak}))$ is Σ-unfavorable. I.e., in the class of Banach spaces the games DG and G' are equivalent.

As shown in Proposition 5.1 of [25] the player Σ has a strategy which is winning for the game $G'((l^\infty, \mathit{weak}))$. This shows that the Banach space l^∞ with the weak topology is not game determined. Moreover, it does not belong to the class of spaces which are Σ-unfavorable for DG.

Remark 2. A generalization of the notion "game determined space" can be obtained if in the definition of this notion one requires that Ω has a strategy which wins plays $(A_i, B_i)_{i\geqslant 1}$ where all A_i (and therefore all B_i) are open subsets of X. We call such spaces "Banach–Mazur determined". They turned out to be useful in the study of the question when is a given semitopological group a topological one.

4. Game determined spaces and extension of minimal mappings

In this section we first give an equivalent definition of game determined spaces. This definition explains the terminology. Then we show that any closed graph minimal mapping $F : Z \to X$ must be upper semicontinuous and compact-valued at many points provided X is a game determined space and Z is a complete metric space (or α-favorable space).

Suppose bX is some compactification of the completely regular space X. Consider in bX a game (of the two players Σ and Ω) in which the plays are as in the game $G(bX)$ but the winning rule is the following: the player Ω is said to have won the play $p = (A_i, B_i)$ if the set $K(p) := \bigcap \overline{A_i}^{bX}$ either does not intersect X or lies entirely in X. Otherwise Σ wins the play p. We will not give this game a separate name and will not introduce a new notation for it because, as the next statement asserts, it is equivalent to the game $DG(X)$. Whenever needed, we will refer to this game as "the game in bX". The words "the game in X" will be used for $DG(X)$.

Proposition 4. *Let X be a completely regular space and bX some compactification of X. The above defined game in bX is equivalent to the game $DG(X)$ in X. In particular, if any of the players Ω or Σ has a winning strategy in one compactification bX of X, then he/she has winning strategy in any other compactification of X.*

Proof. In this proof we will denote by $\overline{C}^b$ the closure in bX of the set $C \subset bX$. The closure of $C \subset X$ in X will be denoted by $\overline{C}$. For the sake of clarity we will denote by ω_b (σ_b) any strategy of Ω (Σ) for the game in bX. For the game in X we use, as above, the notations ω (σ).

The proof rests on the following simple observation.

Lemma 2. *Let $A_1^* \supset \cdots \supset B_{k-1}^* \supset A_k^*$ be any partial play for the game in bX. If there is some open subset $U \subset bX$ such that $A_k^* \cap U \neq \emptyset$ and $U \cap (\overline{A_k^*}^b \cap X) = \emptyset$, then there exists some relatively open set $B_k^* \subset A_k^*$ for which every play in bX starting with $A_1^* \supset \cdots \supset A_k^* \supset B_k^*$ is won by Ω.*

Proof. Find some set V which is open in bX and has the properties: $\overline{V}^b \subset U$ and $B_k^* := A_k^* \cap V \neq \emptyset$. Clearly, B_k^* is relatively open in A_k^*. We also have

$$X \cap \overline{B_k^*}^b \subset X \cap \overline{A_k^*}^b \cap \overline{V}^b \subset X \cap \overline{A_k^*}^b \cap U = \emptyset.$$

This means that, for the game in bX, every play $p^* = (A_i^*, B_i^*)$ which starts with $A_1^* \supset \cdots \supset A_k^* \supset B_k^*$ will be won by Ω because $(\cap \overline{A_i^*}^b) \cap X = \emptyset$. The lemma is proved. □

(The game in X is Σ-favorable) $\Leftrightarrow$ (The game in bX is Σ-favorable). Suppose σ is a winning strategy of Σ for the game in X. This means that for every σ-play $p = (A_i, B_i)$ the non-empty set $\bigcap \overline{A_i}$ is either not compact or it is compact but there is some open set $U \supset \bigcap \overline{A_i}$ such that $A_i \cap (X \setminus U) \neq \emptyset$ for every $i \geqslant 1$. In both cases the compact set $\bigcap \overline{A_i}^b$ intersects not only X but $bX \setminus X$ as well. Thus σ is winning for the game in bX as well.

Let now σ_b be a winning strategy for Σ in bX. We will define a winning strategy σ for the game in X. Let $A_1^* = \sigma_b(bX)$ be the first choice of Σ in bX under the strategy σ_b. Lemma 2 implies that

$$A_1^* \subset \overline{X \cap \overline{A_1}^{*b}}^b.$$

Then the set $A_1 := X \cap \overline{A_1^*}^b$ is not empty. We define $\sigma(X) = A_1$. If B_1 is a relatively open subset of A_1, then there exists some open $U \subset bX$ such that $B_1 = U \cap A_1$. In particular, $U \cap \overline{A_1^*}^b \neq \emptyset$. Then the set $U \cap A_1^*$ is not empty and relatively open in A_1^*. Let V be some open subset of bX such that $\overline{V}^b \subset U$ and $B_1^* := V \cap A_1^*$ is not empty. Apply strategy σ_b to get the set $A_2^* := \sigma_b(A_1^*, B_1^*)$ and define $A_2 = \sigma(A_1, B_1) := X \cap \overline{A_2^*}^b$ (which is again non-empty). Note that

$$A_2 \subset X \cap \overline{B_1^*}^b \subset X \cap \overline{A_1^*}^b \cap \overline{V}^b \subset X \cap \overline{A_1^*}^b \cap U = A_1 \cap U = B_1.$$

Proceeding inductively (and using Lemma 2 many times), we define the strategy σ so that each σ-play $p = (A_i, B_i)$ is accompanied by some σ_b-play $p^* = (A_i^*, B_i^*)$ with $A_i =$

$\overline{A_i^*}^b \cap X$. We prove now that each σ-play $p = (A_i, B_i)$ is won by Σ in X. First note that, by Lemma 2, $\overline{A_i^*}^b = \overline{A_i}^b$ for $i \geqslant 1$ and therefore the set $K(p^*) := \bigcap \overline{A_i^*}^b = \bigcap \overline{A_i}^b$ intersects both X and $bX \setminus X$ (because σ_b is winning in bX). Take some $y_0 \in K(p^*) \cap (bX \setminus X)$ and define $K(p) := \bigcap \overline{A_i}$; it is a subset of X which contains the set $K(p^*) \cap X \neq \emptyset$. If $K(p)$ is not compact, the play p is won by Σ in X and there is nothing to prove. Suppose $K(p)$ is compact. Take some open set U in bX such that $K(p) \subset U$ and $y_0 \notin \overline{U}^b$. Since $y_0 \in \overline{A_i}^b$, $i \geqslant 1$, none of the sets A_i is contained in U. This means that the play p is won by Σ in X.

(The game in X is Ω-favorable) $\Leftrightarrow$ (The game in bX is Ω-favorable). If ω_b is an arbitrary strategy for Ω in bX, then the restriction of ω_b to plays with sets from X is some strategy σ for Ω in X. If ω_b is winning in bX, then its restriction to the subsets of X is a winning strategy in X. Suppose now there exists a strategy ω in X which is winning for Ω. We define a strategy ω_b which will turn out to be winning in bX. Let $A_1^* \neq \emptyset$ be any first move of Σ in bX. If $A_1^* \setminus \overline{\overline{A_1^*}^b \cap X}^b \neq \emptyset$, then Lemma 2 implies that there is an obvious winning strategy for Ω in bX. Hence, without loss of generality, we can assume that $A_1^* \subset \overline{\overline{A_1^*}^b \cap X}^b$. Moreover, for the same reason we will assume that $A_i^* \subset \overline{\overline{A_i^*}^b \cap X}^b$ for all sets A_i^* that appear in the course of defining the strategy ω_b. In particular, the set $A_1 := \overline{A_1^*}^b \cap X$ is not empty and is a possible move for Σ in X. We can apply the strategy ω to get the non-empty set $B_1 = \omega(A_1)$ which is relatively open in A_1. I.e., there is some open set U_1 in bX with $B_1 = A_1 \cap U_1$. Find some open V_1 in bX such that $\overline{V_1}^b \subset U_1$ and $V_1 \cap A_1 \neq \emptyset$. Since $V_1 \cap \overline{A_1^*}^b \neq \emptyset$, the set $A_1^* \cap V_1$ is not empty and relatively open in A_1^*. We define $B_1^* = \omega_b(A_1^*) := A_1^* \cap V_1$. Note that

$$\overline{B_1^*}^b \cap X \subset \overline{A_1^*}^b \cap \overline{V_1}^b \cap X \subset A_1 \cap U_1 = B_1.$$

Hence, for every next move $A_2^* \subset B_2^*$ of Σ in bX the set $A_2 := \overline{A_2^*}^b \cap X \subset B_1$ and we can apply the strategy ω to the partial play $A_1 \supset B_1 \supset A_2$ to get the set $B_2 = \omega(A_1, B_1, A_2) = A_2 \cap U_2$ where U_2 is some open subset of bX. Then, as above, we find some open V_2 with $\overline{V_2}^b \subset U_2$ and $V_2 \cap A_2 \neq \emptyset$. Finally, we define

$$B_2^* = \omega_b(A_1^*, B_1^*, A_2^*) := V_2 \cap A_2^*.$$

Proceeding by induction we construct the strategy ω_b in such a way that every ω_b-play $p^* = (A_i^*, B_i^*)$ is accompanied by some ω-play $p = (A_i, B_i)$ so that

$$A_i = \overline{A_i^*}^b \cap X \quad \text{and} \quad A_i^* \subset \overline{\overline{A_i^*}^b \cap X}^b = \overline{A_i}^b.$$

It follows that $\overline{A_i^*}^b = \overline{A_i}^b$. We show now that Ω wins every ω_b-play $p^* = (A_i^*, B_i^*)$. Suppose that $(\bigcap \overline{A_i^*}^b) \cap X \neq \emptyset$. Then the set $\bigcap \overline{A_i} = \bigcap (\overline{A_i^*}^b \cap X)$ is not empty. Since ω is winning in X, the set $K(p) := \bigcap \overline{A_i}$ is compact. We will show that $K(p^*) := \bigcap \overline{A_i^*}^b$ coincides with $K(p)$. Suppose the contrary and take some point $y_0 \in K(p^*) \setminus K(p)$. Since $K(p)$ is compact, there is some open subset U of bX, $U \supset K(p)$, such that $y_0 \notin \overline{U}^b$. On the other hand, ω is a winning strategy and there exists some integer $n > 0$ with $A_n \subset U$. Then we have the contradiction:

$$y_0 \in K(p^*) \subset \overline{A_i^*}^b = \overline{A_i}^b \subset \overline{U}^b. \qquad \square$$

Remark 3. In connection with Remark 2 we want to mention here that the completely regular space X is Banach–Mazur determined if, and only if, the player Ω has a strategy for the game in bX which wins all plays $(A_i, B_i)_{i\geqslant 1}$ where A_i, $i \geqslant 1$, are open subsets of bX.

We turn now to the extension of minimal mappings. Let us consider a non-empty-valued mapping $F : Z \to X$ from a topological space Z into a completely regular space X. Suppose bX is a compactification of X. The closure of the graph of F in $Z \times bX$ is a graph of some usc compact-valued mapping $\widetilde{F} : Z \to bX$. Such mappings are called usco mappings. It is easy to check that, if F is minimal, then $\widetilde{F}$ is minimal as well. Moreover, the graph of $\widetilde{F}$ does not contain as a proper subset the graph of any other usco mapping with the same domain Z. Thus $\widetilde{F}$ is a minimal usco mapping in the sense of Christensen [7].

Theorem 6. *Let $F : Z \to X$ be a minimal non-empty-valued mapping from the Baire space Z into the game determined space X. Suppose $\widetilde{F} : Z \to bX$ is the set-valued mapping whose graph coincides with the closure in $Z \times bX$ of the graph of F. Then the set $C := \{z \in Z\colon\ \widetilde{F}(z) \subset X\}$ contains a dense G_δ-subset of Z.*

Proof. Denote by ω some winning strategy for Ω in the game $DG(X)$. We consider the Banach–Mazur game in Z and construct a strategy ζ for the player α such that for every ζ-play $(W_i, V_i)_{i\geqslant 0}$ the (possibly empty) set $\bigcap_{i\geqslant 1} W_i$ is contained in C. According to a known theorem of Oxtoby [35] this would suffice to derive that C is residual in Z.

Let $V_0 \neq \emptyset$ be an open subset of Z. Consider the sets $A_1 := F(V_0)$ and $B_1 = \omega(A_1)$. Since B_1 is relatively open in A_1 and F is minimal, there is some open $W_1 \neq \emptyset$, $W_1 \subset V_0$, such that $F(W_1) \subset B_1$. Put $\zeta(V_0) := W_1$. Proceeding inductively we can construct the strategy ζ in such a way that any ζ-play $(W_i, V_i)_{i\geqslant 1}$ be accompanied by some ω-play $p = (A_i, B_i)_{i\geqslant 1}$ so that, for every $i \geqslant 1$,

(a) $A_i = F(V_{i-1})$, and

(b) $F(W_i) \subset B_i$.

Let $z_0 \in \bigcap V_i$. Then $\emptyset \neq F(z_0) \subset \bigcap A_i$. Since ω is a winning strategy in $DG(X)$, the set $K := \bigcap \overline{A_i}$ is compact in X and, for every open $U \subset bX$, $U \supset K$, there is some integer $n > 0$ with $A_n \subset U$. We will show that $\widetilde{F}(z_0) \subset K$. Take some $y_0 \in bX \setminus K$ and find an open set $U \supset K$ such that $y_0 \notin \overline{U}^b$. Let the integer $n > 0$ be such that $A_{n+1} = F(V_n) \subset U$. The set $V_n \times (bX \setminus \overline{U}^b)$ is open in $Z \times bX$, contains the point (z_0, y_0) but does not intersect the graph of F. This shows that $y_0 \notin \widetilde{F}(z_0)$. □

Theorem 7. *Let X be a topological space and bX some compactification of X. The following conditions are equivalent*:

(i) *The game $DG(X)$ is Σ-unfavorable*;

(ii) *For every minimal non-empty-valued mapping $F : Z \to X$ where Z is a complete metric space there exists a point $z_0 \in Z$ for which $\widetilde{F}(z_0) \subset X$*;

(iii) *For every minimal non-empty-valued mapping $F : Z \to X$ where Z is an α-favorable space the set $\{z \in Z\colon\ \widetilde{F}(z) \subset X\}$ is of the second Baire category in any open subset of Z.*

Proof. The proof is very similar to the proof of Theorem 2. We will only briefly outline the essential steps.

(i) $\Rightarrow$ (iii) Let $V_0 \neq \emptyset$ be an open subset of Z and H a first Baire category subset of Z. We will show that $V_0 \setminus H$ contains a point z_0 for which $\widetilde{F}(z_0) \subset X$. To do this we define a strategy σ for the player Σ basing on a strategy ζ for the player α in $BM(Z)$ which "avoids" the set H (see the proof of Theorem 2). Let $W_1 = \zeta(V_0)$. Take the set $A_1 = F(W_1)$ to be the first choice of the strategy σ. For the relatively open subset B_1 of A_1 there is some open subset $V_1 \subset W_1$ such that $F(V_1) \subset B_1$. Consider the sets $W_2 = \omega(V_0, W_1, V_1)$ and $A_2 = F(W_2)$. Define $\sigma(A_1, B_1) = A_2$. Proceeding inductively, one constructs the strategy σ in such a way that every σ-play $p = (A_i, B_i)$ is accompanied by some α-play (W_i, V_i) so that, for every $i \geqslant 1$, we have

(a) $A_i = F(W_i)$;

(b) $F(V_i) \subset B_i$.

Since α is a winning strategy for the Banach–Mazur game in Z, $\bigcap A_i = \bigcap F(W_i) \supset F(\bigcap W_i) \neq \emptyset$ for every σ-play $(A_i, B_i)_{i\geqslant 1}$. Condition (i) implies that some σ-play $(A_i, B_i)_{i\geqslant 1}$ is won by Ω in the game $DG(X)$. Put $K := \bigcap \overline{A_i}$. As in the proof of the previous theorem one shows that, for every $z_0 \in \bigcap W_i$, $\widetilde{F}(z_0) \subset K$.

The implication (iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i) Let σ be any strategy for Σ in the game $DG(X)$. As in the proof of Theorem 2 consider the space P of all σ-plays $p = (A_i, B_i)_{i\geqslant 1}$ endowed with the Baire metric d; $d(p, p') = 0$ if $p = p'$ and, otherwise, $d(p, p') = 1/n$ where $n = \min\{k\colon\ B_k \neq B'_k\}$. (P, d) is a complete metric space. Consider the (set-valued) mapping $F : P \to X$ defined by

$$F\big((A_i, B_i)_{i\geqslant 1}\big) = \bigcap_{i\geqslant 1} \overline{A_i}.$$

Using Lemma 1 it is not difficult to show that $F : P \to X$ is minimal. If for some $p \in P$ we have $F(p) = \emptyset$, then Ω wins the play p and there is nothing to prove. Therefore, without loss of generality, we may assume that $F(p) \neq \emptyset$ for every $p \in P$. By property (ii) there is some point $p_0 \in P$, $p_0 = (A_i^0, B_i^0)_{i\geqslant 1}$, such that $\widetilde{F}(p_0) \subset X$. We will see first that $F(p_0) = \widetilde{F}(p_0)$. The rest will follow from the upper semicontinuity of $\widetilde{F}$. Let $x_0 \in \widetilde{F}(p_0) \subset X$. Suppose that there is some integer $n > 0$ for which $x_0 \notin \overline{A_n^0}^b$. Then for every play $p = (A_i, B_i)_{i\geqslant 1}$ from the set $L := \{p \in P\colon\ d(p, p_0) < n^{-1}\}$ we have $A_n = A_n^0$ and hence $x_0 \notin \overline{A_n}^b = \overline{A_n^0}^b$. This means that the set $L \times (bX \setminus \overline{A_n^0}^b)$ (which is open on $Z \times bX$) contains (p_0, x_0) and does not intersect the graph of F. This contradicts the construction of $\widetilde{F}$. Hence $x_0 \in \overline{A_n}^b \cap X = \overline{A_n}$ for $n \geqslant 1$. Thus $x_0 \in F(p_0)$. $\square$

Remark 4. Let Z and X be completely regular spaces, $F : Z \to X$ a non-empty-valued minimal mapping and bZ, bX some compactifications of Z and X correspondingly. The closure of the graph $G(F)$ in $bZ \times bX$ determines a mapping $F^* : bZ \to bX$ which is minimal and usco. As analogue of Theorem 6 one can prove that the set

$$C(F^*) := \big\{z \in bZ\colon\ F^*(z) \subset X \text{ or } F^*(z) \subset bX \setminus X\big\}$$

contains dense G_δ-subset of bZ provided the space X is game determined. Similarly to Theorem 7 one has: if Z is α-favorable and X is Σ-unfavorable for $DG(X)$, then $C(F^*)$

is of the second Baire category in every non-empty open subset of bZ (and, therefore, in every non-empty open subset of Z).

5. Dense subcontinuity of quasi-continuous mappings

The following notion was introduced by Fuller [11].

Definition 6. The mapping $F:Z\to X$ between the topological spaces Z and X is said to be *subcontinuous* at $z_0\in Z$, if for every net $(z_\alpha,x_\alpha)_{\alpha\in\Lambda}\in Gr(F)$ with $(z_\alpha)_{\alpha\in\Lambda}$ converging to z_0 the net $(x_\alpha)_{\alpha\in\Lambda}$ has a cluster point in X. The map F is said to be subcontinuous if it is subcontinuous at every point of the space Z.

This notion attracted some attention. Recently its single-valued version was used (see [5, 6]) to establish that some semi-topological groups are topological.

It is easy to see that $F:Z\to X\subset bX$ is subcontinuous at some point $z_0\in Z$ if, and only if, $\widetilde{F}(z_0)\subset X$. To derive the next two statements from Theorems 6 and 7 we only need recall Corollary 2 and observe that, if f is a single-valued selection of the minimal (non-empty-valued) mapping F, then the closures in $Z\times bX$ of the graphs of F and f coincide.

Theorem 8. *Let $f:Z\to X$ be a quasi-continuous mapping from the Baire space Z to the game determined space X. Then there exists a dense G_δ-set $Z'\subset Z$ at the points of which f is subcontinuous.*

Theorem 9. *For the topological space X the following conditions are equivalent*:

(i) *$DG(X)$ is Σ-unfavorable*;

(ii) *every quasi-continuous mapping $f:Z\to X$ from the complete metric space Z into X is subcontinuous at at least one point of Z*;

(iii) *every quasi-continuous mapping $f:Z\to X$ from the complete metric space Z into X is subcontinuous at the points of some subset of Z which is of the second Baire category in every non-empty open subset of Z*;

(iv) *every quasi-continuous mapping $f:Z\to X$ from an α-favorable space Z into X is subcontinuous at the points of some subset of Z which is of the second Baire category in every non-empty open subset of Z.*

6. Some examples and applications

In view of what we intend to do in this section it makes sense to consider one more game $\overline{G}(X)$ for the same players Ω and Σ in the topological space X. The difference between the new game and $G(X)$ being that Σ selects only closed subsets of X. All the other components of the game (the moves of Ω and the rule for winning a play) are as in $G(X)$. It is not difficult to see that the two games are equivalent. What was said above implies that, for game determined spaces X, all the three games $G(X)$, $G'(X)$ and $\overline{G}(X)$ are equivalent.

Example 1. There exists a compact space X which is unfavorable for both players Σ and Ω in the game $\overline{G}(X)$.

In [20, Proposition 7(d)], Kalenda constructs a nonfragmentable compact space X for which every minimal usco mapping $F : Z \to X$, where Z is a Čech complete space, must be single-valued at many points. His proof is based on an idea from the paper of Namioka and Pol [32]. We show that the conclusion holds for arbitrary minimal mappings F acting in an α-favorable space Z. Our proof uses the game approach.

Construction of the example. It is based on a generalization of the famous "Double Arrow Space".

Let M be a Bernstein subset of the open interval $I = \{x\colon\ 0 < x < 1\}$. I.e., every continuum cardinality compact subset of I must intersect both M and $I \setminus M$. Note that M is dense in I. Consider the sets

$$X_0 := \left\{(x, 0) \in \mathbb{R}^2\colon\ 0 \leqslant x \leqslant 1\right\},$$
$$X_1 := \left\{(x, 1) \in \mathbb{R}^2\colon\ x \in M\right\}, \quad X := X_0 \cup X_1.$$

Equip X with the topology generated by the lexicographical order in X. This turns X into a compact space. The latter could be derived directly or using the compactness of the Double Arrow space. Note that both X_0 and X_1 (with the topology inherited from X) are fragmented by the Euclidean metric in $\mathbb{R}^2$.

Denote by π the the projection of X on $\mathbb{R}$: $\pi((x, i)) = x$ for $x \in [0, 1]$ and $i = 0, 1$. π is a continuous map. We will show first that X is unfavorable for Σ. Suppose this is not so and denote by σ some winning strategy for Σ. Let $A_1 = \sigma(X)$ be the first closed set selected by Σ. $\pi(A_1)$ is a compact subset of R without isolated points as otherwise there would exist a relatively open subset $B \subset A_1$ containing not more than two points and Ω would easily win any continuation of the partial play $A_1 \supset B$. In particular, $\pi(A_1)$ is infinite. We will use the Cantor set construction to produce a compact C of continuum cardinality and this will help us reach contradiction.

Put $C_1 := A_1$ and construct two disjoint infinite relatively open subsets C_2, C_3 of A_1 that are open intervals in the order inherited by A_1 from X. Moreover, we can assume that $\pi(\overline{C_2}) \cap \pi(\overline{C_3}) = \emptyset$. Denote $A_2 = \sigma(A_1, C_2)$ and $A_3 = \sigma(A_1, C_3)$. The sets $\pi(A_2)$ and $\pi(A_3)$ are compact subsets of $\mathbb{R}$ without isolated points. Therefore in each of the sets A_2, A_3 we can find a pair of infinite relatively open subsets (of interval type with respect to the order in X) which are disjoint. Proceeding inductively we construct a sequence of sets A_i, C_i, $i = 1, 2, 3, \ldots$, so that

(a) A_i is a closed subset of X and $\pi(A_i)$ does not have isolated points;
(b) C_{2i}, C_{2i+1} are infinite open intervals in A_i such that $\pi(\overline{C_{2i}}) \cap \pi(\overline{C_{2i+1}}) = \emptyset$;
(c) $\overline{C_{2i}} \cup \overline{C_{2i+1}} \subset C_i$;
(d) $A_{2i} = \sigma(A_1, \ldots, A_i, C_{2i})$ and $A_{2i+1} = \sigma(A_1, \ldots, A_i, C_{2i+1})$;

(e) for every integer $p \in [2^i, 2^{i+1})$ the diameter of the set $\pi(C_p)$ is less or equal to 2^{-i}.

Since σ is a winning strategy, the intersection of every σ-play appearing in the above construction contains exactly two points which have the same projection on $\mathbb{R}$ (this follows from property (e)). The union C of all such intersections is a compact subset of X which has continuum cardinality. This follows from properties (a)–(d). However $\pi(C) \subset [0,1]$ is compact of continuum cardinality as well. Hence there exists some point $t \in \pi(C) \cap (R \setminus M)$. We see that the set $\pi^{-1}(t) \cap C$ has only one point which belongs to C. This contradiction shows that X is Σ-unfavorable.

One can use the Cantor set construction to establish (as above) that X is Ω-unfavorable as well. This is equivalent to proving that X is not fragmentable. Alternatively, one can use Proposition 3 from [20] where it is shown that the space X (defined as above by means of an arbitrary subset $M \subset I$) is fragmentable if, and only if, M is countable. We prefer to establish this in another way which gives us slightly more.

Example 2. There exist a Baire space M which is a subset of the real line $\mathbb{R}$, a Σ-unfavorable compact space X and a minimal usco mapping $F : M \to X$ which is nowhere single-valued. In particular, X is not fragmentable.

Proof. Let X and M be the spaces from the previous example. It is known that M is a Baire space. Consider the mapping $F : M \to X$ which assigns to every $t \in M$ the set $F(t) := \{(t,0)\} \cup \{(t,1)\}$. It is easy to check that F is a minimal usco mapping. Theorem 1 from the Introduction implies that the space X is not fragmentable. $\square$

Remark 5. Example 2 shows that the space X from Example 1 does not belong to the class of spaces defined by Stegall in [43].

We outline now how the notions and results from this paper could be used in the study of continuity properties of separately continuous mappings. Our goal is not to give an exhaustive list of all possible (and most general) corollaries but just to present a sample of results in this direction. Let $f : Z \times Y \to X$ be a mapping defined in the product of the spaces Z and Y. For every fixed $z \in Z$ ($y \in Y$) one denotes by f_z (f_y) the mapping $f_z : Y \to X$ ($f_y : Z \to X$) defined by $f_z(y) = f(z,y)$ ($f_y(z) = f(z,y)$). f is said to be *separately continuous in* y if, for every $z \in Z$, f_z is continuous. Similarly one defines the notion "separately continuous in z". f is called *separately continuous*, if it is separately continuous both in z and in y. It is known that a separately continuous mapping $f : Z \times Y \to X$ need not be continuous. However, under some relatively mild requirements imposed on the spaces Z, Y, X, it is possible to prove that there exist points of continuity of f. The problem is known as "*joint continuity of separately continuous functions*" and received a lot of attention in the last century (after the famous paper of Baire [2] appeared). Detailed information can be found in the survey papers of Piotrowski [37,38]. Very interesting results are contained also in the papers of Namioka [31], Talagrand [44, 45], Debs [10] and many others. The standard approach to this problem consists in first proving that f is quasi-continuous (by using the properties of the spaces Z and Y) and then establishing that f is continuous at some points of the product $Z \times Y$ (by using metrizability or metrizability-like properties of X). We follow the same scheme of reasoning. Basing upon known results we give some sufficient conditions (in terms of topological games) for the mapping f to be quasi-continuous and then apply Theorem 1 or Theorem 2 to show that there exist points of continuity of f even in cases when X is far from being metrizable. For instance, X could be fragmentable by an appropriate metric or, even less, Σ-unfavorable for $G'(X)$. We formulate also some results concerning points of subcontinuity of separately continuous mappings.

We start with a known fact concerning quasi-continuity of separately continuous mappings. Later we will give another result of the same type.

Proposition 5 (Piotrowski [36]). *Let Z be a Baire space. Suppose the completely regular space Y contains a dense subset of points of countable character, i.e., points with countable base of neighborhoods. Then every separately continuous mapping $f : Z \times Y \to X$ into the completely regular space X is quasi-continuous.*

Here are some cases when the assumptions of Proposition 5 are satisfied.

Proposition 6. *Let the space Y be either*

(a) *Baire and Ω-favorable for the game $G'(Y)$* (*i.e., Baire and fragmentable by a metric which majorizes its topology*);

or

(b) *α-favorable for the game $BM(Y)$ and Σ-unfavorable for the game $G'(Y)$.*

Then Y contains dense subset of points of countable character.

Proof. (a) Let ω be the winning strategy for Ω in $G'(Y)$ and $W_0 \neq \emptyset$ an open subset of Y. We define a strategy s for β in $BM(Y)$. Consider W_0 as a first move of Σ in $G'(Y)$ and put $V_1 := \omega(W_0)$, $s(Y) := V_1$. If the open set W_1 satisfies $\emptyset \neq W_1 \subset V_1$, we put $V_2 = s(V_1, W_1) := \omega(W_0, V_1, W_1)$. Proceeding inductively, we construct the strategy s in such a way that for every s-play $(V_i, W_i)_{i \geqslant 1}$ in $BM(Y)$ the sequence $(W_{i-1}, V_i)_{i \geqslant 1}$ is an ω-play in $G'(Y)$. Since Y is a Baire space, there is some s-play $(V_i, W_i)_{i \geqslant 1}$ for which the set $K := \bigcap V_i \subset W_0$ is non-empty. Since the play $(W_{i-1}, V_i)_{i \geqslant 1}$ is won by Ω, the set K is a singleton with $(V_i)_{i \geqslant 1}$ as countable base of neighborhoods. The first part of Proposition 6 is proved.

(b) Let ζ be a winning strategy for α in $BM(Y)$ and V_1 an open subset of Y. We define a strategy σ for the player Σ in $G'(Y)$. Consider the open set $W_1 = \zeta(V_1)$ as a first move of Σ. I.e., $\sigma(Y) := W_1$. For the non-empty open set $V_2 \subset W_1$ put $W_2 = \zeta(V_1, W_1, V_2)$ and $\sigma(W_1, V_2) := W_2$. Proceeding inductively, we define the strategy σ in such a way that, for every σ-play $(W_i, V_{i+1})_{i \geqslant 1}$ in $G'(Y)$, the sequence $(V_i, W_i)_{i \geqslant 1}$ is a ζ-play in $BM(Y)$. In particular, the set $K = \bigcap V_i \subset V_1$ is non-empty for every play. By the assumptions there exists some σ-play $(W_i, V_{i+1})_{i \geqslant 1}$ which is won by Ω in $G'(Y)$. For such a play the set K is a singleton and $(V_i)_{i \geqslant 1}$ is a base of neighborhoods of K. $\square$

Theorem 10. *Let $f : Z \times Y \to X$ be a separately continuous mapping where Z, Y, X are completely regular spaces such that*

(i) *Z, Y are α-favorable for the game BM and*

(ii) *Y, X are Σ-unfavorable for the game G'.*

Then the set of points in $Z \times Y$ at which f is continuous is of the second Baire category in every open subset of $Z \times Y$.

Proof. Proposition 5 and the second half of Proposition 6 imply that f is quasi-continuous. Since $Z \times Y$ is α-favorable, we can apply Theorem 2 and this completes the proof. □

Applying the first half of Proposition 6 and Theorem 1 instead of Theorem 2 in the last proof we get:

Theorem 11. *Let $f : Z \times Y \to X$ be a separately continuous mapping where Z, Y, X are completely regular spaces such that*

(i) *$Z \times Y$ is a Baire space and*

(ii) *Y, X are Ω-favorable for the game G', i.e., each of these spaces is fragmentable by a metric that majorizes the corresponding topology.*

Then there exists a dense G_δ-subset of $Z \times Y$ at the points of which f is continuous.

Proposition 7. *Let the space Y be either*

(a) *Baire and Ω-favorable for the game $DG(Y)$* (*i.e., Y is Baire and game determined*);

or

(b) *α-favorable for the game $BM(Y)$ and Σ-unfavorable for the game $DG(Y)$.*

Then every non-empty open set $V \subset Y$ contains a non-empty compact K of countable outer base. I.e., there exists a countable family of open sets $(O_i)_{i \geqslant 1}$ such that $K = \bigcap_{i \geqslant 1} O_i$ is a non-empty compact subset of V and for every open $U \supset K$ there exists some integer n with $O_n \subset U$.

The proof is omitted because it is very similar to the proof of Proposition 6.

In the next assertion we follow very closely the proof of Theorem 1 from [6] and Lemma 2.6 from [5].

Proposition 8. *Let Z, Y and X be completely regular spaces and $f : Z \times Y \to X$ a separately continuous mapping. Suppose that every non-empty open subset of Y contains a non-empty compact K with countable outer base and the space Z is either*

(a) *Baire and Ω-favorable for the game DG* (*i.e., Baire and game determined*);

or

(b) *α-favorable for BM and Σ-unfavorable for DG.*

Then f is quasi-continuous.

Proof. Since X is completely regular, it suffices to prove the proposition for the case when X is the real line $\mathbb{R}$. Let $\varepsilon > 0$ and $(z_0, y_0) \in V \times U$, where $V \subset Z$ and $U \subset Y$ are open sets. It suffices to show that there are non-empty open sets $V' \subset V$, $U' \subset U$ such that $|f(V' \times U') - f(z_0, y_0)| \leqslant 3\varepsilon$. For the sake of reaching a contradiction we will suppose that in every such set $V' \times U'$ there is a point (z', y') with $|f(z', y') - f(z_0, y_0)| > 3\varepsilon$.

Since f is separately continuous, there are some open sets V_0 and U_0 such that $z_0 \in V_0 \subset V$, $y_0 \in U_0 \subset U$ and

$$\left|f(V_0, y_0) - f(z_0, y_0)\right| < \varepsilon, \qquad \left|f(z_0, U_0) - f(z_0, y_0)\right| < \varepsilon.$$

There exists a non-empty compact $K \subset U_0$ with countable outer base of open sets $(O_i)_{i \geqslant 1}$. It is enough to prove the proposition for the case when $y_0 \in K$. We begin with the proof of the case (a).

(a) Denote by ω some winning strategy for Ω in $DG(Z)$. We determine a strategy s for the player β in $BM(Z)$ and then use the fact that it is not a winning one. Put $W_0' = Z$ and consider V_0 as a first move of β. I.e., $s(Z) = V_0$. Suppose $W_1' \subset V_0$ is a possible move of α. Consider W_1' as a first move of Σ in $DG(Z)$. Put $W_1 := \omega(W_1')$. Further, consider the set $U_1 := \{y \in O_1\colon\ |f(z_0, y) - f(z_0, y_0)| < 1\}$ which is an open neighbourhood of y_0. By the assumption, there exists a point $(z_1, y_1) \in W_1 \times U_1$ such that $|f(z_1, y_1) - f(z_0, y_0)| > 3\varepsilon$. Define $s(V_0, W_1') = V_1$ to be some non-empty open set such that $\overline{V_1} \subset \{z \in W_1\colon\ |f(z, y_1) - f(z_1, y_1)| < \varepsilon\}$. The first step in the definition of the strategy s is completed. Proceeding inductively, we define the strategy s in such a way that every s-play $(W_i', V_i)_{i \geqslant 0}$ is accompanied by: an ω-play $(W_i', W_i)_{i \geqslant 1}$, a sequence of open sets $(U_i)_{i \geqslant 1}$ and sequences of points $(z_i)_{i \geqslant 0}$, $(y_i)_{i \geqslant 0}$ so that, for $n = 1, 2, 3, \ldots$ we have:

(i) $U_n = \{y \in O_n\colon\ |f(z_k, y) - f(z_k, y_0)| < n^{-1}$ for $k \leqslant n\}$;
(ii) $(z_n, y_n) \in W_n \times U_n$;
(iii) $|f(z, y_n) - f(z_n, y_n)| < \varepsilon$ for $z \in \overline{V_n}$;
(iv) $|f(z_n, y_n) - f(z_0, y_0)| > 3\varepsilon$.

Since Z is a Baire space there is some s-play $(W_i', V_i)_{i \geqslant 0}$ which is won by α: $\bigcap W_i' \neq \emptyset$. The corresponding ω-play $(W_i', W_i)_{i \geqslant 1}$ is won by player Ω in $DG(Z)$. Hence there is a cluster point $z^* \in \bigcap \overline{V_i}$ for the sequence $(z_i)_{i \geqslant 0}$. As $y_n \in O_n$, there is a cluster point y^* of the sequence $(y_i)_{i \geqslant 0}$. From (ii) and (i) we derive that $f(z_k, y^*) = f(z_k, y_0)$ for every $k \geqslant 1$. This implies $f(z^*, y^*) = f(z^*, y_0)$. Since $z^* \in \overline{V_n}$, from (iii) we get $|f(z^*, y_n) - f(z_n, y_n)| < \varepsilon$ for every $n \geqslant 1$. Then, by (iv), we get $|f(z^*, y_n) - f(z_0, y_0)| > 2\varepsilon$ for every $n \geqslant 1$. This leads however to the contradiction:

$$2\varepsilon \leqslant \left|f(z^*, y^*) - f(z_0, y_0)\right| = \left|f(z^*, y_0) - f(z_0, y_0)\right| < \varepsilon.$$

(b) The proof is very similar to the one in case (a). We even use the same notations.

Let ζ be a winning strategy for α in $BM(Z)$. We will construct a strategy σ for the player Σ in $DG(Z)$ and use the fact that it is not winning. Put $W_0' = Z$ and let V_0 (from the beginning of the proof) be the first move of β in $BM(Z)$. Define $W_1' := \zeta(V_0)$ to be the first choice of Σ in $DG(Z)$. I.e., $\sigma(Z) := W_0'$. If W_1 is any open subset of W_1' we find a point $(z_1, y_1) \in W_1' \times U_1$ with $|f(z_1, y_1) - f(z_0, y_0)| > 3\varepsilon$ and select a non-empty open subset V_1 such that $\overline{V_1} \subset \{z \in W_1\colon\ |f(z, y_1) - f(z_1, y_1)| < \varepsilon\}$. Define $W_2' = \sigma(W_1', W_1) := \zeta(V_0, W_1', V_1)$. Proceeding inductively we construct the strategy σ in such a way that every σ-play $(W_i', W_i)_{i \geqslant 1}$ is accompanied by some ζ-play $(W_i', V_i)_{i \geqslant 1}$ and by the sequences $(U_i)_{i\geqslant 1}$, $(z_i)_{i\geqslant 1}$, $(y_i)_{i\geqslant 1}$ so that the properties (i)–(iv) from the proof of case (a) are fulfilled for $n = 1, 2, 3, \ldots$. Since Z is Σ-unfavorable, there is some σ-play $(W_i', W_i)_{i\geqslant 1}$ which is won by Ω. As the corresponding ζ-play $(W_i, V_i)_{i\geqslant 1}$ is won by α, the sequence $(z_i)_{i\geqslant 1}$ has a cluster point $z^* \in \bigcap \overline{V_i}$. The rest of the proof coincides with the one from the case (a). □

This allows one to formulate results concerning joint continuity of separately continuous functions defined in spaces more general than those containing dense subsets of points of countable character.

Corollary 7. *Let Z, Y, X be completely regular spaces which are Σ-unfavorable for $DG(Z)$, $DG(Y)$, $G'(X)$ correspondingly. Suppose Z and Y are α-favorable and $f : Z \times Y \to X$ is a separately continuous function. Then f is continuous at the points of some subset of $Z \times Y$ which is of the second Baire category in every non-empty open subset of $Z \times Y$.*

A related problem which appears here is to find conditions on Z, Y, X under which every separately continuous mapping $F : Z \times Y \to X$ is subcontinuous at some points of $Z \times Y$. Here is a statement of this kind.

Theorem 12. *Let $f : Z \times Y \to X$ be separately continuous where Z, Y, X are completely regular and Σ-unfavorable spaces* (*for the game DG*). *Suppose Z and Y are α-favorable* (*for BM*). *Then f is subcontinuous at the points of a subset which is of the second Baire category in every non-empty open subset $Z \times Y$.*

Proof. In view of Theorem 9 it suffices to show that f is quasi-continuous. This follows from Proposition 8. □

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