Boolean Modules*

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1. INTRODUCTION

Relation algebras were introduced by Tarski in the 1940s to do for the calculus of (binary) relations what Boolean algebras do for the calculus of classes. Tarski's immediate predecessor (by some 45 years) was Ernst Schröder, whose immediate predecessor was C. S. Peirce, whose immediate predecessor was Augustus De Morgan. Although De Morgan was the first to treat the logic of relations it was Peirce who gave the subject an algebraic flavour, and it is his work that forms the starting point of this paper. The development of Peirce's ideas brought him to the concept of relative product of relations (which he did not invent) via the concept of a product between relations and sets (which he did invent). If $R$ and $S$ are relations, and $A$ is a set, the two products are

$$R; S = \{(x, y) | (\exists z)(x, z) \in R \text{ and } (z, y) \in S\},$$
$$R : A = \{x | (\exists y)(x, y) \in R \text{ and } y \in A\}.$$

Tarski has turned the product $R; S$ into algebra by introducing relation algebras. The purpose of this paper is to turn the product $R : A$ into algebra by introducing Boolean modules. Since this product, which I call the Peircean product, combines relations and sets, its algebraic counterpart is a product between elements of a relation algebra and elements of a Boolean algebra. So a Boolean module will be a Boolean algebra with a multiplication from a relation algebra.

Boolean algebras are well known and any equational definition of Boolean algebras will suffice for present purposes. A relation algebra is an algebra $\mathcal{R} = (R + \cdot, e)$ satisfying the following axioms.

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A1: \((R + ')\) is a Boolean algebra;
A2: \((r; s); t = r; (s; t);\)
A3: \((r + s); t = r; t + s; t;\)
A4: \(r; e = r;\)
A5: \(r^{-} = r;\)
A6: \((r + s)^{-} = r^{-} + s^{-};\)
A7: \((r; s)^{-} = s^{-}; r^{-};\)
A8: \(r^{-}; (r; s)^{'} \leq s^{'};\)

This definition is from [11], as modified in [11]. Note that the class of relation algebras is equationally definable. The arithmetic of relation algebras will be taken for granted here; details can be found in [11]. For present purposes it is sufficient to introduce the standard model of the axioms for a relation algebra. A proper relation algebra is a relation algebra \(\mathcal{R} = (R + '; \; ^{-} e)\) where \(R\) is a class of binary relations over some set \(U\), \(+\) is the set-theoretical union of relations, \('\) is the set-theoretical complementation with respect to the universal relation \(V = 1, 0\) is the empty relation \(A\), \(;\) is the set-theoretical relative product, and \(^{-}\) is the set-theoretical conversion of relations. The universal relation need not be a Cartesian square, but it must be an equivalence relation; this follows from the axioms. Also, there must be a relation \(e\) which is an identity element with respect to relative multiplication, but this relation need not be the identity relation over the field of \(V\). If the relation \(e\) in a proper relation algebra \(\mathcal{R}\) is in fact the identity relation over the field of the universal relation \(V\), we call \(\mathcal{R}\) a proper relation algebra with identity. If, in addition, \(V = U^2\), we call \(\mathcal{R}\) a proper relation algebra on the set \(U\).

Contrary to the original intention it turned out that relation algebras do not quite do for the calculus of relations what Boolean algebras do for the calculus of classes. For Boolean algebras a representation result holds: any Boolean algebra is isomorphic to a field of sets. For relation algebras the analogous result does not hold: not every relation algebra is isomorphic to a proper relation algebra [7]. Two further results deserve mention. The first is that any relation algebra is weakly representable: isomorphic to a relation algebra \(\mathcal{S}\), where \(S\) is a class of binary relations contained in an equivalence relation \(V\) which is the universal element of \(\mathcal{S}\), and all operations, with the possible exception of \('\), are as in a proper relation algebra with identity. (See [1, 4, 6].) The second result is that the representable relation algebras can be characterized by an infinite set of equational axioms [8]. It is also known that no finite set of identities will suffice for this purpose [10].
2. Definition, Examples and Arithmetic

Here is the main definition of this paper.

2.1. Definition. Let $\mathcal{R} = (R + \cdot; e)$ be a relation algebra. A (left) Boolean $\mathcal{R}$-module is a Boolean algebra $\mathcal{B} = (B + \cdot)$ together with a mapping $f: R \times B \to B$, where $f(r, a)$ is written $r: a$, subject to the following axioms.

\begin{align*}
M1: & \quad r:(a + b) = r: a + r: b; \\
M2: & \quad (r + s): a = r: a + s: a; \\
M3: & \quad (r; s): a = r: (s: a); \\
M4: & \quad e: a = a; \\
M5: & \quad 0: a = 0; \\
M6: & \quad r': (r: a)' \leq a'.
\end{align*}

An historical digression may be instructive. The product $R: A$ was introduced by Peirce in 1870 in his first major paper on "relatives" (3.68; reference to Peirce's work is made by the numbered paragraphs in his Collected Papers.) It is worth noting that axioms M1–M5 can also be found in that paper. The first three, in particular, Peirce recognized as fundamental; they occur right after his introduction of the Peircean product. The other two are derived in 3.84 and 3.82, respectively. It must be pointed out that after his 1870 paper Peirce said no more about the Peircean product, turning instead to relative product. Schröder then took over the development of relative product in algebraic form, without saying anything about the Peircean product. It is interesting, therefore, to note that the Peircean product crops up independently in Russell and Whitehead's Principia Mathematica: in the Summary to *37 the notation "Rβ" is introduced, which means "the terms which have the relation R to members of β." Not only the concept but some of the axioms can be found in Principia Mathematica. The analogues of M1–M4 can be found in *37.22, *37.221, *37.33, and *50.16, respectively. Many other formulas in the arithmetic of Boolean modules can be found either in Peirce's 1870 paper or in Principia Mathematica. (End of digression.)

Definition 2.1 is strongly reminiscent of the usual definition of an ordinary module. In fact, if we read "ring" for "relation algebra" and "additive Abelian group" for "Boolean algebra," and we scrap M5 and M6, we obtain the definition of an ordinary (unital) module. Here we have defined left Boolean modules; right Boolean modules are defined similarly by taking $f(a, r) = a: r$ to be a mapping of $B \times R$ into $B$ and giving the right-hand version of the axioms. The product in a left Boolean module is intended to be an algebraization of the Peircean product; the product in a right Boolean
module is intended to play the same role for the analogous set-theoretical product

$$A : R = \{ y | (\exists x)[x \in A \text{ and } (x, y) \in R] \}.$$ 

From now on "Boolean module" means "left Boolean $\mathcal{P}$-module" for some relation algebra $\mathcal{P}$. Here are some examples of Boolean modules.

2.2. Example. Let $U$ be any nonempty set. The power set $\mathcal{P}(U)$ consisting of all subsets of $U$ forms a Boolean algebra, and the class $\mathcal{P}(U^2)$ of all relations over $U$ forms a relation algebra, under the standard set-theoretical operations. For any element $R$ of $\mathcal{P}(U^2)$ and any element $A$ of $\mathcal{P}(U)$ the Peircean product $R : A$ is a set contained in $U$; hence the Peircean product defines a mapping of $\mathcal{P}(U^2) \times \mathcal{P}(U)$ into $\mathcal{P}(U)$. Verification of $M_1-M_6$ is straightforward.

2.3. Example. Define a module over a group $\mathcal{G}$, or $\mathcal{G}$-module, as a set $A$ together with a mapping $h : G \times A \rightarrow A$, where $h(g, a)$ is written $g \times a$, subject to the following axioms:

$$(gh) \times a = g \times (h \times a),$$

$$1 \times a = a.$$ 

Let $\mathcal{A}$ be a $\mathcal{G}$-module, then $\mathcal{P}(A)$ forms a Boolean algebra and $\mathcal{P}(G)$ is the base set of the complex algebra of $\mathcal{G}$, which is a relation algebra under the operations $H; K = \{hk|h \in H \text{ and } k \in K\}$ and $H^{-1} = \{h^{-1}|h \in H\}$ and with identity element $\{1\}$. Now define a multiplication : between elements of $\mathcal{P}(G)$ and elements of $\mathcal{P}(A)$ by

$$K : X = \{k \times x|k \in K \text{ and } x \in X\} \quad \text{for every } K \subseteq A.$$ 

This multiplication defines a mapping from the relation algebra $\mathcal{P}(\mathcal{G})$ to the Boolean algebra $\mathcal{P}(\mathcal{A})$. It is not difficult to see that $M_1-M_6$ are satisfied.

2.4. Example. Let $\mathcal{B}$ be any Boolean algebra. The class of all $n \times 1$ matrices with elements from $B$ forms a Boolean algebra $\mathcal{B}$ under componentwise Boolean operations. Similarly, the class of all $n \times n$ matrices with elements from $B$ forms a relation algebra $\mathcal{B}$ under componentwise Boolean operations, matrix multiplication, transposition and the identity matrix. Ordinary matrix multiplication is a multiplication between elements of $\mathcal{B}$ and elements of $\mathcal{B}$, and this multiplication satisfies $M_1-M_6$.

In the standard model of the axioms for a Boolean $\mathcal{P}$-module $\mathcal{B}$ the elements of $\mathcal{B}$ are binary relations and the elements of $\mathcal{B}$ are sets. More precisely:
2.5. Definition. A Boolean \( R \)-module \( B \) is called a proper Boolean \( R \)-module iff \( B \) is a field of sets, \( R \) is a proper relation algebra with identity, and the operation : is the Peircean product. If the universal set of \( B \) coincides with the field of the universal relation of \( R \), then \( B \) is called a proper Boolean \( R \)-module over the field of \( R \).

Thus the Boolean module of Example 2.2 is proper, the other two are not. Two points concerning proper Boolean modules deserve mention. First, if \( B \) is a proper Boolean \( R \)-module, then the universal set \( U \) of \( B \) must be contained in the field of the universal relation \( V \) of \( R \). For, by M4 we must have \( I : U = U \), where \( I \) is the identity relation over \( F \), the field of \( V \). Then

\[
U = I : U = \{x | (\exists y) [(x, y) \in I \text{ and } y \in U] \}
\]

\[
= \{x | (\exists y) [x \in F, y \in F, x = y, \text{ and } y \in U] \}
\]

\[
= \{x | x \in F \text{ and } x \in U \}
\]

\[
= F \cap U.
\]

It follows that \( U \) is contained in \( F \). So we see that if \( B \) is a field of sets with universal set \( U \), and \( R \) is a proper relation algebra with universal relation \( V \), then the condition that \( U \) is contained in the field of \( V \) is a necessary condition for \( B \) to be a proper Boolean \( R \)-module. (Note: it is not a sufficient condition.) A second point to note is that in a proper Boolean \( R \)-module \( B \) it is possible that for some relations \( R \) and \( S \) we may have \( R : A = S : A \) for every element \( A \) of \( B \), without having \( R = S \). For a simple example, let \( B \) be the field of sets whose only elements are \( \emptyset \) and \( \{b\} \), and let \( R \) be the proper relation algebra with identity whose elements are \( A \), \( \{(a, a)\}, \{(b, b)\} \), and \( V = \{(a, a), (b, b)\} \). Then \( B \) is a proper Boolean \( R \)-module. Let \( S \) be the relation \( \{(b, b)\} \), then

\[
V: \emptyset = \emptyset = S: \emptyset,
\]

\[
V: \{b\} = \{b\} = S: \{b\}.
\]

Hence \( V : A = S : A \) for every \( A \) in \( B \), even though \( V \neq S \). Boolean modules for which this situation cannot arise deserve a name of their own. (The terminology will be explained later.)

2.6. Definition. A Boolean \( R \)-module \( B \) is bijective iff

\[
(\forall a \in B)[r : a = s : a] \Rightarrow r = s.
\]
T1. If $a \leq b$, then $r: a \leq r: b$.

Proof. $a < b$ implies $a + b = b$; hence $r: a + r: b = r: (a + b) = r: b$ by M1, so $r: a \leq r: b$.

T2. If $r \leq s$, then $r: a \leq s: a$.

Proof. Similar to that of T1.

T3. $r: (a \cdot b) \leq (r: a) \cdot (r: b)$.

Proof. $r: (a \cdot b) \leq r: a$ by T1. Similarly $r: (a \cdot b) \leq r: b$; hence $r: (a \cdot b) \leq (r: a) \cdot (r: b)$.

T4. $(r \cdot s): a \leq (r: a) \cdot (s: a)$.

Proof. Similar to that of T3.

T5. $(r: a) \cdot b = 0$ iff $(r^{-}: b) \cdot a = 0$.

Proof. If $(r: a) \cdot b = 0$, then $b \leq (r: a)'$; hence $r^{-}: b \leq r^{-}: (r: a)' \leq a'$ by M6, and so $(r^{-}: b) \cdot a = 0$. Similarly for the converse.

By M1, multiplication by any element of $R$ is an additive operation. The next result shows that it is also completely additive.

T6. If $\sum_{i \in I} a_i$ exists, then so does $\sum_{i \in I} r: a_i$, and $r: \sum_{i \in I} a_i = \sum_{i \in I} r: a_i$.

Proof. $a_i \leq \sum_{i \in I} a_i$ for every $i \in I$. Hence, by T1,

$$r: a_i \leq r: \sum_{i \in I} a_i \quad \text{for every } i \in I.$$  \hspace{1cm} (1)

Now let $x \in B$ such that

$$r: a_i \leq x \quad \text{for every } i \in I.$$  \hspace{1cm} (2)

Then $(r: a_i) \cdot x' = 0$; hence $(r^{-}: x') \cdot a_i = 0$ by T5, and hence $a_i \leq (r^{-}: x')'$ for every $i \in I$. Therefore $\sum_{i \in I} a_i \leq (r^{-}: x')'$, so that $(\sum_{i \in I} a_i) \cdot (r^{-}: x') = 0$; hence $(r: \sum_{i \in I} a_i) \cdot x' = 0$ by T5, and so

$$r: \sum_{i \in I} a_i \leq x.$$  \hspace{1cm} (3)

By (1) and the fact that (2) implies (3), T6 is proved.

T7. $r: 0 = 0$.

Proof. $(r^{-}: 1) \cdot 0 = 0$; hence $(r: 0) \cdot 1 = 0$ by T5.
T8. \( 1: 1 = 1 \).

Proof. \( e \leq 1 \); hence \( e: 1 \leq 1: 1 \) and hence \( 1 \leq 1: 1 \).

T9. \( (r: 1)' \leq r': 1 \).

Proof. \( r: 1 + r': 1 = (r + r'): 1 = 1: 1 = 1 \) by T8. Hence \( (r: 1)' \leq r': 1 \).

T10. \( (r: a) \cdot b \leq r: (a \cdot (r^*: b)) \).

Proof. \( r: a = r: (a \cdot 1) = r: (a \cdot ((r^*: b) + (r^*: b)')) \)
\[ = r: (a \cdot (r^*: b)) + r: (a \cdot (r^*: b)') \] by M1
\[ \leq r: (a \cdot (r^*: b)) + (r: a) \cdot (r: (r^*: b)') \] by T3
\[ \leq r: (a \cdot (r^*: b)) + (r: a) \cdot b' \] by M6.

Hence \( (r: a) \cdot b \leq (r: (a \cdot (r^*: b))) \cdot b + 0 \)
\[ \leq r: (a \cdot (r^*: b)) \).

T11. \( a \leq 1: a \).

Proof. \( a - e: a \leq 1: a \) by M4 and T2.

The reader familiar with the arithmetic of relation algebras will have noticed that it is largely paralleled by the arithmetic of Boolean modules. Extrapolating from this observation we introduce the concept of an ideal element in a Boolean module: an element \( a \) such that \( 1: a = a \). These elements behave in much the same way as ideal elements in a relation algebra; we will see that they furnish us with ideals in a Boolean module. By T7 and T8 any nontrivial Boolean module has at least two ideal elements, namely, 0 and 1. An ideal element \( a \neq 1 \) is called a proper ideal element. T11 shows that \( 1: a \leq a \) is a necessary and sufficient condition for an element \( a \) to be an ideal element.

T12. If \( a \) is an ideal element, so is \( a' \).

Proof. By hypothesis \( 1: a = a \); hence \( (1: a) \cdot a' = 0 \), so \( (1^*: a') \cdot a = 0 \) by T5. Hence \( (1: a') \cdot a = 0 \) and so \( 1: a' \leq a' \).

T13. If \( a \) and \( b \) are ideal elements, so is \( a + b \).

Proof. \( 1: (a + b) = 1: a + 1: b = a + b \)

Let \( B^* \) denote the (nonempty) class of all ideal elements in a Boolean \( \mathcal{B} \)-module \( \mathcal{B} \). T12 and T13 establish:

2.7. Theorem. The class \( B^* \) of ideal elements in a Boolean \( \mathcal{B} \)-module \( \mathcal{B} \) forms a subalgebra \( B^* \) of \( \mathcal{B} \) considered as a Boolean algebra.
The class of ideal elements will, in general, not itself be a Boolean module since \( r: a \) is not necessarily an ideal element when \( a \) is an ideal element. Here are a few more results concerning ideal elements.

**T14.** \( 1: a \) is an ideal element.

**Proof.** \( 1: (1: a) = (1; 1): a = 1: a. \)

**T15.** If \( a \) is an ideal element, then \( (r: b) \cdot a = r: (b \cdot a). \)

**Proof.** \( r: (b \cdot a) \leq (r: b) \cdot (r: a) \leq (r: b) \cdot (1: a) = (r: b) \cdot a \) by T3, T2, and the fact that \( a \) is an ideal element. Conversely, \( r': a \leq 1: a = a; \) hence \( b \cdot (r': a) \leq b \cdot a \) and so \( r: (b \cdot (r': a)) \leq r: (b \cdot a). \) But \( (r: b) \cdot a \leq r: (b \cdot (r': a)) \) by T10; hence \( (r: b) \cdot a \leq r: (b \cdot a). \)

**T16.** If \( a \) is an ideal element, then \( r: a = (r: 1) \cdot a. \)

**Proof.** By T15.

By T14, \( 1: a \) is an ideal element for every element \( a \) of a Boolean \( \mathcal{B} \)-module \( \mathcal{B}. \) Multiplication by 1 can be regarded as an operation \( c \) on \( B, \) defined by \( c(a) = 1: a \) for every \( a \in B. \) This operation maps elements onto ideal elements; it has certain familiar properties. T7, T11, M1, and T14 show that

1. \( c(0) = 0, \)
2. \( a \leq c(a), \)
3. \( c(a + b) = c(a) + c(b), \)
4. \( c(c(a)) = c(a). \)

These are precisely the Kuratowski axioms for a closure operator on a topological space, so \( (B + c) \) is a closure algebra, as defined in [9]. Moreover, since \( 1: a \) is an ideal element, so is \( (1: a)', \) hence \( 1: (1: a)' = (1: a)'. \) That is:

5. \( c(c(a)') = c(a)'. \)

Conditions (1)–(5) are the conditions under which Halmos calls a mapping of a Boolean algebra into itself an existential quantifier. So \( (\mathcal{B}, c) \) is a monadic algebra, as defined in [2].

This section is concluded with a result on duality. To each Boolean algebra \( \mathcal{B} = (B +) \) there corresponds a dual algebra \( \mathcal{B}_d = (B \cdot). \) To each relation algebra \( \mathcal{R} \) there corresponds three relation algebras:

\[
\mathcal{R}_1 = (R \cdot + d) \quad \text{where} \quad r \cdot s = (r'; s')' \quad \text{and} \quad d = e', \\
\mathcal{R}_2 = (R + \circ e) \quad \text{where} \quad r \circ s = s; r, \\
\mathcal{R}_3 = (R \cdot + d) \quad \text{where} \quad r \cdot s = s \cdot r.
\]
2.8. **Theorem.** Let $B$ be a Boolean $R$-module under a multiplication :, and let

$$r \cdot a = (r' \cdot a')' \hspace{1cm} r : a = r^c : a \hspace{1cm} r \Box a = r^c \Box a$$

Then $\mathcal{P}_d$ is an $R_1$-module under $\Box$, $\mathcal{P}$ is an $R_2$-module under $:$, and $\mathcal{P}$ is an $R_3$-module under $\Box$.

The proof is straightforward.

3. **Universal-Algebraic Results**

As defined in Section 2 a Boolean $\mathcal{A}$-module $\mathcal{B}$ is not an abstract algebra. This is inconvenient but rectifiable. For, to every element $r \in R$ there corresponds a unary operation $f_r$ on $B$, defined by $f_r(a) = r : a$. This suggests that a Boolean $\mathcal{A}$-module can be considered as a Boolean algebra endowed with a family of unary operations, one for each element of $R$, satisfying appropriately rewritten versions of M1–M6. By M1 each of the operations $f$ is additive; we thus come to regard a Boolean module as a Boolean algebra with operators (see [5] in the same way as an ordinary module can be regarded as a group with operators. By distinguishing the operations $f_r$, the terminology of Definition 2.6 is explained: a Boolean $\mathcal{A}$-module $\mathcal{B}$ is bijective iff there is a bijection between the elements $r \in R$ and the unary operations $f_r$.

The most useful consequence of viewing Boolean modules as algebras is that they are *equationally definable* algebras, with all the advantages pertaining to this notion. Universal-algebraic concepts such as submodule (the subalgebra of a Boolean module), homomorphism, congruence relation, and direct and subdirect product can be defined in the usual way. In each case a little rewriting of the original universal-algebraic definition shows that the new concept is just this same concept as applied to Boolean algebras with an extra condition added to take care of multiplication. Thus, a submodule of a Boolean $\mathcal{A}$-module $\mathcal{B}$ is a subalgebra of $\mathcal{B}$ considered as a Boolean algebra which is closed under multiplication by elements of $\mathcal{A}$; a homomorphism between Boolean modules is a homomorphism between Boolean algebras with the additional property that the image of $r : a$ is $r$ times the image of $a$, and so on. The additional properties are not negligible. For example, $\{0, 1\}$ is always a subalgebra of a nontrivial Boolean algebra, but it is not always a submodule of a nontrivial Boolean $\mathcal{A}$-module. For, if $\{0, 1\}$ is to be a submodule, then it must hold for every $r \in R$ that $r : 1 = 0$ or $r : 1 = 1$. But this is not always the case. Thus, in the $\mathcal{P}(\mathcal{U}^2)$-module $\mathcal{P}(\mathcal{U})$ of Example 2.2 we have $R : U = D(R)$, but the domain $D(R)$ of a relation need not be either empty or the universal set $U$. 


Since, for any given relation algebra \( R \), the class of Boolean \( R \)-modules is an equationally definable class, a theorem due to Birkhoff informs us that submodules, homomorphic images and direct products of Boolean modules are again Boolean modules. Further, standard methods suffice to show that all congruence relations over a Boolean module commute. This means that the structure lattice of a Boolean module is always modular, and in this lattice there is a simple characterization of joins: \( \theta \vee \chi = \theta ; \chi \). It is more instructive, however, to move away from the universal-algebraic context and talk about ideals instead of congruence relations.

3.1. **Definition.** An ideal in a Boolean \( R \)-module \( B \) is a nonempty subset \( J \) of \( B \) such that:

1. \( a + b \in J \) for all \( a, b \in J \),
2. \( a \cdot b \in J \) for all \( a \in J \) and \( b \in B \)
3. \( r \cdot a \in J \) for all \( a \in J \) and \( r \in R \).

So an ideal in a Boolean \( R \)-module \( B \) is just an ideal in \( B \) considered as a Boolean algebra (a Boolean ideal) with the additional property of being closed under multiplication by elements of \( R \). As with Boolean algebras the concept of a filter in a Boolean module can be defined by dualization (a Boolean filter closed under the operation \( r \sqcap a = (r' : a')' \)). Again as with Boolean algebras, if \( J \) is an ideal and \( J_d \) is the set dual to \( J \) (i.e., consisting of all complements of elements of \( J \)), then \( J_d \) is a filter, and conversely. But not all properties of Boolean ideals and filters carry over to ideals and filters in a Boolean module. Here are two examples. First, in a Boolean algebra \( J \cup J_d \) is a subalgebra for any ideal \( J \), but in a Boolean module \( J \cup J_d \) is not always a submodule. For example, \( \{0\} \) is always an ideal in a Boolean module, and \( \{1\} \) is a filter, but, as pointed out above, \( \{0, 1\} \) is not always a submodule. Second, for any element of a Boolean algebra \( A \) there is an ideal, namely,

\[ S(a) = \{b | b \leq a\}, \]

the principal ideal generated by \( a \). This does not hold for Boolean modules, for which we have:

3.2. **Theorem.** \( S(a) \) is an ideal in a Boolean \( R \)-module \( B \) iff \( a \in B \) is an ideal element.

**Proof.** If \( S(a) \) is an ideal, then \( 1 : a \in S(a) \) since \( a \in S(a) \); hence \( 1 : a \leq a \). Conversely, \( S(a) \) is known to be a Boolean ideal. Let \( b \in S(a) \) and \( r \in R \) arbitrarily, then \( r : b \leq r : a \) since \( b \leq a \) (T2.1). But \( r : a \leq 1 : a = a \) (T2.2), and so \( r : b \in S(a) \).
A Boolean ideal in a Boolean module is, therefore, not necessarily a module ideal. It turns out, however, that there is a correlation between Boolean ideals and module ideals in a Boolean \( \mathcal{A} \)-module \( \mathcal{B} \) if we take as the Boolean algebra in question not \( \mathcal{B} \) itself, but \( \mathcal{B}^* \), the Boolean algebra consisting of all ideal elements in \( \mathcal{B} \). The development is as follows. The class \( I(\mathcal{B}) \) of all ideals in a Boolean \( \mathcal{A} \)-module \( \mathcal{B} \) is partially ordered by set inclusion. Every subset of \( I(\mathcal{B}) \) has an infimum, namely, the intersection of all the ideals in this set. It follows that \( I(\mathcal{B}) \), together with the partial order \( \subseteq \), is a complete lattice \( \mathcal{J}(\mathcal{B}) \). Similarly, the class of ideals in the Boolean algebra \( \mathcal{B}^* \) form a complete lattice \( \mathcal{J}(\mathcal{B}^*) \).

3.3. **Lemmag.** Let \( \mathcal{B} \) be a Boolean \( \mathcal{A} \)-module.

1. If \( J \) is an ideal in \( \mathcal{B} \), then \( J \cap \mathcal{B}^* \) is an ideal in the Boolean algebra \( \mathcal{B}^* \).

2. If \( L \) is an ideal in \( \mathcal{B}^* \), then \( L^+ = \{ a \in \mathcal{B} | \exists c \in L | a \leq c \} \) is an ideal in the Boolean module \( \mathcal{B} \).

3. For any ideal \( J \) in \( \mathcal{B} \), \( (J \cap \mathcal{B}^*)^+ = J \).

4. For any ideal \( L \) in \( \mathcal{B}^* \), \( L^+ \cap \mathcal{B}^* = L \).

5. For any ideals \( J \) and \( K \) in \( \mathcal{B} \), \( J \subseteq K \) iff \( J \cap \mathcal{B}^* \subseteq K \cap \mathcal{B}^* \).

**Proof.**

1. Easy.

2. Let \( a, b \in L^+ \), say \( a \leq c \) and \( b \leq d \). Then \( a + b \leq c + d \in L \); hence \( a + b \in L^+ \). Second, let \( a \in L^+ \), say \( a \leq c \in L \), and let \( b \in B \). Then \( a \cdot b \leq c \cdot b \leq c \in L \); hence \( a \cdot b \in L^+ \). Third, let \( a \in L^+ \), say \( a \leq c \in L \), and let \( r \in R \). Then \( r : a \leq r : c = (r : 1) \cdot c \leq c \in L \) by T2.16; hence \( r : a \in L^+ \).

3. If \( a \in (J \cap \mathcal{B}^*)^+ \), then \( a \leq c \) for some \( c \in J \cap \mathcal{B}^* \subseteq J \); hence \( a \in J \). Conversely, if \( a \in J \), then \( a \leq 1 : a \in J \cap \mathcal{B}^* \) and so \( a \in (J \cap \mathcal{B}^*)^+ \).

4. If \( a \in L^+ \cap \mathcal{B}^* \), then \( a \in \mathcal{B}^* \) and \( a \leq c \) for some \( c \in L \); hence \( a \in L \). Conversely, if \( a \in L \subseteq \mathcal{B}^* \), then \( a \in L^+ \) since \( a \leq a \); hence \( a \in L^+ \cap \mathcal{B}^* \).

5. Clearly \( J \subseteq K \) implies that \( J \cap \mathcal{B}^* \subseteq K \cap \mathcal{B}^* \). Now suppose that \( J \cap \mathcal{B}^* \subseteq K \cap \mathcal{B}^* \) and let \( a \in J \). Then \( 1 : a \in J \) and \( 1 : a \in \mathcal{B}^* \), hence \( 1 : a \in J \cap \mathcal{B}^* \subseteq K \cap \mathcal{B}^* \). Hence, since \( a \leq 1 : a \), we get \( a \in K \). □

3.4. **Theorem.** The lattice of all ideals in a Boolean \( \mathcal{A} \)-module \( \mathcal{B} \) is isomorphic to the lattice of all ideals in the associated Boolean algebra \( \mathcal{B}^* \).

**Proof.** Define a mapping \( f: I(\mathcal{B}) \to I(\mathcal{B}^*) \) by

\[
 f(J) = J \cap \mathcal{B}^*.
\]

Then \( f(J) \in I(\mathcal{B}^*) \) by (1) of Lemma 3.3. From (2) and (4) it follows that,
given any element $L$ of $I(\mathcal{B})$, there is an element $L^+$ of $I(\mathcal{B})$ such that $f(L^+) = L$; hence the mapping $f$ is onto. Also, if $f(J) = f(K)$, then by (3)

$$J = (J \cap B^+)^+ = f(J)^+ = f(K)^+ = (K \cap B^+)^+ = K$$

and hence $f$ is one-one. By (5), $f$ is isotone, and the inverse of $f$ is also isotone. (For, if $L \subseteq M$ in $I(\mathcal{B})$, then $L^+ \cap B^+ = L \subseteq M = M^+ \cap B^+$ by (4), hence $L^+ \subseteq M^+$ by (5). But $L^+ = f^{-1}(L)$ and $M^+ = f^{-1}(M)$.) Thus, $f$ is an isotone bijection between lattices, with an isotone inverse. Any such mapping is known to be a lattice isomorphism; hence $f$ is an isomorphism between $\mathcal{I}(\mathcal{B})$ and $\mathcal{I}(\mathcal{B})$. 

We still have to make sure that the ideals we are talking about are linked to congruence relations in the usual way. Here is a sketch of the development. Given an ideal $J$ in a Boolean $\mathcal{R}$-module $\mathcal{B}$, define a relation $\mu(J)$ by

$$a \equiv b(\mu[J]) \quad \text{iff} \quad a + d = b + d \quad \text{for some} \quad d \in J.$$

Also, given a congruence relation $\theta$ over $\mathcal{B}$, define the set $M[\theta]$ by

$$M[\theta] = \{a | a \equiv 0(\theta)\}.$$

Then $\mu(J)$ is a congruence relation for any ideal $J$, and $M[\theta]$ is an ideal for any congruence relation $\theta$. Moreover, $M[\mu(J)] = J$, $\mu[M[\theta]] = \theta$ and for any ideals $J$ and $K$ of $\mathcal{B}$, $J \subseteq K$ iff $\mu[J] \subseteq \mu[K]$. These facts ensure that the mapping $f$ defined by

$$f(\theta) = M[\theta]$$

is a lattice isomorphism between the structure lattice of $\mathcal{B}$ and the lattice of all its ideals. (The proof is similar to that of Lemma 3.3: $f$ is an isotone bijection between lattices, with an isotone inverse.)

The isomorphism between lattices and ideals means that we can now effect a translation between ideals and congruence relations. For example, from our knowledge of the structure lattice of a Boolean $\mathcal{R}$-module $\mathcal{B}$ we can now conclude that the lattice of ideals in $\mathcal{B}$ is modular, and joins are given by

$$J \lor K = \{c + d | c \in J \text{ and } d \in K\}.$$  

Something else we can do is to take the universal-algebraic versions of the Homomorphism Theorem and the Isomorphism Theorems, stated in terms of congruence relations over an arbitrary algebra, and translate these into the language of ideals of Boolean modules. The development is unproblematic. 

A few words on quotient modules are required here. For any ideal $J$ in a
Boolean $R$-module $B$ the quotient module $B/J$ exists, its elements are equivalence classes $a + J$, and its operations are given by

$$(a + J) + (b + J) = (a + b) + J, \quad (a + J)' = a' + J, \quad r: (a + J) = r: a + J.$$ 

The zero element of $B/J$ is $0 + J$, which is just $J$. The universal element is

$$1 + J = \{ b | b \equiv 1(J) \} = \{ b | (\exists d \in J)[b + d = 1] \}.$$ 

We know that two equivalence classes $a + J$ and $b + J$ coincide iff $a \equiv b(J)$. From this we derive two consequences:

$$a + J = J \quad \text{iff} \quad a \in J,$$

$$a + J = 1 + J \quad \text{iff} \quad a' \in J.$$ 

These two facts will be used in Theorem 4.3.

4. Simplecty, Direct and Subdirect Decompositions

A nontrivial abstract algebra $A$ is simple iff the identity relation $I_A$ over $A$ and the complete relation $W_A = A^2$ are the only congruence relations over $A$. For Boolean algebras the concept of simplicity is disappointing: $\{0, 1\}$ is the only simple Boolean algebra. For Boolean modules simplicity is more interesting. We have:

4.1. Theorem. The following conditions on a Boolean $R$-module $B$ are equivalent:

(1) $B$ is simple.

(2) $B$ has no nontrivial proper ideals.

(3) $B^1$ is the two-element Boolean algebra.

(4) $a \neq 0 \Rightarrow 1:a = 1$.

Proof. The equivalence of (1) and (2) follows from the correspondence between congruence relations and ideals in $B$. Condition (2) implies (3) since for any ideal element $a$, $S(a)$ is an ideal; hence $S(a) = \{0\}$ or $S(a) = B$, which means $a = 0$ or $a = 1$. Condition (3) implies (4) since $1:a \in B^1 = \{0, 1\}$ and $1:a \neq 0$ (by T2.11). And (4) implies (2) since, for any nonzero element of a nontrivial ideal $J$, $1:a = 1 \in J$. □
It follows that, unlike Boolean algebras, a simple Boolean module may have more than two elements. For example, let $U$ be any set with more than one element, and consider the proper Boolean $\mathcal{P}(U^2)$-module $\mathcal{P}(U)$, which has more than two elements. The universal element of this module is $U$ itself, and the universal element of $\mathcal{P}(U^2)$ is the complete relation $U^2$. Now let $A$ be any nonempty element of $\mathcal{P}(U)$, then a little calculation shows that $V: A = U$; hence, by condition (4), $\mathcal{P}(U)$ is simple.

In connection with simplicity it is interesting to look at maximal ideals. For Boolean algebras, we know that an ideal is maximal iff the corresponding quotient algebra is the two-element Boolean algebra. An analogous result for Boolean modules is established by means of:

4.2. **Lemma.** An ideal $J$ in a Boolean module is maximal iff, for each ideal element $a$ in $B$, either $a \in J$ or $a' \in J$, but not both.

The proof is a straightforward extension of the same result for Boolean algebras (see, e.g., [3]). We now get:

4.3. **Theorem.** An ideal $J$ of a Boolean $\mathcal{B}$-module $\mathcal{B}$ is maximal iff the quotient module $\mathcal{B}/J$ is simple.

**Proof.** Let $J$ be a maximal ideal in $\mathcal{B}$ and let $a + J$ be an ideal element in $\mathcal{B}/J$, so that $a + J = 1: (a + J) = 1: a + J$. But $1: a$ is an ideal element in $\mathcal{B}$; hence $1: a \in J$ or $(1:a)' \in J$, by Lemma 4.2. In the first case $a + J = 1: a + J = J$; in the second case $a + J = 1: a + J = 1 + J$. $\mathcal{B}/J$ has only the zero element $J$ and the universal element $1 + J$ as ideals; hence it is simple. Conversely, let $\mathcal{B}/J$ be simple and let $a$ be any ideal element in $\mathcal{B}$. Suppose $a \in J$, then $1: (a + J) = 1: a + J = a + J = J$; hence $1: (a + J) = 1 + J$ since $1: (a + J)$ is an ideal element in $\mathcal{B}/J$, which is simple. Thus, $a + J = 1 + J$ and so $a' \in J$. Hence either $a \in J$ or $a' \in J$ for any ideal element $a$. Also, not both $a$ and $a'$ can be in $J$ since then $a + a' = 1 \in J$; hence $J = B$ and so $\mathcal{B}/J$ is trivial, contradicting the fact that $\mathcal{B}/J$ is simple. So, by Lemma 4.2, $J$ is a maximal ideal.

We can also connect the concept of simplicity of Boolean modules with that of subdirect irreducibility. For this we need some terminology. Let $\{\mathcal{B}_i\}_{i \in I}$ be a family of Boolean $\mathcal{B}$-modules, and let $\mathcal{B}$ be their direct product. For every $i \in I$ the projection mapping $e_i$, which maps any element of $\mathcal{B}$ onto its $i$th component, is an epimorphism. If $\mathcal{B}$ is isomorphic to some other $\mathcal{B}$-module $\mathcal{A}$, then $\mathcal{B}$ is a direct decomposition of $\mathcal{A}$. A subdirect product of the $\mathcal{B}_i$'s is a submodule of $\mathcal{B}$, say $\mathcal{C}$, such that the image of $\mathcal{C}$ under the projection mapping $e_i$ is $\mathcal{B}_i$ for every $i \in I$. If $\mathcal{C}$ is isomorphic to some other $\mathcal{B}$-module $\mathcal{D}$, then $\mathcal{C}$ is a subdirect decomposition of $\mathcal{D}$. If $h$ is the isomorphism between $\mathcal{D}$ and $\mathcal{C}$, then each of the mappings $g_i$ defined by
Booleans

$g_i(a) = e_i(h(a))$ is an epimorphism from $D$ to $D_i$, called the natural epimorphism. A subdirect decomposition in which at least one of the natural epimorphisms is an isomorphism is trivial; if none of the natural epimorphisms is an isomorphism, the subdirect decomposition is nontrivial. A Boolean module is said to be subdirectly irreducible if it is a nontrivial module which has no nontrivial subdirect decompositions.

To establish now a connection between simplicity and subdirect irreducibility, let $R$ be any Boolean $R$-module and consider the algebra

$$R_A = (S(a) + s),$$

where $x^* = x' \cdot a$ and $S(a) = \{ b \mid b \leq a \}$.

4.4. Lemma. If $a$ is a nonzero ideal element of an $R$-module $R$, then $R_A$ is an $R$-module and a homomorphic image of $R$.

Proof. Verification of the axioms adopted for a Boolean algebra shows that $S(a)$ is a Boolean algebra. $S(a)$ is closed under multiplication by elements of $R$ since $r \cdot x \leq r \cdot a \leq 1 : a = a$ for any $x \in S(a)$ and $r \in R$, $a$ being an ideal element. M1–M5 hold in $S(a)$ as in $B$. For M6 we have

$$r^* : (r : x)^* = r^* : ((r : x') : a)$$
$$\leq (r^* : (r : x') : (r^* : a)) \quad \text{by T2.3}$$
$$\leq r' \cdot (r^* : a) \quad \text{by M6}$$
$$\leq r' \cdot (1 : a) = r' \cdot a = x^*.$$

Thus $R_A$ is an $R$-module. To show that it is a homomorphic image of $R$ define a mapping $f : B \rightarrow S(a)$ by $f(x) = x \cdot a$ for every $x \in B$. Then $f$ is onto, and it is a homomorphism since

$$f(x + y) = (x + y) \cdot a = x \cdot a + y \cdot a = f(x) + f(y),$$
$$f(x') = x' \cdot a = x^*,$$
$$f(r : x) = (r : x) \cdot a = r : (x \cdot a) = r : f(x) \quad \text{by T2.15}.$$

Hence $f$ is an epimorphism, and so $R_A$ is a homomorphic image of $R$.

4.5. Lemma. If $a$ is a nonzero ideal element of an $R$-module $R$, then $R$ is isomorphic to $R_A \times R_A$.

Proof. The elements of $R_A \times R_A$ are pairs $(u, v)$ such that $u \leq a$ and $v \leq a'$. We map any element $x$ of $B$ onto the pair $(x \cdot a, x \cdot a')$; this mapping proves to be an isomorphism. Given any pair $(u, v)$, let $x = u + v$. Then $f(x) = (u, v)$ since $u \leq a$ and $v \leq a'$; this shows that $f$ is onto. It is one–one.
since if \( f(x) = f(y) \), then \( x \cdot a = y \cdot a \) and \( x \cdot a' = y \cdot a' \); hence \( x \cdot a + x \cdot a' = y \cdot a + y \cdot a' \) and so \( x = y \). And it is a homomorphism since \( f \) is easily seen to preserve addition and complementation, while for multiplication we have
\[
f(r \cdot x) = (r \cdot x) \cdot a, ((r \cdot x) \cdot a')
\]
\[
= (r \cdot (x \cdot a), r \cdot (x \cdot a')) \quad \text{by T2.15}
\]
\[
= r \cdot (x \cdot a, x \cdot a')
\]
\[
= r \cdot f(x).
\]
The desired connection between simplicity and subdirect irreducibility now follows:

4.6. **Theorem.** A Boolean \( \mathcal{B} \)-module \( \mathcal{B} \) is subdirectly irreducible iff it is simple.

**Proof.** We know that any simple algebra is subdirectly irreducible, so the "if" part holds. For the "only if" part, let \( \mathcal{B} \) be subdirectly irreducible. Then \( \mathcal{B} \) is nontrivial and so it has at least two ideal elements, namely, 0 and 1. Suppose now that \( \mathcal{B} \) is not simple, then by (3) of Theorem 4.1 there is a nonzero proper ideal element \( a \) in \( \mathcal{B} \); hence \( \mathcal{B} \) is isomorphic to \( \mathcal{B}_a \times \mathcal{B}_{a'} \), with the isomorphism given by \( f(x) = (x \cdot a, x \cdot a') \). \( \mathcal{B}_a \times \mathcal{B}_{a'} \), being a direct product, is itself a subdirect product; hence \( B \) has a subdirect decomposition. The natural epimorphisms here are \( f_1(x) = x \cdot a \) and \( f_2(x) = x \cdot a' \). Since \( a \) and \( a' \) are proper, neither of these epimorphisms is an isomorphism (e.g., \( f_1(a) = a = f_1(1) \), but \( a \neq 1 \)); hence the subdirect decomposition is nontrivial—contradicting the fact that \( \mathcal{B} \) is subdirectly irreducible. So \( \mathcal{B} \) must be simple.

The correspondence between simplicity of Boolean algebras and of Boolean modules can now be set out as follows:

<table>
<thead>
<tr>
<th>A Boolean algebra ( \mathcal{B} ) is simple</th>
<th>A Boolean ( \mathcal{B} )-module ( \mathcal{B} ) is simple</th>
</tr>
</thead>
<tbody>
<tr>
<td>iff ( \mathcal{B} ) is subdirectly irreducible</td>
<td>iff ( \mathcal{B} ) is subdirectly irreducible</td>
</tr>
<tr>
<td>iff ( \mathcal{B} ) is the two-element Boolean algebra</td>
<td>iff ( \mathcal{B}^* ) is the two-element Boolean algebra</td>
</tr>
<tr>
<td>iff ( a \neq 0 \Rightarrow a = 1 ) for every ( a \in B ).</td>
<td>iff ( a \neq 0 \Rightarrow 1: a = 1 ) for every ( a \in B ).</td>
</tr>
</tbody>
</table>

By a theorem due to Birkhoff we know that any abstract algebra has a decomposition as a subdirect product of subdirectly irreducible algebras. Further, an abstract algebra is said to be **semisimple** iff it has a decomposition as a subdirect product of simple algebras. By combining Birkhoff's theorem with Theorem 4.6 we get the gratifying result:
4.7. **Theorem.** Any Boolean module is semisimple.

So much for subdirect decompositions—what about direct decompositions? Necessary and sufficient conditions are given by:

4.8. **Theorem.** A Boolean $\mathcal{R}$-module $\mathcal{B}$ is isomorphic to a direct product of a family $\{\mathcal{B}_i\}_{i \in I}$ of $\mathcal{R}$-modules iff there is a family $\{a_i\}_{i \in I}$ of elements of $A$ such that:

1. $a_i$ is an ideal element for every $i \in I$.
2. $a_i \cdot a_j = 0$ whenever $i \neq j$.
3. $\sum_{i \in I} a_i = 1$.
4. For any family $\{x_i\}_{i \in I}$ of elements of $A$, $\sum_{i \in I} x_i \cdot a_i \in A$.
5. $\mathcal{A}_a$ is isomorphic to $\mathcal{B}_i$ for every $i \in I$.

The proof (too lengthy to give here) is an application to Boolean modules of the proof of a similar result for relation algebras [5]. This theorem is much stronger than Lemma 4.5, which is in fact a consequence of it. By means of Theorem 4.8 we obtain sufficient conditions for an $\mathcal{R}$-module $\mathcal{B}$ to have a decomposition as a direct product of simple modules. We first note that if $a$ is an atom in $\mathcal{B}$, then $sa$ is simple. Indeed, if $g_0$ is not simple, then there is a nonzero proper ideal element $b$ in $g_0$ (Theorem 4.1), which is also an ideal element in $L^0$ since the condition $1: b = b$ holds in $\mathcal{B}$ as in $g_0$. But $b \in S(a)$ means $b < a$, and this contradicts the fact that $a$ is an atom in $\mathcal{B}$, so $L^0$ must be simple.

4.9. **Theorem.** Given an $\mathcal{R}$-module $\mathcal{B}$, let $At$ be the set of all atoms in $\mathcal{B}$. If $\mathcal{B}$ is atomic, and $\sum_{a \in At} (x_a \cdot b) \in B$ for any family $\{x_a\}_{a \in At}$ of elements of $B$ indexed by $At$, then $\mathcal{B}$ has a decomposition as a direct product of simple $\mathcal{R}$-modules.

**Proof.** Application of Theorem 4.8 shows that $\mathcal{B}$ is isomorphic to the direct product of the $\mathcal{B}_a$'s, $a \in At$. Since every element of $At$ is an ideal element, (1) holds; since the elements of $At$ are atoms in $\mathcal{B}$, (2) holds; since $\mathcal{B}$ is atomic, (3) holds; (4) holds by hypothesis, and (5) holds trivially. Further, each of the $\mathcal{B}_a$'s is simple.

5. **A Representation Result**

A Boolean module is a Boolean algebra which exhibits some of the features of a relation algebra. All Boolean algebras are representable, but not all relation algebras are representable. Are all Boolean modules represen-
table? This question requires a definition of representability of Boolean modules.

For a Boolean $\mathcal{B}$-module $\mathcal{R}$ to be a proper $\mathcal{B}$-module, $\mathcal{R}$ must be a proper relation algebra. So the representability of an $\mathcal{B}$-module $\mathcal{R}$ does not consist simply in being isomorphic to a proper $\mathcal{B}$-module: if $\mathcal{R}$ is not proper, there simply will not be any proper $\mathcal{B}$-modules for $\mathcal{R}$ to be isomorphic to. This is no great problem if $\mathcal{R}$ is representable since given a proper relation algebra $\mathcal{I}$ isomorphic to $\mathcal{R}$, we can transform $\mathcal{R}$ into an $\mathcal{I}$-module by letting the elements of $R$ go proxy for the elements of $S$. (If $g: \mathcal{B} \rightarrow \mathcal{I}$ is the isomorphism, define the mapping $f: S \times B \rightarrow B$ by $f(s, a) = g^{-1}(s): a$.) Once this is done one can look for a field of sets $\mathcal{A}$ which is a proper $\mathcal{I}$-module and is isomorphic to $\mathcal{B}$ regarded as an $\mathcal{I}$-module. $\mathcal{A}$ would then be the representation of $\mathcal{B}$. However, $\mathcal{B}$ may not be representable; therefore such an $\mathcal{I}$ may not be forthcoming. There are now two strategies available: one can either relax the requirements on the isomorphic copy $\mathcal{I}$ (e.g., just require $S$ to be a set of relations) or one can strengthen the requirements on $\mathcal{R}$ (e.g., require $\mathcal{R}$ to be representable). Of these two strategies I adopt the first. The strongest requirement one can impose on $\mathcal{I}$ which will simultaneously ensure its existence and its consisting of relations is that $\mathcal{I}$ must be a weak representation of $\mathcal{R}$. I therefore define:

5.1. Definition. A Boolean $\mathcal{B}$-module $\mathcal{B}$ is representable iff there is a relation algebra $\mathcal{I}$ and a field of sets $\mathcal{A}$ such that $\mathcal{I}$ is a weak representation of $\mathcal{B}$, $\mathcal{A}$ is an $\mathcal{I}$-module under the Peircean product, and $\mathcal{A}$ is module-isomorphic to $\mathcal{B}$ when the latter is regarded as an $\mathcal{I}$-module.

I now proceed to prove that any bijective Boolean module is representable.

The first step is to narrow the representation problem down to Boolean modules which are complete and atomic. In [5] the concepts of completeness and atomicity are defined for Boolean algebras with operators as follows: a Boolean algebra with operators is atomic iff it is atomic as a Boolean algebra, and it is complete iff it is complete as a Boolean algebra and each of the additional operations is completely additive. By T2.6 each of the operations $f_r$ on a Boolean module is completely additive; it follows that a Boolean module is complete and atomic iff it is complete and atomic as a Boolean algebra. In [5] it is shown that any Boolean algebra with operators $\mathcal{B}$ can be extended to a complete and atomic Boolean algebra with operators $\mathcal{A}$ in such a way that $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and $\mathcal{A}$ belongs to any equationally definable class of algebras to which $\mathcal{B}$ belongs. Since Boolean modules are equationally definable, it follows that any Boolean module can be extended to a Boolean module which is complete and atomic.

It must now be shown that any bijective Boolean module which is
complete and atomic is representable. The method followed here originated with a consideration of some technicalities in [5], where it is shown that every (normal) Boolean algebra with operators is isomorphic to a subalgebra of the complex algebra of some relational system. First a digression concerning the Peircean product. For any nonempty set $U$, $\mathcal{P}(U)$ is a Boolean $\mathcal{P}(U)$-module, and to every relation $R$ over $U$ there corresponds a mapping $f_R: \mathcal{P}(U) \rightarrow \mathcal{P}(U)$, namely, $f_R(X) = R: X$, where the last colon indicates the Peircean product. By T2.6, $f_R$ is completely additive. This procedure of obtaining a completely additive mapping of $\mathcal{P}(U)$ into itself from a relation over $U$ is reversible, in the following sense. For any mapping $F: \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ define a relation $\rho[F]$ over $U$ by

$$\rho[F] = \{(x, y) | x \in F(\{y\})\}.$$ 

5.2. THEOREM. For any relation $R$ over $U$, and any completely additive mapping $F: \mathcal{P}(U) \rightarrow \mathcal{P}(U)$,

(1) $\rho[f_R] = R$,
(2) $\rho[f_{\rho[F]}] = F$.

Proof. (1) $(x, y) \in \rho[f_R]$ iff $x \in f_R(\{y\})$, iff $\exists z(x, z) \in R$ and $z \in \{y\}$, iff $\exists z(x, z) \in R$ and $z = y$, iff $(x, y) \in R$.

(2) Consider any atom in $\mathcal{P}(U)$, say $\{b\}$. Then $x \in f_{\rho[F]}(\{b\})$ iff $x \in \rho[F]: \{b\}$, iff $\exists y(x, y) \in \rho[F]$ and $y \in \{b\}$, iff $\exists y(x, y) \in F(\{y\})$ and $y = b$, iff $x \in F(\{b\})$. Hence $f_{\rho[F]}(\{b\})$ coincides with $F(\{b\})$ for every atom $\{b\}$. But any element in $\mathcal{P}(U)$ is a union of such atoms, and both $f_{\rho[F]}$ and $F$ are completely additive. Hence $f_{\rho[F]}(X)$ and $F(X)$ coincide for every $X$ in $\mathcal{P}(U)$, and so $f_{\rho[F]} = F$. (End of digression.)

Now let $\mathcal{A}$ be a bijective Boolean $\mathcal{P}$-module which is complete and atomic. Since $\mathcal{A}$ is then complete and atomic as a Boolean algebra, it is isomorphic to a power set Boolean algebra $\mathcal{P}(U)$ under an isomorphism $h: \mathcal{A} \rightarrow \mathcal{P}(U)$. Each operation $f_r$ on $A$ induces an operation $F_r$ on $\mathcal{P}(U)$, namely,

$$F_r(X) = h(f_r(h^{-1}(X))) \quad \text{for every } X \in \mathcal{P}(U).$$

Since each $f_r$ is isotone and completely additive, so is each $F_r$. Since each $F_r$ is a mapping of $\mathcal{P}(U)$ into itself, the relation $\rho[F_r]$ corresponds to it.

5.3. LEMMA. (1) $\rho[F_0]$ is the empty relation $\Lambda$.
(2) The field of $\rho[F_1]$ is $U$.
(3) $\rho[F_{r+1}] = \rho[F_r] \cup \rho[F_r]$.
(4) $\rho[F_e]$ is the identity relation over $U$. 
(5) \( \rho[F_r] = (\rho[F_r])^* \).

(6) \( \rho[F_{r+s}] = \rho[F_r]; \rho[F_s] \).

Proof: (1) \( F_0(X) = h(0) \) by definition and by M5. But \( h(0) = A \) since \( h \) is an isomorphism.

(2) \( \rho[F_1] \) is a relation over \( U \), hence its field is contained in \( U \). Now let \( x \in U \), then \( h^{-1}(\{x\}) \subseteq A \). But \( h^{-1}(\{x\}) \leq f_1(h^{-1}(\{x\})) \) by T2.11; hence \( \{x\} \subseteq h(f_1(h^{-1}(\{x\}))) = F_1(\{x\}) \); hence \( x \in F_1(\{x\}) \) and so \( (x, x) \in \rho[F_1] \).

Thus \( x \) is an element of the field of \( \rho[F_1] \).

(3) follows from the definition of \( \rho[F_{r+s}] \), M2, and the fact that \( h \) is a homomorphism.

(4) By M4 and by definition, \( F_e(X) = X \). Now let \( (x, x) \in I_u \), the identity relation over \( U \), then \( x \in \{x\} = F_e(\{x\}) \), and so \( (x, x) \in \rho[F_e] \). Conversely, let \( (x, y) \in \rho[F_e] \), then \( x \in F_e(\{y\}) = \{y\} \); hence \( x = y \) and so \( (x, y) \in I_u \).

(5) follows from T2.5 and the fact that \( h \) and \( h^{-1} \) are homomorphisms.

(6) It is easy to show that \( (x, y) \in \rho[F_{r+s}] \) iff \( x \in F_r(F_s(\{y\})) \). Also, \( (x, y) \in \rho[F_r]; \rho[F_s] \) iff \( x \in F_r(\{z\}) \) and \( z \in F_s(\{y\}) \) for some \( z \in U \). Now, if this last condition is satisfied, then \( \{y\} \subseteq F_r(\{y\}) \); hence \( F_r(\{z\}) \subseteq F_r(F_s(\{y\})) \) since \( F_r \) is isotone, and so \( x \in F_r(F_s(\{y\})) \). Conversely, if \( x \in F_r(F_s(\{y\})) \), let \( Z = F_s(\{y\}) \); then

\[
F_r(Z) = F_r \left( \bigcup_{z \in Z} \{z\} \right) = \bigcup_{z \in Z} F_r(\{z\})
\]

since \( F_r \) is completely additive. But \( x \in F_r(Z) \); hence there is at least one \( z \in Z \subseteq U \) such that \( x \in F_r(\{z\}) \). So \( x \in F_r(\{z\}) \) and \( z \in F_s(\{y\}) \).

Consider now the class \( S = \{\rho[F_r] | r \in R \} \) of relations over \( U \), and define a unary operation \(*\) on \( S \) by

\[
\rho[F_r]^* = \rho[F_r].
\]

5.4. Lemma. \( \mathcal{S} = (S \cup *; \sim I_u) \) is a relation algebra and a weak representation of \( \mathcal{R} \).

Proof: Since \( \mathcal{R} \) is closed under +, \( \mathcal{S} \) is closed under unions by Theorem 5.3(3). Since \( \mathcal{R} \) is closed under ', \( \mathcal{S} \) is closed under *. Verification of the axioms shows that \( \mathcal{S} \) is a Boolean algebra, with zero element \( A \) (Theorem 5.3(1)) and universal element \( \rho[F_1] \). By (4)-(6) of Theorem 5.3, \( \mathcal{S} \) is closed under set-theoretical conversion, relative product, and the identity relation. So the axioms A1–A7 are immediately satisfied: they hold.
as in a proper relation algebra. And a straightforward computation verifies
A8; hence \( \mathcal{S} \) is a relation algebra. To show now that \( \mathcal{S} \) is a weak repre-
sentation of \( \mathcal{R} \), define the mapping \( g: R \to S \) by

\[
g(r) = \rho[F_r] \quad \text{for every } r \in R.
\]

By definition, \( g \) is a surjection; to show that it is also an injection let
\( g(r) = g(s) \). Then \( \rho[F_r] = \rho[F_s] \); hence \( f_{\rho[F_r]} = f_{\rho[F_s]} \) and so \( F_r = F_s \) by
Theorem 5.2(2).

Hence \( F_r(X) = F_s(X) \) for every \( X \in \mathcal{P}(U) \).
Hence \( h(f_r(h^{-1}(X))) = h(f_s(h^{-1}(X))) \) for every \( X \in \mathcal{P}(U) \).
Hence \( f_r(h^{-1}(X)) = f_s(h^{-1}(X)) \) for every \( X \in \mathcal{P}(U) \).
Hence \( f_r(x) = f_s(x) \) for every \( x \in A \).

And so \( r = s \) since \( \mathcal{R} \) is bijective.

Thus, \( g \) is a bijection, and Theorem 5.3 shows that it is a homomorphism, so
\( \mathcal{R} \) is isomorphic to \( \mathcal{S} \) under \( g \). The universal element of \( S \) is \( \rho[F_1] \), and its
field is \( U \). The proof of Theorem 5.3(2) shows that \( \rho[F_1] \) is reflexive; it is
symmetric since \( \rho[F_1] = \rho[F_1] = \rho[F_1] \); and it is transitive since \( \rho[F_1] \); \( \rho[F_1] = \rho[F_{11}] = \rho[F_1] \). Thus, \( \rho[F_1] \) is an equivalence relation. So, in the
relation algebra \( \mathcal{S} \) the base set \( S \) is a class of binary relations contained in
the equivalence relation \( \rho[F_1] \). \( I_U \) is the identity relation over the field of
\( \rho[F_1] \), and all the operations of \( \mathcal{S} \) except complementation have their usual
set-theoretic meanings. Hence \( \mathcal{S} \) is a weak representation of \( \mathcal{R} \).  

5.5. LEMMA. \( \mathcal{P}(\mathcal{U}) \) is a Boolean \( \mathcal{S} \)-module under the Peircean product,
and it is module-isomorphic to \( \mathcal{S} \) regarded as an \( \mathcal{S} \)-module.

Proof. The Peircean product \( \rho[F_r]: X \) is defined for any element of \( S \) and
any element of \( \mathcal{P}(U) \) since the former are relations and the latter sets. Also,
\( \rho[F_r]: X \) is an element of \( \mathcal{P}(U) \) since \( \rho[F_r]: X \) is contained in the domain of
\( \rho[F_r] \), which is contained in the domain of \( \rho[F_1] \), which is \( U \). Since all the
operations of \( \mathcal{S} \) and \( \mathcal{P}(\mathcal{U}) \) which appear in the axioms for a Boolean
module are the usual set-theoretic ones, verification of the axioms is
straightforward. As has been pointed out at the beginning of this section, \( \mathcal{S} \)
can be regarded as an \( \mathcal{S} \)-module by defining the product.

\[
\rho[F_r]: a = g^{-1}(\rho[F_r]): a = r: a \quad \text{for every } r \in R \text{ and } a \in A,
\]

where \( g: R \to \mathcal{S} \) is the isomorphism of Lemma 5.4. \( \mathcal{S} \) is already
isomorphic to \( \mathcal{P}(\mathcal{U}) \) as a Boolean algebra, under the isomorphism
\( h: \mathcal{S} \to \mathcal{P}(\mathcal{U}) \). But \( h \) is in fact also a module-isomorphism since
h(ρ[F_r]; a) = h(r; a) = h(f_r(a))

= h(f_r(h^{-1}(h(a))))

= F_r(h(a))

= f_{ρ[F_r]}(h(a)) \quad \text{by Theorem 5.2(2)}

= ρ[F_r]; h(a). \quad \blacksquare

5.6. THEOREM. Any bijective Boolean module is representable.

Proof. Let $\mathcal{B}$ be a Boolean $\mathcal{B}$-module and let $\mathcal{A}$ be a complete and atomic extension of $\mathcal{B}$. Since $\mathcal{B}$ is bijective, so is $\mathcal{A}$. (For, if $r; a = s; a$ for every $a \in A$, then $r; a = s; a$ for every $a \in B$; hence $r = s$ since $\mathcal{B}$ is bijective.) Lemmas 5.3–5.5 show that there is a relation algebra $\mathcal{I}$ and a power set Boolean algebra $\mathcal{P}(\mathcal{U})$ such that $\mathcal{I}$ is a weak representation of $\mathcal{B}$, $\mathcal{P}(\mathcal{U})$ is a Boolean $\mathcal{I}$-module under the Peircean product, and $\mathcal{P}(\mathcal{U})$ is module-isomorphic to $\mathcal{I}$ when the latter is regarded as an $\mathcal{I}$-module. If $h: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{U})$ is the isomorphism, then $h[\mathcal{B}]$, the image of $\mathcal{B}$ under $h$, is a submodule of $\mathcal{P}(\mathcal{U})$ since $\mathcal{B}$ is a submodule of $\mathcal{A}$. Hence $h[\mathcal{B}]$ is a field of sets and it is an $\mathcal{I}$-module under the Peircean product. $\mathcal{B}$ is isomorphic to $h[\mathcal{B}]$; hence $h[\mathcal{B}]$ is a representation of $\mathcal{B}$. \quad \blacksquare

Theorem 5.6 shows that bijectivity is a sufficient condition for representability. It is not a necessary condition: there are representable Boolean modules which are not bijective. One such a module is the example given in Section 2 of a nonbijective Boolean module: it is representable because it is proper. It remains an open problem whether all Boolean modules are representable in the sense of Definition 5.1.

We have been concerned here with the first of the two strategies (mentioned at the beginning of this section) concerning the representability of Boolean modules. As regards the second strategy there are two lines of attack. First, one can ask a hypothetical question: given a Boolean $\mathcal{B}$-module $\mathcal{B}$, if $\mathcal{B}$ is representable, say by $\mathcal{I}$, is there a field of sets which is a proper Boolean $\mathcal{I}$-module under the Peircean product and is isomorphic to $\mathcal{B}$? Second, one can attempt to eliminate the dependence of the representability of a Boolean $\mathcal{B}$-module $\mathcal{B}$ on the representability of $\mathcal{B}$ by redefining the concept of a Boolean module. One could, for example, define a Boolean module to be a Boolean algebra linked by multiplication to a representable relation algebra, as defined by the axioms of [8]. Whether either or both of these approaches will result in a general representation result for Boolean modules is an open problem.
References