The transmission problem in domains with a corner point for the Laplace operator in weighted Hölder spaces

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Abstract

We prove the existence and uniqueness of a solution to the elliptic transmission problem in nonsmooth domains in the weighted Hölder space. The coercive estimates of the solution are given.

1. Introduction

We consider a boundary value problem for an elliptic equation in the two-dimensional case. The problem under investigation is peculiar in two ways. First, it has discontinuous coefficients. A curve where the coefficients experience a discontinuity is called an interface. Second, the interface is non-regular. Specifically, it has an angular point.

In the case of the smooth interfaces a substantial amount of results related to the problem with discontinuous coefficients (the transmission problem) was obtained in the papers of O.A. Ladyzhenskaia et al. [10], O.A. Oleinik [14,15], M. Schechter [17], Ja.A. Roitberg and Z.G. Sheftel [16], Z.G. Sheftel [18] for the $W^{1,p}$ and $C^{1+\alpha}$ spaces.

The study of boundary value problems in domain with singularities is of great interest in applications. The state of the theory of boundary value problems on nonsmooth domains, as it was about
twenty years ago, is described in detail in the well-known survey of V.A. Kondratiev and O.A. Oleinik [9]. In particular, elliptic boundary value problems in domain with conical or dihedral singularities have been extensively studied, starting from Kondratiev’s famous paper [8]. In the field of research it should be also noted the works of P. Grisvard [6], V.G. Maz’ya and B.A. Plamenevskii [12], V.A. Solonnikov and E.V. Frolova [19], V.A. Solonnikov and V. Zaionchkovskii [20]. The results of those investigations are formulated in both $W^{1,p}$ and $C^{1+\alpha}$ spaces and corresponding weighted classes, where the desired functions, together with their derivatives, are bounded with some power weights in the vicinity of a corner point.

Later R.B. Kellogg [7], A. Ben M’Barek and M. Mérigot [3], K. Lemrabet [11], M. Dauge and S. Nicaise [5], S. Nicaise [13] have studied the transmission problems (also called interface problem) for elliptic equations with a nonsmooth interface. In these works it has been shown that a variational solution of the transmission problem for the second order elliptic equation can be decomposed into a regular part from an appropriate space $W^{1,p}$, and a singular part of the form $\sum k_i S_i$. The singular functions $S_i$ depend only on the interface, and coefficients $k_i$ are defined by the problem data.

In contrast to the previous researches we study the two-dimensional transmission problem with an angular point on the interface and determine the sufficient conditions for the existence of a unique solution in the corresponding weighted Hölder classes. We use the local approach in the investigation of this problem. In the corresponding model problems we get the integral representation of solutions in the corresponding weighted Hölder classes. We use the local approach in the investigation of boundary value problems with nonsmooth boundaries is study of behavior to the solution in the neighborhood of the boundary singularities. The weighted Hölder classes which are used in the paper allow us to obtain the exponent of the solution decrease rate near a corner point.

Let the domain $\Omega \subset \mathbb{R}^2$ have the boundary $\Gamma_0 \in C^{2+\beta}$ where $C^{2+\beta}$ is the standard Hölder space, $\beta \in (0, 1)$, $l$ is a nonnegative integer. The closed curve $\Gamma \subset \Omega$ splits $\Omega$ onto parts $\Omega_1$ and $\Omega_2$, so that $\Omega = \Omega_1 \cup \Omega_2$, $\Gamma_0 \subset \Omega_2$. We assume that the origin of coordinates $O \in \Gamma$ is the corner point and in the vicinity of $O$ the curve $\Gamma$ consists of two intersecting segments with an angular opening $\alpha \in (0, \pi)$, and $\Gamma \in C^{2+\beta}$ outside of the neighborhood of $O$ (see Fig. 1).

We look for the functions $u_i(y), y=(y_1, y_2), y \in \Omega_i, i=1, 2$, by the following conditions

$$\Delta u_i = g_i(y) \quad \text{in } \Omega_i,$$

$$u_1 = g_1(y) \quad \text{on } \Gamma_0; \quad u_1 - u_2 = \psi_0(y), \quad \frac{\partial u_1}{\partial n} - k \frac{\partial u_2}{\partial n} = \psi(y) \quad \text{on } \Gamma. \quad (1.1)$$

Here $n(y)$ is the outer normal to the domain $\Omega_1$; $g_i(y)$ and $\psi(y), \psi_0(y)$ are the given functions, $k$ is a constant. In what follows we shall consider the case of positive $k$ and $k \neq 1$.

Let us introduce the functional spaces. Let $r(y)$ be the distance from the point $y \in \Omega$ to the origin, $r = \min(r(y), r(\tilde{y})), \tilde{y}, y \in \Omega, \beta \in (0, 1)$ and

$$\langle v \rangle_{y, \mu, \Omega}^{(\beta)} = \sup_{y, \tilde{y} \in \Omega, |y - \tilde{y}| < r/2} r^{-\mu} \frac{|v(y) - v(\tilde{y})|}{|y - \tilde{y}|^\beta},$$

where $\mu$ is some constant.
Let \( l \) be a nonnegative integer and \( s \) be a constant. We will say that a function \( v(y) \in E^{1+\beta}_2(\Omega) \) if the following norm
\[
\|v\|_{E^{1+\beta}_2(\Omega)} = \sum_{|m|=0}^{l} \sup_{\Omega} r^{-s+|m|}(y) |D^{m}_y v(y)| + \sum_{|m|=l} (D^{m}_y v(y))^{(\beta)}_{y, s-1-\beta, \Omega} \tag{1.2}
\]
is finite, where \(|m| = m_1 + m_2\).

In a domain \( \Omega/\cup_{\varepsilon} U_i \), where \( U_\varepsilon \) is a ball of the radius \( \varepsilon \) with the center in \( O \), the space \( E_s^{1+\beta}(\Omega/\cup_{\varepsilon} U_i) \) is the same as \( C^{1+\beta}(\Omega/\cup_{\varepsilon} U_i) \).

The principal result of our paper is

**Theorem 1.1.** Let \( g_i \in E_s^{1+\beta}(\Omega_i), i = 1, 2, \psi \in E_1^{1+\beta}(\Gamma), \psi_0 \in E_2^{2+\beta}(\Gamma), g_3 \in E_2^{2+\beta}(\Gamma_0), \alpha \in (0, \pi), 2 + \sigma \in (\frac{\pi}{2}, \frac{\pi}{2s-\alpha}) \). There exists a unique solution of problem (1.1) \( u_i(y) \in E^{1+\beta}_2(\Omega_i) \) and
\[
\|u_i\|_{E^{1+\beta}_2(\Omega_i)} \leq C(\|g_1\|_{E^{1+\beta}_2(\Omega_1)} + \|g_2\|_{E^{1+\beta}_2(\Omega_2)} + \|g_3\|_{E^{2+\beta}_2(\Omega_0)} + \|\psi\|_{E^{1+\beta}_1(\Gamma)} + \|\psi_0\|_{E^{2+\beta}_2(\Gamma_0)}), \tag{1.3}
\]
where \( C \) is a constant independent of \( u_i \).

Henceforward letter \( C \) will be used to denote different constants encountered in our formulae.

As known from the classical paper by V.A. Kondratiev [8], the behavior of solutions of the classical boundary value problems for the Laplace equation near singularities is controlled by the eigenvalues of the eigenvalue problem for the Laplace–Beltrami operator. In other words these eigenvalues specify certain restrictions on the weight in the weighted classes where the solution of the problem is searched. In our case, \( 2 + \sigma \in (\frac{\pi}{2}, \frac{\pi}{2s-\alpha}) \), and the interval \( (\frac{1}{2}, \frac{\pi}{2s-\alpha}) \) does not contain eigenvalues of the corresponding spectral problem (see problem (3.1) below). The analogous conditions appear in the work of S. Nicaise [13] if the weighted spaces are used. In addition, the condition \( 2 + \sigma \in (\frac{1}{2}, \frac{\pi}{2s-\alpha}) \) in Theorem 1.1 assures both the uniqueness of the solution to problem (1.1) and the smoothness of this solution near the angular point. It is also possible to get the greater exponent of the solution decrease rate near a corner point under the condition that the angular opening \( \alpha \) is sufficiently small (see Remark 3.1).

This paper is organized as follows. In the second section we study some model boundary value problems for the Laplace operator. The main difficulties under the investigation deal with the study of the solvability to the transmission problem for the Laplace operator when \( \Omega_i, i = 1, 2, \) are plane corners, \( \Omega_1 \cup \Omega_2 = R^2 \). The main results of Section 2 are given in Theorem 2.1 and Remark 2.1. Section 3 is devoted to the proof of Theorem 1.1 and can be divided into two parts. In the beginning we apply results of [13] and [18] to get the existence and uniqueness of the weak solution in problem (1.1). After that, based on Theorem 2.1 and Remark 2.1, we obtain the corresponding estimates in the weighted Hölder spaces. Appendix A contains the proofs of some auxiliary assertions which are applied in Section 2.

### 2. Model problem

#### 2.1. The statement of the problem

Let \((r, \varphi)\) be the polar coordinates in the plane \(R^2\), and the domains \(\Omega_i\) have the forms: \(\Omega_1 = \{(r, \varphi): r > 0, \alpha < \varphi < 2\pi\}, \Omega_2 = \{(r, \varphi): r > 0, 0 < \varphi < \alpha\}, \alpha \in (0, \pi)\); \(\Gamma = \Gamma_1 \cup \Gamma_2 = \{(r, \varphi): r \geq 0, \varphi = \alpha\} \cup \{(r, \varphi): r \geq 0, \varphi = 0\}\). Hereafter, we identify the set \(\{(r, \varphi): r \geq 0, \varphi = 0\}\) with the set \(\{(r, \varphi): r \geq 0, \varphi = 2\pi\}\).
In the model problem it is necessary to find the functions \( u_1(y) \) and \( u_2(y) \) by the conditions:

\[
\Delta u_i = g_i(y) \quad \text{in } \Omega_i, \quad i = 1, 2, \\
u_1 = u_2 \quad \text{on } \Gamma,
\]

\[
\frac{\partial u_1}{\partial n}_{\varphi=\alpha} - k \frac{\partial u_2}{\partial n}_{\varphi=\alpha} = \psi_1(y), \\
\frac{\partial u_1}{\partial n}_{\varphi=2\pi} - k \frac{\partial u_2}{\partial n}_{\varphi=0} = \psi_2(y).
\] (2.1)

where \( g_i(y) \) and \( \psi_i(y) \) are finite functions; \( g_i(y) \in E_\sigma^\beta(\Omega_i) \) and \( \psi_i(y) \in E_{1+\sigma}^\beta(\Gamma_i) \).

We introduce the new independent variables

\[
x_2 = \varphi, \quad x_1 = \ln r,
\] (2.2)

so that the image of \( \Omega_i \) is the strip \( G_i, \ i = 1, 2, \)

\[
G_1 = \{(x_1, x_2) : x_1 \in (-\infty, \infty), \ x_2 \in (\alpha, 2\pi)\}, \\
G_2 = \{(x_1, x_2) : x_1 \in (-\infty, \infty), \ x_2 \in (0, \alpha)\};
\]

and \( \Gamma \) goes to \( \hat{\Gamma} : \)

\[
\hat{\Gamma} = \hat{\Gamma}_1 \cup \hat{\Gamma}_2 = \{(x_1, x_2) : x_1 \in (-\infty, \infty), \ x_2 = \alpha\} \\
\hspace{2cm} \cup \{(x_1, x_2) : x_1 \in (-\infty, \infty), \ x_2 = 0\}.
\]

Simple calculations transform problem (2.1) to the form:

\[
\frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} = e^{2x_1} g_i(x) \quad \text{in } G_i, \ i = 1, 2, \\
u_1 = u_2 \quad \text{on } \hat{\Gamma},
\]

\[
\frac{\partial u_1}{\partial x_2}_{x_2=\alpha} - k \frac{\partial u_2}{\partial x_2}_{x_2=\alpha} = -e^{x_1} \psi_1(x_1), \\
\frac{\partial u_1}{\partial x_2}_{x_2=2\pi} - k \frac{\partial u_2}{\partial x_2}_{x_2=0} = e^{x_1} \psi_2(x_1).
\] (2.3)

Here we keep the former designations for the desired functions.

In the next step we introduce the new unknown functions

\[
v_i(x) = e^{-(2+\sigma)x_1} u_i(x),
\] (2.4)

where the value of \( \sigma \) will be chosen later, and rewrite problem (2.3) as

\[
\Delta v_i + 2(2 + \sigma) \frac{\partial v_i}{\partial x_1} + (2 + \sigma)^2 v_i = e^{-(\sigma-1)x_1} g_i(x) \equiv f_i(x) \quad \text{in } G_i, \ i = 1, 2, \\
v_1 = v_2 \quad \text{on } \hat{\Gamma},
\]

\[
\frac{\partial v_1}{\partial x_2}_{x_2=\alpha} - k \frac{\partial v_2}{\partial x_2}_{x_2=\alpha} = -e^{-(\sigma+1)x_1} \psi_1(x_1) \equiv q_1(x_1), \\
\frac{\partial v_1}{\partial x_2}_{x_2=2\pi} - k \frac{\partial v_2}{\partial x_2}_{x_2=0} = e^{-(\sigma+1)x_1} \psi_2(x_1) \equiv q_2(x_1).
\] (2.5)
We will search a solution of problem (2.5) \( v_i(x), i = 1, 2, \) in the classes \( C^{2+\beta}(\hat{G}_i), \) when \( f_i(x) \in C^{\beta}(\hat{G}_i), \) \( q_i(x_1) \in C^{1+\beta}(\hat{F}_i). \)

### 2.2. The coercive estimates in the case of the homogeneous boundary conditions

First of all we study problem (2.5) in the case of \( q_i(x_1) = 0. \) Let \( \tilde{w}(\lambda, x_2) \) be the Fourier transformation of the function \( w(x_1, x_2), \) i.e.

\[
\tilde{w}(\lambda, x_2) = \int_{-\infty}^{\infty} w(x_1, x_2) e^{-i\lambda x_1} dx_1.
\]

If \( q_i(x_1) = 0, \) the Fourier transformation in (2.5) leads to the problem

\[
\frac{d^2 \tilde{v}_i}{dx_2^2} + i(2 + \sigma)\lambda \tilde{v}_i + ((2 + \sigma)^2 - \lambda^2) \tilde{v}_i = \tilde{f}_i,
\]

\[
\tilde{v}_i|_{x_2=\alpha} = \tilde{v}_2|_{x_2=\alpha}, \quad \tilde{v}_i|_{x_2=2\pi} = \tilde{v}_2|_{x_2=0},
\]

\[
\left. \frac{d\tilde{v}_1}{dx_2} \right|_{x_2=\alpha} - k \left. \frac{d\tilde{v}_2}{dx_2} \right|_{x_2=\alpha} = 0, \quad \left. \frac{d\tilde{v}_1}{dx_2} \right|_{x_2=2\pi} - k \left. \frac{d\tilde{v}_2}{dx_2} \right|_{x_2=0} = 0.
\] (2.6)

One can easily check that the following functions solve the equations in (2.6)

\[
\tilde{v}_1 = c_1^{(1)} \sin \rho x_2 + c_2^{(1)} \cos \rho x_2 + \int_{\alpha}^{x_2} \frac{\tilde{f}_1(\lambda, \xi)}{\rho} \sin \rho (x_2 - \xi) d\xi,
\]

\[
\tilde{v}_2 = c_1^{(2)} \sin \rho x_2 + c_2^{(2)} \cos \rho x_2 + \int_{0}^{x_2} \frac{\tilde{f}_2(\lambda, \xi)}{\rho} \sin \rho (x_2 - \xi) d\xi.
\] (2.7)

where \( c_1^{(i)} \) and \( c_2^{(i)}, i = 1, 2, \) are arbitrary constants and \( \rho = i\lambda + s, s = 2 + \sigma. \)

Substituting (2.7) in the boundary conditions from (2.6), one can get the linear system of the algebraic equations to find the unknown coefficients \( c_i^{(j)}, i, j = 1, 2. \) It is easy to check that the determinant of this system is \( (k^2 - 1) \sin \rho\alpha \sin \rho(2\pi - \alpha). \) If this determinant does not vanish for all \( \lambda \in R, \) the homogeneous problem corresponding to (2.6) will only have a trivial solution. As is well known for boundary value problems, problem (2.6) is well posed if and only if the only solution of the corresponding homogeneous boundary value problem is zero. Therefore, to ensure well-posedness of problem (2.6), we will chose below the parameter \( s \) in such a way that \( (k^2 - 1) \sin \rho\alpha \sin \rho(2\pi - \alpha) \neq 0 \) for all \( \lambda \in R. \)

Now we begin to study the behavior of the solutions \( v_i(x_1), i = 1, 2, \) on the boundary \( \hat{F}. \) To that end we introduce new unknown functions \( M_i(x_1), i = 1, 2, \) as

\[
M_i(x_1) = v_i(x_1, x_2)|_{\hat{F}}.
\] (2.8)

The Fourier transformation in (2.8) gives

\[
\tilde{M}_1(\lambda) = \tilde{v}_1(\lambda, 2\pi) = \tilde{v}_2(\lambda, 0), \quad \tilde{M}_2(\lambda) = \tilde{v}_1(\lambda, \alpha) = \tilde{v}_2(\lambda, \alpha).
\] (2.9)
Then, using the boundary conditions in (2.6), we can conclude that \( \tilde{M}_1(\lambda) \) and \( \tilde{M}_2(\lambda) \) satisfy the system of the algebraic equations

\[
\begin{align*}
\tilde{M}_1 a_1(\lambda) - \tilde{M}_2 a_2(\lambda) &= F_1(\lambda), \\
\tilde{M}_1 a_2(\lambda) - \tilde{M}_2 a_1(\lambda) &= F_2(\lambda),
\end{align*}
\]

(2.10)

where

\[
a_1(\lambda) = \cot \rho(2\pi - \alpha) + k \cot \rho \alpha, \quad a_2(\lambda) = \frac{1}{\sin \rho(2\pi - \alpha)} + k \frac{1}{\sin \rho \alpha},
\]

\[
F_1(\lambda) = -\int_\alpha^{2\pi} \frac{\tilde{f}_1(\lambda, \xi) \sin \rho(\xi - \alpha)}{\rho \sin \rho(2\pi - \alpha)} \, d\xi - k \int_0^\alpha \frac{\tilde{f}_2(\lambda, \xi) \sin \rho(\alpha - \xi)}{\rho \sin \rho \alpha} \, d\xi,
\]

\[
F_2(\lambda) = \int_\alpha^{2\pi} \frac{\tilde{f}_1(\lambda, \xi) \sin \rho(2\pi - \xi)}{\rho \sin \rho(2\pi - \alpha)} \, d\xi + k \int_0^\alpha \frac{\tilde{f}_2(\lambda, \xi) \sin \rho \xi}{\rho \sin \rho \alpha} \, d\xi.
\]

Let us study the determinant of system (2.10)

\[
D(\lambda) = -a_1^2 + a_2^2 = (1 + k)^2 + 4k \frac{\sin^2 \rho(\pi - \alpha)}{\sin \rho \alpha \sin \rho(2\pi - \alpha)}.
\]

(2.11)

The function \( D(\lambda) \) vanishes if

\[
0 = \frac{\sin^2 \rho(\pi - \alpha)}{\sin \rho \alpha \sin \rho(2\pi - \alpha)} - l,
\]

(2.12)

where \( l \equiv -\frac{(1+k)^2}{4k} \). When the real and imaginary parts in the right-hand side of (2.12) are considered, one can conclude that Eq. (2.12) is equivalent to the following system of the algebraic equations

\[
(l + 1) \cos 2s(\pi - \alpha) \cosh 2\lambda(\pi - \alpha) - 1 = l \cos 2\pi s \cosh 2\lambda \pi,
\]

(2.13)

\[
(l + 1) \sin 2s(\pi - \alpha) \sinh 2\lambda(\pi - \alpha) = l \sin 2\pi s \sinh 2\lambda \pi.
\]

(2.14)

We will find such value \( s \) that for every \( \lambda \in R \) system (2.13), (2.14) will be unsolvable, i.e. \( D(\lambda) \neq 0 \).

Note that \( 1 + l = \frac{-(k-1)^2}{4k} \) and \( 0 < \frac{l+1}{l} < 1 \). In the case of \( \sin 2\pi s = 0 \) and \( \lambda \neq 0 \), Eq. (2.14) gives

\[
\frac{l + 1}{l} \frac{\sin 2s(\pi - \alpha)}{\sin 2\pi s} = \frac{\sinh 2\pi \lambda}{\sinh 2\lambda(\pi - \alpha)}.
\]

(2.15)

It will be impossible to satisfy equality (2.15) if

\[
\frac{\sin 2s(\pi - \alpha)}{\sin 2\pi s} < 0.
\]

(2.16)

It is easy to show that condition (2.16) can be satisfied if

\[
\begin{align*}
2\pi m < 2s(\pi - \alpha) < 2\pi m + \pi, & \quad m = 0, 1, \ldots, \\
2\pi m + \pi < 2s\pi < 2\pi m + 2\pi,
\end{align*}
\]
or
\[
\max\left(\frac{\pi m}{\pi - \alpha}, m + 1/2\right) < s < \min\left(m + 1, \frac{2\pi m + \pi}{2(\pi - \alpha)}\right).
\] (2.17)

Note that \(\lambda = 0\) is the solution of (2.14), and in this case Eq. (2.13) can be represented as
\[(l + 1) \cos 2s(\pi - \alpha) - 1 = l \cos 2\pi s,
\]
or
\[\sin^2 s(\pi - \alpha) = l \sin(\alpha s) \sin(2\pi - \alpha).\]

This equation does not have any solutions if
\[\sin(\alpha s) \sin(2\pi - \alpha) > 0,\] (2.18)
as \(l < 0\).

Condition (2.18) can be satisfied if \(s\) is a solution to the following system of inequalities:
\[
\begin{cases}
0 < \alpha s < \pi, \\
2\pi m < s(2\pi - \alpha) < 2m\pi + \pi,
\end{cases}
\]
which is equivalent to the inequality
\[
\max\left(0, \frac{2\pi m}{2\pi - \alpha}\right) < s < \min\left(\frac{\pi}{\alpha}, \frac{2\pi m + \pi}{2\pi - \alpha}\right).
\] (2.19)

It is now clear that (2.17) and (2.19) ensure \(D(\lambda) \neq 0\) if
\[
\max\left(\frac{\pi m}{\pi - \alpha}, m + 1/2\right) < s < \min\left(m + 1, \frac{2\pi m + \pi}{2\pi - \alpha}\right), \quad 0 < \alpha < \frac{\pi}{m + 1}, \quad m = 0, 1, \ldots
\] (2.20)

In other words \(D(\lambda) \neq 0\) if \(\alpha \in (0, \frac{\pi}{2m + 1})\), and
\[m + 1/2 < s < \frac{2\pi m + \pi}{2\pi - \alpha}, \quad m = 0, 1, \ldots ;\] (2.21)

but in the case of \(\alpha \in \left(\frac{\pi}{2m + 1}, \frac{\pi}{m + 1}\right)\), we have \(D(\lambda) \neq 0\) if
\[\frac{\pi m}{\pi - \alpha} < s < \frac{2\pi m + \pi}{2\pi - \alpha}, \quad m = 1, 2, \ldots .\] (2.22)

Below we will assume that either (2.21) or (2.22) is fulfilled for some \(m\).

Let us return to system (2.10) and find \(\tilde{M}_i, i = 1, 2:\)
\[
\tilde{M}_1 = \frac{1}{D(\lambda)}(-a_1 F_1 + a_2 F_2), \quad \tilde{M}_2 = \frac{1}{D(\lambda)}(a_1 F_2 - a_2 F_1).
\] (2.23)
Then, for example, the full expression for $\tilde{M}_2$ has the form

$$
\tilde{M}_2 = \frac{1}{D(\lambda)} \left\{ \int_{\alpha}^{2\pi} \tilde{f}_1(\lambda, \xi) \rho \left[ \cos(2\pi - \xi) \sin(2\pi - \alpha) + k \frac{\sin(2\pi - \xi) \cos \rho + \sin \rho (\xi - \alpha)}{\sin(\rho \alpha) \sin(2\pi - \alpha)} \right] d\xi + k \int_{0}^{\alpha} \tilde{f}_2(\lambda, \xi) \rho \left[ k \cos(2\pi - \alpha) \sin(\rho \alpha) + \cos(2\pi - \xi) \sin(\rho \xi) + \sin(\rho(\alpha - \xi)) \right] d\xi \right\}.
$$

To obtain the corresponding estimates of the solution to problem (2.5) with $q_i = 0$, it is enough to describe the properties of the functions:

$$
M_i(x_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{M}_i(\lambda) e^{ix_1\lambda} d\lambda, \quad i = 1, 2.
$$

To this end, we consider the function

$$
S(x_1) = \frac{2\pi}{\alpha} \int_{-\infty}^{\infty} f(x_1 - z, \xi) G(z, \xi) dz.
$$

with $f(x_1, x_2) \in C^\beta((\bar{G}_1))$.

$$
\tilde{G}(\lambda, \xi) = \frac{1}{\rho D(\lambda)} \cos(2\pi - \xi) \frac{\sin(2\pi - \alpha)}{\sin(2\pi - \alpha)}, \quad G(z, \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(\lambda, \xi) e^{iz\lambda} d\lambda, \quad \rho = i\lambda + s.
$$

Note that $\tilde{G}(\lambda, \xi)$ belongs to the main part of the kernel in the representation of $\tilde{M}_2(\lambda)$. One can see that the functions $M_i(x_1)$, $i = 1, 2$, can be estimated similarly to $S(x_1)$.

In Proposition 2.1 we describe the properties of $G(z, \xi)$. Its proof is represented in Appendix A.

**Proposition 2.1.** Let $\mu_2 = \frac{\alpha(m+1/2)}{4\pi m \pi - \alpha}$, $\mu_2 + \frac{1}{2} + m < s < \frac{2\pi + \pi}{2 \pi - \alpha}$, $0 < \alpha < \frac{\pi}{m+1}$, $m = 0, 1, \ldots$, and $\delta$ be some nonnegative constant,

$$
J(\xi) := \int_{0}^{\infty} \frac{\mu_1 e^{-\mu_1 (\xi - \alpha)}}{\mu_1^2 + (s - \mu_2)^2} d\mu_1.
$$

Then

$$
|G(z, \xi)| \leq Ce^{-\mu_2|z|} \left( 1 + J(\xi) \right),
$$

$$
\int_{\alpha}^{2\pi} J(\xi) d\xi \leq C,
$$

$$
\left| \frac{\partial G}{\partial z}(z, \xi) \right| \leq Ce^{-\mu_2|z|} \left( 1 + J(\xi) + \frac{\xi - \alpha + |z|}{z^2 + (\xi - \alpha)^2} \right),
$$
\[
\left| \int_\alpha^{2\pi} \frac{\partial G}{\partial z}(z, \xi) \bigg|_{|z|=\delta} \right| \leq C, \tag{2.30}
\]
\[
\left| \frac{\partial^2 G}{\partial z^2}(z, \xi) \right| + \left| \frac{\partial^3 G}{\partial z^3}(z, \xi) \right| \leq C e^{-\mu|z|} \left( 1 + f(\xi) + \frac{1 + \xi - \alpha + |z|}{\delta^2 + (\xi - \alpha)^2} \right), \tag{2.31}
\]

where the constants \(C\) depend on the value of \(\alpha\) and are bounded for every fixed \(\alpha \in (0, \pi)\).

**Proposition 2.2.** Let \(f(x_1, x_2) \in C(\tilde{G}_1)\), then
\[
\max_R |S(x_1)| \leq C \max_{\tilde{G}} |f|. \tag{2.32}
\]

The proof of Proposition 2.2 follows from (2.27) and (2.28).

**Proposition 2.3.** Let \(f(x_1, x_2) \in C^{\beta}(\tilde{G}_1)\), then
\[
\max_R \left| \frac{d^2 S(x_1)}{dx_1^2} \right| \leq C \|f\|_{C^{\beta}(\tilde{G}_1)}. \tag{2.33}
\]

**Proof.** To obtain estimate (2.33), we use the following representation (the analogous representation is used in the investigation of the volume potential in the elliptic theory of boundary value problems, see, e.g., Chapter 3 in [10])
\[
\frac{d^2 S(x_1)}{dx_1^2} = \int_\alpha^{2\pi} d\xi \int_{|x_1-z|\geq\delta} f(z, \xi) \frac{\partial^2 G(x_1 - z, \xi)}{\partial x_1^2} dz + \int_\alpha^{2\pi} d\xi \int_{|x_1-z|\leq\delta} (f(z, \xi) - f(x_1, \alpha)) \frac{\partial^2 G(x_1 - z, \xi)}{\partial x_1^2} dz + f(x_1, \alpha) \int_\alpha^{2\pi} d\xi \frac{\partial G(x_1 - z, \xi)}{\partial x_1} \bigg|_{|x_1-z|=\delta} = i_1 + i_2 + i_3, \tag{2.34}
\]
where \(\delta\) is a positive constant. Note that the estimate of \(i_3\) is the simplest and follows from (2.30):
\[
|i_3| \leq C \max_{\tilde{G}_1} |f|. \tag{2.35}
\]

We use inequalities (2.31), (2.28) to evaluate \(i_1\), and obtain
\[
|i_1| \leq \text{const.} \max_{\tilde{G}_1} \|f\| \int_\alpha^{2\pi} d\xi \int_{\delta}^{+\infty} e^{-\mu y} \left( 1 + f(\xi) + \frac{1 + \xi - \alpha + y}{\delta^2 + (\xi - \alpha)^2} \right) dy \leq C \max_{\tilde{G}_1} |f|. \tag{2.36}
\]
Finally,
This representation together with inequalities (2.31), \(|y| \leq \delta\) and \(\xi - \alpha \leq C\) lead to

\[
|i_2| \leq C (\beta) \left\{ \int_{\alpha}^{2\pi} d\xi \int_{|y| \leq \delta} |y|^\beta \left[ 1 + J(\xi) + (y^2 + (\xi - \alpha)^2)^{-1} \right] dy \\
+ \int_{\alpha}^{2\pi} d\xi \int_{|y| \leq \delta} (\xi - \alpha)^\beta \left[ 1 + J(\xi) + (y^2 + (\xi - \alpha)^2)^{-1} \right] dy \right\}. 
\]

Due to

\[
\int_{|y| \leq \delta} dy \int_{\alpha}^{2\pi} |y|^\beta (y^2 + (\xi - \alpha)^2)^{-1} d\xi = \int_{|y| \leq \delta} |y|^\beta -1 \arctan \frac{\xi - \alpha}{y} \bigg|_{\xi = \alpha}^{\xi = 2\pi} dy \leq C
\]

and

\[
\int_{\alpha}^{2\pi} (\xi - \alpha)^\beta d\xi \int_{|y| \leq \delta} (y^2 + (\xi - \alpha)^2)^{-1} dy \leq \text{const.} \int_{\alpha}^{2\pi} (\xi - \alpha)^\beta -1 d\xi \leq C,
\]

one can easily obtain

\[
|i_2| \leq C (\beta) .
\] (2.37)

Thus, the proof of this proposition follows from estimates (2.35)–(2.37).

\textbf{Proposition 2.4.} Let \(f(x_1, x_2) \in C^\beta (\bar{G}_1)\), then

\[
\langle S_{x_1, x_2}^{(\beta)} \rangle_{x_1, R} \leq C (\beta) .
\] (2.38)

\textbf{Proof.} For \(x_1, \tilde{x}_1 \in R\), we estimate the difference

\[
S_{x_1, x_1}(x_1) - S_{x_1, \tilde{x}_1}(\tilde{x}_1) \\
= \int_{\alpha}^{2\pi} d\xi \int_{|x_1 - z| \geq 2|\tilde{x}_1 - x_1|} \left[ f(z, \xi) - f(x_1, \alpha) \right] \left[ G_{x_1, x_1}(x_1 - z, \xi) - G_{x_1, \tilde{x}_1}(\tilde{x}_1 - z, \xi) \right] dz \\
+ \int_{\alpha}^{2\pi} d\xi \int_{|x_1 - z| \leq 2|\tilde{x}_1 - x_1|} \left[ f(z, \xi) - f(x_1, \alpha) \right] G_{x_1, x_1}(x_1 - z, \xi) dz.
\]
Due to Propositions 2.1–2.4 we conclude that

$$\|M_i(x_1)\|_{C^{2+\beta}([\alpha,\beta])} \leq C\left(\|f_1\|_{C^\beta(\tilde{G}_1)} + \|f_2\|_{C^\beta(\tilde{G}_2)}\right), \quad i = 1, 2.$$  

(2.43)
Because of (2.8) the functions \( v_i(x_1, x_2) \) in (2.5) meet the conditions

\[
\Delta v_i + 2s \frac{\partial v_i}{\partial x_1} + s^2 v_i = f_i(x), \quad x \in G_1,
\]

\[
v_1|_{x_2=\alpha} = M_2(x_1), \quad v_1|_{x_2=2\pi} = M_1(x_1);
\]

\[
v_2|_{x_2=\alpha} = M_2(x_1), \quad v_2|_{x_2=0} = M_1(x_1).
\]

Then estimate (2.43) together with the results from Chapter 3 of [10] give the next assertion.

**Lemma 2.1.** Let \( f_i(x) \in C^\beta(\tilde{G}_1) \), \( q_i(x) = 0 \), \( i = 1, 2 \), then there exists a unique solution of (2.5) \( v_i \in C^{2+\beta}(\tilde{G}_1) \) and

\[
\|v_i\|_{C^{2+\beta}(\tilde{G}_1)} \leq C\left(\|f_1\|_{C^\beta(\tilde{G}_1)} + \|f_2\|_{C^\beta(\tilde{G}_2)}\right)
\]

(2.44)

with the constant \( C \) independent of \( v_i \).

### 2.3. Solvability and coercive estimates in the general case

Here we study problem (2.5) in the case of \( f_i(x) = 0 \) and \( q_i(x_1) \in C^{1+\beta}(\tilde{F}_1) \). The solution of this problem can be represented by the Fourier transformation as

\[
\tilde{v}_1(\lambda, x_2) = \tilde{N}_1(\lambda) \frac{\sin \rho(x_2 - \alpha)}{\sin \rho(2\pi - \alpha)} + \tilde{N}_2(\lambda) \frac{\sin \rho(2\pi - x_2)}{\sin \rho(2\pi - \alpha)},
\]

\[
\tilde{v}_2(\lambda, x_2) = \tilde{N}_1(\lambda) \frac{\sin \rho(\alpha - x_2)}{\sin \rho \alpha} + \tilde{N}_2(\lambda) \frac{\sin \rho x_2}{\sin \rho \alpha}, \quad \rho = i\lambda + s, \tag{2.45}
\]

where \( \tilde{N}_1(\lambda) = \tilde{v}_1(\lambda, 2\pi) = \tilde{v}_2(\lambda, 0) \), \( \tilde{N}_2(\lambda) = \tilde{v}_1(\lambda, \alpha) = \tilde{v}_2(\lambda, \alpha) \).

The transmission conditions in (2.5) give the next system to find the functions \( \tilde{N}_1(\lambda), \tilde{N}_2(\lambda) \):

\[
\begin{cases}
\tilde{N}_1(\lambda)a_2(\lambda) - \tilde{N}_2(\lambda)a_1(\lambda) = \frac{\tilde{q}_1(\lambda)}{\rho}, \\
\tilde{N}_1(\lambda)a_1(\lambda) - \tilde{N}_2(\lambda)a_2(\lambda) = \frac{\tilde{q}_2(\lambda)}{\rho},
\end{cases} \tag{2.46}
\]

where the functions \( a_1(\lambda) \) and \( a_2(\lambda) \) are given in (2.10). Thus,

\[
\tilde{N}_1 = \frac{1}{D(\lambda)\rho} \left\{ \tilde{q}_1(\lambda) \left( \frac{1}{\sin \rho(2\pi - \alpha)} + k \frac{1}{\sin \rho \alpha} \right) + \tilde{q}_2(\cot \rho(2\pi - \alpha) + k \cot \rho \alpha) \right\},
\]

\[
\tilde{N}_2 = \frac{1}{D(\lambda)\rho} \left\{ \tilde{q}_1(\lambda)(\cot \rho(2\pi - \alpha) + k \cot \rho \alpha) - \tilde{q}_2 \left( \frac{1}{\sin \rho(2\pi - \alpha)} + k \frac{1}{\sin \rho \alpha} \right) \right\},
\]

where \( D(\lambda) \) is defined in (2.11) and its properties have been studied in Section 2.2.

To describe the properties of the desired functions \( N_1(x_1) \) and \( N_2(x_1) \), which are the inverse Fourier transformation of \( \tilde{N}_1(\lambda) \) and \( \tilde{N}_2(\lambda) \), we consider for the sake of simplicity the following function

\[
N(x_1) = \int_{-\infty}^{\infty} q(z)K(x_1 - z) dz, \tag{2.47}
\]
with
\[ \tilde{K}(\lambda) = \frac{1}{D(\lambda)\rho} a_1(\lambda) = \frac{1}{D(\lambda)\rho} \left( \cot \rho (2\pi - \alpha) + k \cot \rho \alpha \right). \]

The properties of the kernel \( K(z) \) in (2.47) are given in Proposition 2.5 and its proof is given in Appendix A.2.

**Proposition 2.5.** Let \( \mu_2 = \frac{\alpha(m+1/2)}{2\pi} - \frac{\alpha}{m+1}, \mu_2 + \frac{1}{2} + m < s < \frac{2m\pi + \pi}{2\pi} - \alpha, 0 < \alpha < \frac{\pi}{m+1}, m = 0, 1, \ldots, \) and \( \delta \) be a some nonnegative constant. Then
\[
|K(z)| \leq Ce^{-\mu_2|z|} \left( 1 + \frac{1}{|z|} \right), \tag{2.48}
\]
\[
\left| \frac{\partial K(z)}{\partial z} \right| \leq Ce^{-\mu_2|z|} \left( 1 + \frac{1}{|z|} \right), \tag{2.49}
\]
\[
\left| \frac{\partial^2 K(z)}{\partial z^2} \right| \leq Ce^{-\mu_2|z|} \left( 1 + \frac{1}{|z|} + \frac{1}{|z|^2} \right), \tag{2.50}
\]
\[
\left| \int_{-\delta}^{\delta} \frac{\partial K(z)}{\partial z} dz \right| \leq C, \tag{2.51}
\]

where the constants \( C \) depend on the value of \( \alpha \) and are bounded for every fixed \( \alpha \in (0, \pi) \).

**Proposition 2.6.** Let \( q(x_1) \in C^{1+\beta}(R) \), then
\[
\sup_R \left| N(x_1) \right| \leq C \| q \|_{C^{\beta}(R)}, \tag{2.52}
\]
\[
\sup_R \left| N_{x_1} q(x_1) \right| \leq C \| q \|_{C^{\beta}(R)}, \tag{2.53}
\]
\[
\left( N_{x_1} q(x_1) \right)_{x_1} \leq C \| q \|_{C^{\beta}(R)}. \tag{2.54}
\]

**Proof.** The function of \( N(x_1) \) can be rewritten as
\[
N(x_1) = \int_{-\infty}^{\infty} \left[ q(x_1 - z) - q(x_1) \right] K(z) dz + q(x_1) \int_{-\infty}^{\infty} K(z) dz.
\]

The second item in this representation is easily estimated due to
\[
\int_{-\infty}^{\infty} K(z) dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{K}(\lambda) d\lambda \int_{-\infty}^{\infty} e^{iz\lambda} dz = \tilde{K}(0) = C.
\]

The last equality is ensured by the choice of \( s \). Then application of inequality (2.48) to the first term in the representation of \( N(x_1) \) gives estimate (2.52).
To evaluate the function $N_{x_1}(x_1)$, we use the following equality

$$N_{x_1}(x_1) = \int_{-\infty}^{\infty} \left[ q_{x_1}(x_1 - z) - q_{x_1}(x_1) \right] K_z(z) \, dz + q_{x_1}(x_1) \int_{-\infty}^{\infty} K_z(z) \, dz,$$

(2.55)

that follows from (2.47). Thanks to the fact that

$$\int_{-\infty}^{\infty} K_z(z) \, dz = \int_{-\infty}^{\infty} \frac{i\mu_1}{D(\mu_1 + i\mu_2)} \left[ \cot(i\mu_1 + s - \mu_2)(2\pi - \alpha) + k \cot(i\mu_1 + s - \mu_2)\alpha \right] d\mu_1 \int_{-\infty}^{\infty} e^{iz\mu_1} \, dz,$$

the second term in (2.55) is zero.

The estimate of the first term in (2.55) follows from (2.49) and the inclusion $q_{x_1} \in C^{1+\beta}(R)$. Thus, inequality (2.53) has been proved.

To obtain inequality (2.54), we consider the difference

$$N_{x_1}(x_1) - N_{x_1}(\tilde{x}_1) = \int_{|z-x_1|\leq 2|\tilde{x}_1-x_1|} \left[ q_{x_1}(z) - q_{x_1}(\tilde{x}_1) \right] K_z(z) \, dz$$

$$- \int_{|z-x_1|\leq 2|\tilde{x}_1-x_1|} \left[ q_{x_1}(z) - q_{x_1}(\tilde{x}_1) \right] K_z(\tilde{x}_1 - z) \, dz$$

$$+ \int_{|z-x_1|\geq 2|\tilde{x}_1-x_1|} \left[ q_{x_1}(z) - q_{x_1}(\tilde{x}_1) \right] [K_z(x_1 - z) - K_z(\tilde{x}_1 - z)] \, dz$$

$$+ \left[ q_{x_1}(\tilde{x}_1) - q_{x_1}(x_1) \right] \int_{|y|\geq 2|\tilde{x}_1-x_1|} K_y(y) \, dy \equiv j_1 + j_2 + j_3 + j_4.$$

In the case of $j_4$ we get

$$\int_{|y|\geq 2|\tilde{x}_1-x_1|} K_y(y) \, dy = \int_{-\infty}^{\infty} K_y(y) \, dy - \int_{-2|\tilde{x}_1-x_1|}^{2|\tilde{x}_1-x_1|} K_y(y) \, dy = - \int_{-2|\tilde{x}_1-x_1|}^{2|\tilde{x}_1-x_1|} K_y(y) \, dy,$$

and (2.51) leads to

$$\left| \int_{|y|\geq 2|\tilde{x}_1-x_1|} K_y(y) \, dy \right| \leq C.$$

Then

$$|j_4| \leq C|\tilde{x}_1 - x_1|^\beta (q_{x_1})^{(\beta)}_{x_1, R}.$$
To estimate \( j_1 \) and \( j_2 \), inequality (2.49) can be applied as \( |K_z(z)| \leq Ce^{-\mu_2|z|^2}z^{-1} \) which gives

\[
|j_1| + |j_2| \leq C|\bar{x}_1 - x_1|\beta(q_{x_1})_{x_1, R}^{(\beta)}.
\]

At last, the estimate of \( j_3 \) can be obtained using the mean value theorem and inequality (2.50) in the form \( |K_{zz}(z)| \leq Ce^{-\mu_2|z|^2}z^{-2} \) with \( \mu_2 = \mu_2 - \varepsilon, \mu_2 > \varepsilon > 0 \). Then

\[
|j_3| \leq C|\bar{x}_1 - x_1|\beta(q_{x_1})_{x_1, R}^{(\beta)} \int_{|y| \geq 2|x_1 - x|} y^{\beta - 2} dy \leq C|\bar{x}_1 - x_1|\beta(q_{x_1})_{x_1, R}^{(\beta)}.
\]

Collecting the estimates for \( j_k, k = 1, 4 \), one gets inequality (2.54). This completes the proof of the proposition. \( \square \)

Due to Propositions 2.5 and 2.6 we conclude that

\[
\|N(x_1)\|_{C^{2+\beta}(\mathcal{R})} \leq C\|q\|_{C^{1+\beta}(\mathcal{R})}.
\]

The estimates of the functions \( N_i(x_1), i = 1, 2 \), are obtained in the same way, so that

\[
\|N_i(x_1)\|_{C^{2+\beta}(\mathcal{R})} \leq C\left(\|q_1\|_{C^{1+\beta}(\bar{\mathcal{R}})} + \|q_2\|_{C^{1+\beta}(\bar{\mathcal{R}})}\right), \quad i = 1, 2.
\]

After that, returning to the functions \( v_i(x_1, x_2), i = 1, 2 \), from (2.5) where \( f_i(x_1, x_2) \equiv 0 \), we can get the following result.

**Lemma 2.2.** Let \( f_1(x) = 0, q_1(x_1) \in C^{1+\beta}(\bar{\mathcal{R}}) \). Then there exists a unique solution to problem (2.5) \( v_j \in C^{2+\beta}(\bar{\mathcal{G}}), i = 1, 2 \), and

\[
\|v_i\|_{C^{2+\beta}(\bar{\mathcal{G}})} \leq C\left(\|q_1\|_{C^{1+\beta}(\bar{\mathcal{R}})} + \|q_2\|_{C^{1+\beta}(\bar{\mathcal{R}})}\right),
\]

where the constant \( C \) is independent of \( v_i \).

The proof of Lemma 2.2 is similar to the arguments of Lemma 2.1.

Note that if the functions \( v_i(x_1, x_2) = e^{-\alpha|x_1|}u_i(x_1, x_2) \in C^{2+\beta}(\bar{\mathcal{G}}) \), then \( u_i(y_1, y_2) \in E^{2+\beta}_{2+\sigma}(\tilde{T}_i) \) where the mapping \( x = x(y) \) is given by (2.2).

Thus, Lemmas 2.1 and 2.2 ensure the following assertions.

**Theorem 2.1.** Let the finite functions \( g_i(y) \in E^{1+\beta}_{2+\sigma}(\tilde{T}_i), \psi_i(y) \in E^{1+\beta}_{1+\sigma}(\Gamma_i), i = 1, 2, \alpha \in (0, \pi), 2 + \sigma \in \left(\frac{\pi}{2}, \frac{2\pi - \sigma}{2}\right) \). There exists a unique solution \( u_i(y) \in E^{2+\beta}_{2+\sigma}(\tilde{T}_i) \) to problem (2.1) and

\[
\|u_i\|_{E^{2+\beta}_{2+\sigma}(\tilde{T}_i)} \leq C\left(\|g_1\|_{E^{1+\beta}_{2+\sigma}(\tilde{T}_i)} + \|g_2\|_{E^{1+\beta}_{2+\sigma}(\tilde{T}_i)} + \|\psi_1\|_{E^{1+\beta}_{1+\sigma}(\Gamma_i)} + \|\psi_2\|_{E^{1+\beta}_{1+\sigma}(\Gamma_i)}\right),
\]

where the constant \( C \) is independent of \( u_i \).

**Remark 2.1.** Theorem 2.1 is true in the cases (see (2.21), (2.22)): \( 2 + \alpha \in (m + 1, \frac{2\pi + m + \pi}{2\pi - \alpha}) \) for \( \alpha \in (0, \frac{\pi}{2m + 1}); 2 + \sigma \in (\frac{m}{m + 1}, \frac{2\pi + m + \pi}{2\pi - \alpha}) \) for \( \alpha \in (0, \frac{\pi}{2m + 1}), \frac{\pi}{m + 1}); m = 1, 2, \ldots \).
2.4. Auxiliary model problems

In addition to model problem (2.1) we will need the analogous model transmission problem with
\( G_1 = R_+^2 = \{(y_1, y_2) : y_2 > 0 \} \), \( G_2 = R_-^2 = \{(y_1, y_2) : y_2 < 0 \} \):

\[
\Delta v_i = g_i(y), \quad y \in R_\pm^2, \\
v_1 = v_2, \quad \frac{\partial v_1}{\partial y_2} - k \frac{\partial v_2}{\partial y_2} = \psi(y) \quad \text{on} \quad y_2 = 0.
\] (2.59)

**Proposition 2.7.** Let the finite functions \( g_i(y) \in C^\beta(\bar{G}_i), \ \psi(y) \in C^{1+\beta}(R), \ i = 1, 2 \). There exists a unique solution of problem (2.59) \( v_i(y) \in C^{2+\beta}(\bar{G}_i) \) and

\[
\|v_i\|_{C^{2+\beta}(\bar{G}_i)} \leq C \left( \|g_1\|_{C^\beta(\bar{G}_1)} + \|g_2\|_{C^\beta(\bar{G}_2)} + \|\psi\|_{C^{1+\beta}(R)} \right),
\]

where the constant \( C \) is independent of \( v_i \).

This result follows from [18].

Finally, we will use the well-known results on the solvability for the Dirichlet problem in a half-space.

**Proposition 2.8.** Let the functions \( g(y) \) and \( g_3(y_1) \) be finite and \( g(y) \in C^\beta(R_+^2), \ g_3(y) \in C^{2+\beta}(R) \). There exists a unique solution \( u(y) \) of the problem

\[
\Delta u = g(y), \quad y \in R_+^2, \quad u|_{y_2=0} = g_3(y_1),
\] (2.60)

such that

\[
\|u\|_{C^{2+\beta}(R_+^2)} \leq C \left( \|g\|_{C^\beta(R^2_+)} + \|g_3\|_{C^{2+\beta}(R)} \right),
\]

where the constant \( C \) is independent of \( u \).

3. The proof of Theorem 1.1

In the beginning we prove Theorem 1.1 in the case of \( \psi_0(y) = 0 \). We will use the following notations and definitions which are represented below.

Denote by \( \lambda_n, \ n = 1, 2, \ldots, \) eigenvalues of the spectral problem for the functions \( z_i(\varphi), \ i = 1, 2, \)

\[
\frac{d^2 z_1}{d\varphi^2} + \lambda^2 z_1 = 0, \quad \varphi \in (\alpha, 2\pi), \sin \varphi = (0, \alpha), \sin \varphi = (0, \alpha),
\]

\[
z_1(\alpha) - z_2(\alpha) = 0, \quad z_1(2\pi) - z_2(0) = 0, \quad z_1(\alpha) - z_2(\alpha) = 0, \quad z_1(2\pi) - z_2(0) = 0,
\]

\[
\frac{dz_1}{d\varphi}(\alpha) - k \frac{dz_2}{d\varphi}(\alpha) = 0, \quad \frac{dz_1}{d\varphi}(2\pi) - k \frac{dz_2}{d\varphi}(0) = 0.
\] (3.1)

Notice that the problem arises when one looks for the nontrivial solutions to homogeneous problem (2.1). It is known [13] that there exists the countable increasing set of \( \lambda_n > 0, \ n = 1, 2, \ldots, \lambda_1 \in (0, 1) \).
Let $W^{l,p}(\Omega)$ be the standard Sobolev space with the norm $\|u\|_{l,p;\Omega}$ where, for example, for an integer $l > 0$

$$\|u\|_{l,p;\Omega} = \left(\sum_{|\beta| \leq l} \int_{\Omega} |D^\beta u(x)|^p \, dx\right)^{1/p},$$

and let $W^{l,p}_\gamma(\Omega)$ be the weighted Sobolev space with the norm $\|u\|_{l,p;\gamma,\Omega}$ where in the case of an integer $l > 0$

$$\|u\|_{l,p;\gamma,\Omega} = \left(\sum_{|\beta| \leq l} \int_{\Omega} r^{(\gamma + |\beta| - l)p} |D^\beta u(x)|^p \, dx\right)^{1/p}.$$

The case of arbitrary $l$ can be found in [1,13].

Now we apply some results from [13], in particular those that follow from Theorem 3.12 [13] for the Dirichlet condition on $\Gamma_0$, to problem (1.1) using some evident renaming in the notation.

Let in problem (1.1) $\lambda_n \neq 2 - \gamma - \frac{2}{p}$ for all $n$, and $1 \neq 2 - \gamma - \frac{2}{p} > 0$; $g_i \in W^{0,p}_\gamma(\Omega_i)$, $i = 1, 2$, $\psi \in W^{1-1/p,p}_\gamma(\Gamma)$, $g_3 \in W^{2-1/p,p}_\gamma(\Gamma_0)$. Then by Theorem 3.12 [13] there exists a unique solution $u_i \in W^{1,2}(\Omega_i)$ to problem (1.1), which admits the expansion:

$$u_i(y) = u_{0i}(y) + \sum_{\lambda_n \in (0, 2 - \gamma - 2/p)} T_n(g_1, g_2, \psi, g_3) S_n(y), \quad (3.2)$$

where $u_{0i} \in W^{2,p}_\gamma(\Omega_i)$ and $T_n$ is the bounded functional on the corresponding space, $S_n(y)$ is some given function which contains the factor $r^{\lambda_n}$, and

$$\|u_{0i}\|_{2,p;\gamma,\Omega_1} + \|u_{02}\|_{2,p;\gamma,\Omega_2} + \max_{\Omega} \sum_{\lambda_n \in (0, 2 - \gamma - 2/p)} |T_n S_n| \leq C D_1(g_1, g_2, \psi, g_3) \quad (3.3)$$

with

$$D_1(g_1, g_2, \psi, g_3) = \|g_1\|_{0,p;\gamma,\Omega_1} + \|g_2\|_{0,p;\gamma,\Omega_2} + \|\psi\|_{1-1/p,p;\gamma,\Gamma} + \|g_3\|_{2-1/p,p;\gamma,\Gamma_0}.$$

We use estimate (3.3) to evaluate $\max_{\Omega_i} |r^{-\sigma} u_i|$. The latter quantity is necessary for a priori estimate of the solution in Theorem 1.1. One can easily check that for $\gamma \leq -\sigma$ and $p > 1$

$$\max_{\Omega_i} |r^{-\sigma} u_{0i}| \leq \text{const.} \|r^{-\sigma} u_{0i}\|_{2,p;\Omega_i} \leq C \|u_{0i}\|_{2,p;\gamma,\Omega_i}, \quad (3.4)$$

where the first inequality follows from the embedding theorem, and the second is directly testable by the definition of the spaces.

Let, in addition,

$$-\sigma - 1/p < \gamma \leq -\sigma, \quad (3.5)$$

then
\[ D_1(g_1, g_2, \psi, g_3) \leq C \left( \|g_1\|_{E^\beta(\Omega_1)} + \|g_2\|_{E^\beta(\Omega_2)} + \|g_3\|_{E^{2\beta}(\partial_0)} + \|\psi\|_{E^{1+\beta}(\overline{\Gamma})} \right) \]

\[ \equiv CD_2(g_1, g_2, \psi, g_3). \]  

(3.6)

Inequalities (3.4), (3.3) and (3.6) lead to

\[ \max_{\Omega_i} \left| r^{-\sigma} u_{0i} \right| \leq CD_2(g_1, g_2, \psi, g_3). \]

(3.7)

Let \( p \) and \( \gamma \) satisfy the inequalities \( 0 < 2 - \gamma - 2/p < \lambda_1 \). Since \(-\sigma - 1/p < \gamma\) then

\[ 0 < 2 - \gamma - 2/p < 2 + \sigma - 1/p < \lambda_1. \]

(3.8)

Thus one can see that the assumption of Theorem 3.12 [13] is satisfied if inequalities (3.8) hold.

It is easy to obtain from (3.8) that

\[ 1/p > 2 + \sigma - \lambda_1, \]

(3.9)

\[ \gamma < 2 - 2/p. \]

(3.10)

If \( 2 + \sigma - \lambda_1 > 0 \), then, as it follows from (3.9),

\[ p < \frac{1}{2 + \sigma - \lambda_1}, \]

but for \( 2 + \sigma - \lambda_1 \leq 0 \) estimate (3.9) holds for all \( p \).

Inequality (3.10) together with (3.5) mean

\[ -\sigma - 1/p < \gamma < \min(-\sigma, 2 - 2/p). \]

These inequalities are true if \( p > \frac{1}{2 + \sigma} \).

Summing up the all written above, we can conclude that the conditions of Theorem 3.12 [13] are satisfied if either \( 2 + \sigma > \lambda_1 \) and \( p \in \left( \frac{1}{2 + \sigma}, \frac{1}{2 + \sigma - \lambda_1} \right) \), or \( 2 + \sigma \leq \lambda_1 \) and \( p \in \left( \frac{1}{2 + \sigma}, \infty \right) \). Note that, due to \( 2 - \gamma - 2/p < \lambda_1 \), the second term in (3.2) is absent. Thus the existence and uniqueness of the solution to problem (1.1) are a consequence of Theorem 3.12 [13].

Finally, it remains to check that the solution \( u_i, i = 1, 2 \), of problem (1.1) satisfies inequality (1.3).

Let us remark that the smoothness of the functions \( u_i(y) \) inside \( \Omega_i \), e.g., \( u_i \in C^{2+\beta}(\Omega_i) \), follows from the general theory for the elliptical boundary value problems. Therefore, to finish the proof of Theorem 1.1 for \( \psi_0 = 0 \), we apply the partition of unity together with the local estimates from Theorem 2.1, Propositions 2.7 and 2.8.

Now we show how problem (1.1) can be transformed to the case of \( \psi_0(y) = 0 \). For this purpose we consider the Dirichlet problem

\[ \Delta u = 0, \quad y \in \Omega_2, \quad u|_{\Gamma} = \psi_0(y). \]  

(3.11)

Problem (3.11) can be studied with the same technique as we used above. The main difficulties arise during the investigation of the model problem in the plane corner \( G = \{(y_1, y_2): y_1 \in (0, \infty), \quad 0 < y_2 < y_1 \tan \alpha\} \) with the boundary \( \Gamma = \{(y_1, y_2): y_1 \in (0, \infty), \quad y_2 = 0\} \cup \{(y_1, y_2): y_1 \in (0, \infty), \quad y_2 = y_1 \tan \alpha\} \):

\[ \Delta u = f(y), \quad y \in G, \quad u|_{\Gamma} = \psi_0(y). \]  

(3.12)
The solution of problem (3.12) can be represented with the volume and double-layer potentials. The corresponding estimates of $u$ can be got with methods similar to those used in Section 2. In addition, we use results [4] on the one-to-one solvability of problem (3.11) in the weighted Sobolev space. A priori estimates in problem (3.12) allow us to get the following: a unique solution $u(y) \in E^{2+\beta}_{2+\sigma}(\tilde{Q}_2)$ of (3.11) satisfies

$$\|u\|^{2+\beta}_{E^{2+\sigma}_{2+\sigma}(\tilde{Q}_2)} \leq C \|\psi_0\|^{2+\beta}_{E^{2+\sigma}_{2+\sigma}(I')}.$$  \hfill (3.13)

This completes the proof of Theorem 1.1.

**Remark 3.1.** Theorem 1.1 is true in the cases (see (2.21), (2.22)): $2 + \sigma \in (m + \frac{1}{2}, \frac{2m + \pi}{2\pi - \alpha})$ for $\alpha \in (0, \frac{\pi}{2m + 1})$; $2 + \sigma \in (\frac{\pi}{2m + 1}, \frac{2m + \pi}{2\pi - \alpha})$ for $\alpha \in (\frac{\pi}{2m + 1}, \frac{\pi}{m + 1})$; $m = 1, 2, \ldots$.

The proof of Remark 3.1 repeats the arguments of Theorem 1.1 and uses essentially the results of Remark 2.1. However, it is necessary to check that the assumptions of Theorem 3.12 [13] are fulfilled in this case. For instance, if $m = 2$ and $\alpha \in (0, \pi/5)$, we can put $\gamma = -\sigma$ and $p > 1$. Then it easy to show inequalities (3.4) and (3.6) hold. If, in addition, we choose $p \neq 2/\gamma$ and $p \in (\frac{2}{\gamma - 2\pi - 1}, \frac{2}{\gamma - 2\pi - 2})$, then the conditions from Theorem 3.12 [13] are satisfied and, as a result, inequality (3.7) is fulfilled.

**Appendix A**

In this section we give the proofs of some estimates used in the arguments above.

**A.1. The proof of Proposition 2.1**

For the sake of definiteness we assume $z > 0$. To estimate the function $G(z, \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(\lambda, \xi) \times e^{iz\lambda} \, d\lambda$, it is useful to calculate the integral along the shifted contour: $\lambda = \mu_1 + i\mu_2$. Then

$$G(z, \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\mu_1 z - \mu_2 z}}{D(\mu_1 + i\mu_2)(\mu_1 + s - \mu_2)} \cos(i\mu_1 + s - \mu_2)(2\pi - \xi) \sin(i\mu_1 + s - \mu_2)(2\pi - \alpha) \, d\mu_1,$$

where the denominator of the integrand is nonzero owing to the assumption $\frac{1}{2} + m < s - \mu_2 < \frac{2m + \pi}{2\pi - \alpha}$.

The asymptotics of the function $\tilde{G}(\mu_1 + i\mu_2, \xi)$ is

$$\tilde{G}(\mu_1 + i\mu_2, \xi) \sim \begin{cases} \mp iCe^{-|\mu_1|}(\xi - \alpha) e^{i(s - \mu_2)(\xi - \alpha)} \big/ i(\mu_1 + s - \mu_2), & \mu_1 \to \pm \infty, \\ C \cos(s - \mu_2)(2\pi - \xi) \sin(s - \mu_2)(2\pi - \alpha), & \mu_1 \to 0. \end{cases} \hfill (A.1)$$

To obtain (2.27), it suffices to estimate the following integral

$$B(z, \xi) = ie^{-\mu_2 z} \int_{-\infty}^{\infty} e^{i\mu_1(\xi - \alpha)} e^{-i(s - \mu_2)(\xi - \alpha)} \frac{e^{iz\mu_1}}{i\mu_1 + s - \mu_2} \, d\mu_1$$

$$- ie^{-\mu_2 z} \int_{-\infty}^{\infty} e^{-i\mu_1(\xi - \alpha)} e^{i(s - \mu_2)(\xi - \alpha)} \frac{e^{iz\mu_1}}{i\mu_1 + s - \mu_2} \, d\mu_1$$

$$+ e^{-\mu_2 z} \int_{-\infty}^{\infty} \cos(s - \mu_2)(2\pi - \xi) \sin(s - \mu_2)(2\pi - \alpha) e^{iz\mu_1} \, d\mu_1, \hfill (A.2)$$

where $N$ is some fixed positive number.
By integrating, we get

\[ B_1(z, \xi) = e^{-\mu_2 z} \int_0^\infty e^{-\mu_1 (\xi - \alpha)} e^{i(s-\mu_2)(\xi - \alpha)} \frac{e^{iz\mu_1}}{i\mu_1 + s - \mu_2} \, d\mu_1, \tag{A.3} \]

since

\[ \left| \int_0^N e^{-\mu_1 (\xi - \alpha)} e^{i(s-\mu_2)(\xi - \alpha)} \frac{e^{iz\mu_1}}{i\mu_1 + s - \mu_2} \, d\mu_1 \right| \leq C. \]

Then

\[ |B_1(z, \xi)| \leq e^{-\mu_2 z} \left\{ \int_0^\infty \frac{i\mu_1 e^{-\mu_1 (\xi - \alpha) - iz}}{\mu_1^2 + (s - \mu_2)^2} \, d\mu_1 + \int_0^\infty \frac{(s - \mu_2) e^{-\mu_1 (\xi - \alpha) - iz}}{\mu_1^2 + (s - \mu_2)^2} \, d\mu_1 \right\} \]

\[ \leq Ce^{-\mu_2 z} (J(\xi) + 1). \]

The first term in (A.2) can be studied by the same arguments. That finishes the proof of (2.27).

It is easy to check inequality (2.28). Indeed,

\[ \int^\infty_0 \frac{d\mu_1}{\mu_1^2 + (s - \mu_2)^2} \left( 1 - e^{-\mu_1 (2\pi - \alpha)} \right) \leq C. \]

Let us represent the function \( \frac{\partial G}{\partial z}(z, \xi) \) as

\[ \frac{\partial G}{\partial z}(z, \xi) = -sG(z, \xi) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\mu_1 z - \mu_2 z}}{D(\mu_1 + i\mu_2)} \cos(i\mu_1 + s - \mu_2)(2\pi - \xi) \, d\mu_1 \]

\[ = i_1 + i_2. \tag{A.4} \]

Keeping in mind the asymptotics for the functions \( \cos(x + iy), \sin(x + iy) \), with real \( x \) and \( y \), as \( y \to \pm \infty \), \( y \to 0 \), we will prove estimate (2.29) if we repeat argument used to prove inequality (2.27) and consider the following integral

\[ B_2(z, \xi) = e^{-\mu_2 z} \left\{ \int_{-\infty}^0 e^{(\xi - \alpha)(\mu_1 - i(s - \mu_2))} e^{iz\mu_1} \, d\mu_1 - \int_0^\infty e^{(\xi - \alpha)(i(s - \mu_2) - \mu_1)} e^{iz\mu_1} \, d\mu_1 \right\}. \]

By integrating, we get

\[ |B_2(z, \xi)| = \left| 2ie^{-\mu_2 z} \int_{\xi - \alpha}^{\xi - \alpha + iz} e^{-i(s - \mu_2)(\xi - \alpha)} \frac{\im e^{-i(s - \mu_2)(\xi - \alpha)}}{\xi - \alpha + iz} \right| \leq Ce^{-\mu_2 z} \frac{\xi - \alpha + z}{(\xi - \alpha)^2 + z^2}. \tag{A.5} \]
Thus, the second term in the right-hand side of (A.4) is evaluated as

$$|i_2| \leq C e^{-\mu z} \left[ 1 + \frac{\xi - \alpha + z}{(\xi - \alpha)^2 + z^2} \right].$$

Note that the estimate of $i_1$ is a simple consequence of (2.27). Hence the last inequality together with (2.27) give (2.29).

To prove (2.30) we consider the integral

$$I = \int_{\alpha}^{\infty} B_2(\delta, \xi) d\xi = e^{-\mu_2 \delta} \int_{\alpha}^{2\pi} d\xi \int_{-\infty}^{0} e^{(\xi-\alpha)(\mu_1 - i(s-\mu_2))} e^{i\beta \mu_1} d\mu_1$$

$$- e^{-\mu_2 \delta} \int_{\alpha}^{2\pi} d\xi \int_{0}^{\infty} e^{i\beta \mu_1} e^{(\xi-\alpha)(-\mu_1 + i(s-\mu_2))} d\mu_1$$

$$= e^{-\mu_2 \delta} \int_{-\infty}^{0} e^{i\beta \mu_1} d\mu_1 \int_{\alpha}^{2\pi} e^{(\xi-\alpha)(\mu_1 - i(s-\mu_2))} d\xi$$

$$- e^{-\mu_2 \delta} \int_{0}^{\infty} e^{i\beta \mu_1} d\mu_1 \int_{\alpha}^{2\pi} e^{(\xi-\alpha)(-\mu_1 + i(s-\mu_2))} d\xi$$

$$= e^{-\mu_2 \delta} \int_{-\infty}^{0} e^{i\beta \mu_1} \frac{e^{(2\pi - \alpha)(\mu_1 - i(s-\mu_2))} - 1}{\mu_1 - i(s-\mu_2)} d\mu_1$$

$$- e^{-\mu_2 \delta} \int_{0}^{\infty} e^{i\beta \mu_1} \frac{e^{(2\pi - \alpha)(-\mu_1 + i(s-\mu_2))} - 1}{-\mu_1 + i(s-\mu_2)} d\mu_1.$$ \hspace{1cm} (A.6)

Here we evaluate the following difference from the right-hand side of (A.6)

$$I = \int_{0}^{\infty} e^{i\beta \mu_1} \frac{d\mu_1}{-\mu_1 + i(s-\mu_2)} - \int_{-\infty}^{0} e^{i\beta \mu_1} \frac{d\mu_1}{\mu_1 - i(s-\mu_2)}. \hspace{1cm} (A.7)$$

Let us remark that the uniform estimates with respect to $\delta$ of the remaining terms on the right-hand side of (A.6) are evident due to the presence of the factor $e^{-(2\pi - \alpha)|\mu_1|}$. We have

$$I = \int_{0}^{\infty} \left\{ \frac{e^{i\beta \mu_1}}{-\mu_1 + i(s-\mu_2)} + \frac{e^{-i\beta \mu_1}}{\mu_1 + i(s-\mu_2)} \right\} d\mu_1 = 2i \int_{0}^{\infty} \text{Im} \frac{e^{-i\beta \mu_1}}{\mu_1 + i(s-\mu_2)} d\mu_1$$

$$= 2i \int_{0}^{\infty} \frac{e^{-i\beta \mu_1}(\mu_1 - i(s-\mu_2))}{\mu_1^2 + (s-\mu_2)^2} d\mu_1 = 2i \int_{0}^{\infty} \frac{-\mu_1 \sin(\mu_1 \delta) - (s-\mu_2) \cos(\mu_1 \delta)}{\mu_1^2 + (s-\mu_2)^2} d\mu_1$$

$$= -2i(j_1 + j_2). \hspace{1cm} (A.8)$$
It is obvious that \(|j_2| \leq C\) for all \(\delta\). As for \(j_1\), in the case of \(\delta \geq 0\) we get

\[
j_1 = \int_0^\infty \frac{\mu \sin(\mu \delta)}{\mu^2 + (s - \mu_2)^2} d\mu_1 = \int_0^\infty \frac{y \sin y}{y^2 + \delta^2(s - \mu_2)^2} dy = \int_0^{\pi/2} \frac{y \sin y}{y^2 + \delta^2(s - \mu_2)^2} dy
\]

\[+ \int_\pi^{2\pi} \frac{y \sin y}{y^2 + \delta^2(s - \mu_2)^2} dy.\]

One can easily show the first integral is uniformly restricted with respect to \(\delta\). The uniform boundedness of the second integral can be proved with integrating by parts. In the case of \(\delta < 0\) the estimate of \(j_1\) is studied in the same way.

Now we study the function \(\frac{\partial i_2}{\partial z}(z, \xi)\) (see (A.4)). Note that it is enough to estimate \(\frac{\partial i_2}{\partial z}(z, \xi)\). It is not difficult to see the calculations similar to those performed above (see the proof of (A.5)) give

\[
\left|\frac{\partial i_2}{\partial z}(z, \xi)\right| \leq C e^{-\mu_2 z} \left(1 + \frac{1 + \xi - \alpha + z}{z^2 + (\xi - \alpha)^2}\right).
\]

Due to presence of \(e^{-\mu_2 z}\) in the representation \(i_2(z, \xi)\), one can obtain the estimate of \(\left|\frac{\partial^2 i_2}{\partial z^2}(z, \xi)\right|\) like (A.9). This leads to the inequality for \(\frac{\partial^2 i_2}{\partial z^2}(z, \xi)\) in (2.31).

### A.2. The proof of Proposition 2.5

There are the following asymptotic representations of the function \(\tilde{K}(\lambda)\)

\[
\tilde{K}(\lambda) \sim \begin{cases}
C(\cot s(2\pi - \alpha) + k \cot s\alpha), & \lambda \to 0, \\
\mp \frac{i}{(1 + k)^2(\lambda + s)}, & \lambda \to \pm \infty.
\end{cases}
\]

(A.10)

The behavior of \(\tilde{K}(\lambda)\) for \(|\lambda| \to \infty\) and the absence of poles in \(\tilde{K}(\lambda)\) due to selection of \(s\) allow us to change the variables: \(\lambda = \mu_1 + i\mu_2\) (so that \(\lambda + s = i\mu_1 + s - \mu_2\)) in the calculation of \(K(z)\).

For the sake of clarity we will consider the case of \(\mu \geq 0\), then

\[
K(z) = \frac{e^{-\mu_2 z}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iz\mu_1} d\mu_1}{(i\mu_1 + s - \mu_2)D(\mu_1 + i\mu_2)}
\]

\[\times \left[\cot(i\mu_1 + s - \mu_2)(2\pi - \alpha) + k \cot(i\mu_1 + s - \mu_2)\alpha\right].\]

As it follows from the asymptotics of \(\tilde{K}(\lambda)\) (see (A.10)), it is enough to prove Proposition 2.5 for the following function

\[
K_1(z) = e^{-\mu_2 z} \int_{-\infty}^{\infty} e^{iz\mu_1} \left[\cot s(2\pi - \alpha) + k \cot s\alpha\right] d\mu_1 + \int_{0}^{\infty} \frac{e^{iz\mu_1} d\mu_1}{(1 + k)^2(\mu_1 + s - \mu_2)}
\]

\[\mp \frac{i e^{-\mu_2 z}}{(1 + k)^2} \int_{-\infty}^{0} \frac{e^{iz\mu_1} d\mu_1}{i\mu_1 + s - \mu_2} - \frac{i e^{-\mu_2 z}}{(1 + k)^2} \int_{0}^{\infty} \frac{e^{iz\mu_1} d\mu_1}{i\mu_1 + s - \mu_2}.
\]
\[ + \frac{ie^{-\mu_2 z}}{(1+k)^{\frac{1}{2}}} \int_{-\infty}^{0} \frac{e^{iz\mu_1} d\mu_1}{i\mu_1 + s - \mu_2} \equiv e^{-\mu_2 z}(i_1 + i_2 + i_3), \]  
\quad \text{(A.11)}

where \( n \) is a fixed number, and \( i_1 \) is the sum of the first three integrals in (A.11).

It is clear that
\[ |i_1(z)| \leq C. \]  
\quad \text{(A.12)}

Thereafter
\[ i_2 + i_3 = \text{const.} \text{Im} \int_{0}^{\infty} \frac{e^{iz\mu_1} d\mu_1}{i\mu_1 + s - \mu_2} \]
\[ = C \left\{ \int_{0}^{\infty} \frac{(s - \mu_2) \sin z\mu_1}{\mu_1^2 + (s - \mu_2)^2} d\mu_1 - \int_{0}^{\infty} \frac{\mu_1 \cos z\mu_1}{\mu_1^2 + (s - \mu_2)^2} d\mu_1 \right\} . \]

Then, integrating by parts in the second term gives
\[ |i_2 + i_3| \leq C(1 + z^{-1}). \]  
\quad \text{(A.13)}

Thus, inequalities (A.12) and (A.13) lead to estimate (2.48).

Now we estimate \( \frac{\partial K_1(z)}{\partial z} \):
\[ \frac{\partial K_1(z)}{\partial z} = -\mu_2 K_1(z) + e^{-z\mu_2} \left( \frac{\partial i_1(z)}{\partial z} + \frac{\partial i_2(z)}{\partial z} + \frac{\partial i_3(z)}{\partial z} \right) . \]

It is easy to see that
\[ \left| \frac{\partial i_1(z)}{\partial z} \right| \leq C, \]  
\quad \text{(A.14)}

and
\[ \frac{\partial i_2(z)}{\partial z} + \frac{\partial i_3(z)}{\partial z} = \text{const.} \text{Im} \int_{0}^{\infty} \frac{i\mu_1 e^{iz\mu_1} d\mu_1}{i\mu_1 + s - \mu_2} \]
\[ = \text{const.} \text{Im} \int_{0}^{\infty} e^{iz\mu_1} d\mu_1 - \text{const.} \text{Im} \int_{0}^{\infty} \frac{(s - \mu_2) e^{iz\mu_1} d\mu_1}{i\mu_1 + s - \mu_2} \]
\[ = \text{const.} \text{Im}(\pi \delta(-z) + iz^{-1}) - \text{const.}(s - \mu_2)(i_2 + i_3), \]
where we used Lemma 5.3 from [2], and \( \delta(z) \) is the delta-function. This representation gives the following estimate
\[ \left| \frac{\partial i_2(z)}{\partial z} + \frac{\partial i_3(z)}{\partial z} \right| \leq C(1 + z^{-1}). \]  
\quad \text{(A.15)}

Thus, estimate (2.49) immediately follows from (A.14) and (A.15).
Statement (2.50) follows from the boundedness of \( \frac{\partial^2 (i_1 + i_3)}{\partial z^2} \) and the estimate of \( \frac{\partial (i_2 + i_3)}{\partial z} \) through \( \partial (i_2 + i_3) \) and \( z^{-2} \).

To prove (2.51), it is enough to integrate the functions \( \frac{\partial (i_2 + i_3)}{\partial z} \). Thus, we obtain

\[
\int_{-\delta}^{\delta} \frac{\partial (i_2 + i_3)}{\partial z} \, dz = \text{const.} \int_0^\infty \frac{d\mu_1}{i \mu_1 + s - \mu_2} \int_{-\delta}^{\delta} i \mu_1 e^{iz\mu_1} \, dz
\]

\[
= \text{const.} \int_0^\infty \frac{i \mu_1}{i \mu_1 + s - \mu_2} \, d\mu_1 = \text{const.} \int_0^\infty \frac{\sin(\mu_1 \delta)}{\mu_1^2 + (s - \mu_2)^2} \, d\mu_1 \leq C.
\]

References