Discrete Mathematics 311 (2011) 817–826

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

M-alternating paths and the construction of defect n-extendable bipartite graphs with different connectivities

Xuelian Wen [a,*], Zan-Bo Zhang [b], Dingjun Lou [c]

[a] School of Economics and Management, South China Normal University, Guangzhou 510006, PR China
[b] Department of Computer Engineering, Guangdong Industry Technical College, Guangzhou 510300, PR China
[c] Department of Computer Science, Sun Yat-Sen University, Guangzhou 510275, PR China

ARTICLE INFO

Article history:
Received 14 August 2009
Received in revised form 28 January 2011
Accepted 28 January 2011
Available online 2 March 2011

Keywords:
A near perfect matching
Defect n-extendable
An M-alternating path

ABSTRACT

A **near perfect matching** is a matching covering all but one vertex in a graph. Let G be a connected graph and $n \leq (|V(G)| - 2)/2$ be a positive integer. If any n independent edges in G are contained in a near perfect matching, then G is said to be **defect n-extendable**. In this paper, we first characterize defect n-extendable bipartite graph G with $n = 1$ or $\kappa(G) \geq 2$ respectively using M-alternating paths. Furthermore, we present a construction characterization of defect n-extendable bipartite graph G with $n \geq 2$ and $\kappa(G) = 1$. It is also shown that these characterizations can be transformed to polynomial time algorithms to determine if a given bipartite graph is defect n-extendable.

1. Terminology and introduction

We only consider bipartite graphs. All graphs considered in this paper are undirected, finite and simple.

A matching covering all but d vertices in a graph G is a **defect *d* matching** in G. A defect 0 matching is also called a **perfect matching** and a defect 1 matching is also called a **near perfect matching**. Let G be a connected graph and $n \leq (|V(G)|-2)/2$ be a positive integer. If any n independent edges in G are contained in a perfect matching of G, then G is ***n*-extendable**. If any n independent edges in G are contained in a near perfect matching of G, then G is **defect *n*-extendable**. Particularly, if G has a perfect matching, then G is **0-extendable** and if G has a near perfect matching, then G is **defect 0-extendable**.

We use $\boldsymbol{G} = (\boldsymbol{U}, \boldsymbol{W})$ to denote a bipartite graph G with bipartition U, W. Let A and B be two sets. Then $\boldsymbol{A \Delta B}$ denotes the symmetric difference of A and B. Let G be a graph and $S \subseteq V(G)$. Then $\boldsymbol{\Gamma_G(S)}$ denotes the neighbor set of S in G, and the minimum degree and the connectivity of G are denoted by $\boldsymbol{\delta(G)}$ and $\boldsymbol{\kappa(G)}$ respectively. Throughout this paper, $\boldsymbol{p}$ and $\boldsymbol{q}$ denote the number of vertices and edges of the given graph respectively.

A path from vertex x to vertex y is called an ***xy*-path**. If a path P contains vertices u and v, then we use ***uPv*** to denote the path from u to v in P. Let G be a graph and M be a matching of G. An ***M*-alternating path (cycle)** of G is a path (cycle) in G where edges in M and edges in $E(G) \setminus M$ appear on the path (cycle) alternately. In this paper, we only consider alternating paths starting from an edge not in the given matching. In other words, when we say that $P = a_1a_2 \ldots a_k$ is an M-alternating a_1a_k-path in a graph G, it always means that $a_ia_{i+1} \in E(G) \setminus M$ if i is odd and $a_ia_{i+1} \in M$ if i is even.

For the other terminology and notations not defined in this paper, the reader is referred to [2].

The concept of defect n-extendable graph was introduced by Lou and Wen [10]. They showed that the connectivity of a defect n-extendable graph can be any positive integer. While Plummer [7] proved that the connectivity of an n-extendable

* Corresponding author.
E-mail address: wen_sysu@yahoo.com.cn (X. Wen).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2011.01.028

graph is not less than $n+1$, which implies that the results on defect n-extendable graphs may not be deduced trivially from those of n-extendable graphs.

In fact, a few results on defect n-extendable graphs have been established until now.

In [3], Little et al. gave two characterizations of defect 1-extendable graphs which were called 1-covered graph in their paper.

A graph is said to be ***k*-critical** if deleting any k of its vertices, the remaining subgraph has a perfect matching. To combine the concept of n-extendable graphs and k-critical graphs, Liu and Yu [4] introduced (k, n, d)-graphs. Let G be a graph. Let k, n, and d be non-negative integers such that $k + 2n + d \leq |V(G)| - 2$ and $|V(G)| - k - d$ is even. Then G is called a **$(\boldsymbol{k}, \boldsymbol{n}, \boldsymbol{d})$-graph** if deleting any k of its vertices, the subgraph contains a matching of size n and every matching of size n can be extended to a defect d matching. It is not difficult to see that $(0, n, 1)$-graphs are the same as defect n-extendable graphs. They gave a Tutte style characterization and a property of (k, n, d)-graphs which can directly deduce a characterization of defect n-extendable graphs.

Lou and Wen found the path decomposition of defect 1-extendable bipartite graphs in [11] and gave the characterization of defect n-extendable bipartite graphs with different connectivities in [10] (see Theorems 1.1 and 1.2).

Theorem 1.1 (*Wen and Lou [10]*). *Let $G = (U, W)$ be a bipartite graph with $\kappa(G) = 1$ and $|W| = |U| + 1$, x be a cut vertex of G and $H = (X, Y)$ be a component in $G - x$. Let n be a positive integer with $n \leq (|V(G)| - 2)/2$. Then G is defect n-extendable if and only if the following statements hold:*

(1) $||X| - |Y|| \leq 1$.
(2) *Either there are exactly two odd components and no even component in $G - x$, or all components in $G - x$ are even.*
(3) *If $|X| = |Y| = m$, then H is s-extendable and $G[V(H) \cup \{x\}]$ is defect t-extendable where $s = \min\{n-1, m-1\}$ and $t = \min\{n, m-1\}$.*
(4) *If $|X| = |Y| + 1 = m + 1$, then*
  (4.1) *$x \in U$, $Y \subseteq U$ and $X \subseteq W$.*
  (4.2) *if $m \geq 1$, then H is defect s-extendable where $s = \min\{n, m-1\}$.*
  (4.3) *for any $w \in V(H)$ such that $xw \in E(G)$, each component $H' = (X', Y')$ in $H - w$ with $|X'| = m'$ is t-extendable where $t = \min\{n-1, m'-1\}$.*
(5) *If $G - x$ has odd components and $|\Gamma_G(x) \cap V(H)| < |X|$ holds for each odd component $H = (X, Y)$ in $G - x$ with $|X| = |Y| + 1$, then $d_G(x) \geq n + 1$.*

Theorem 1.2 (*Wen and Lou [10]*). *Let $G = (U, W)$ be a bipartite graph with $|W| = |U| + 1$ and $\kappa(G) \geq 2$. Then G is defect n-extendable if and only if for any $S \subseteq W$ and $2 \leq |S| \leq |W| - n$, $|\Gamma_G(S)| \geq |S| + n - 1$.*

In spite of the considerable amount of work on the characterizations of defect n-extendable graphs, the fundamental problem: is there a polynomial time algorithm to determine if a graph G is a defect n-extendable graph (even bipartite graph) is not solved yet. In this paper, we solve this problem for bipartite graphs by characterizing defect n-extendable bipartite graph G using M-alternating path theory when $n = 1$ or $\kappa(G) \geq 2$ in Sections 3 and 4, and by giving a construction characterization of G when $\kappa(G) = 1$ and $n \geq 2$ in Section 5.

2. Preliminary results

In this section, we introduce some known results which will be used in the proof of the main results of this paper.

Lemma 2.1 (*Wen and Yang [12]*). *A bipartite graph $G = (U, W)$ with $|W| = |U| + 1$ and $\kappa(G) \geq 2$ is defect n-extendable if and only if for any $S \subseteq U$ and $1 \leq |S| \leq |U| - n$, $|\Gamma_G(S)| \geq |S| + n$.*

Lemma 2.2 (*Wen and Yang [12]*). *Let n be a positive integer and $G = (U, W)$ be a defect n-extendable bipartite graph with $|W| = |U| + 1$ and $\kappa(G) \geq 2$. Then for any $S \subseteq U$ and $|U| - n + 1 \leq |S| \leq |U|$, $|\Gamma_G(S)| \geq |W| - 1$.*

Lemma 2.3 (*Lou et al. [5]*). *Let $G = (U, W)$ be a bipartite graph with $|U| = |W|$, which has a perfect matching. Let $x \in U$ and $y \in W$. Let M and M_0 be perfect matchings of G. If G has k internally disjoint M_0-alternating xy-paths, then G also has k internally disjoint M-alternating xy-paths.*

Lemma 2.4 (*Liu and Yu [4]*). *A defect n-extendable graph is also defect $(n-1)$-extendable.*

Lemma 2.5 (*Wen and Lou [10]*). *Let n be an integer and $G = (U, W)$ be a defect n-extendable bipartite graph with $|W| = |U| + 1$. Then for any $w \in W$, each component in $G - w$ is k-extendable where $k = \min\{\kappa(G) - 1, n - 1\}$.*

Lemma 2.6 (*Aldred et al. [1]*). *Let $G = (U, W)$ be a bipartite graph which has a perfect matching. Then G is n-extendable if and only if for any perfect matching M and for each pair of vertices $x \in U$ and $y \in W$, there are n internally disjoint M-alternating xy-paths.*

Lemma 2.7 (*Plummer [8]*). *Let $G = (U, W)$ be a bipartite graph with $|W| = |U|$. Then G is n-extendable if and only if for any $S \subseteq U$ and $1 \leq |S| \leq |U| - n$, $|\Gamma_G(S)| \geq |S| + n$.*

Lemma 2.8 (*Plummer [7]*). *Let n be a positive integer. If G is n-extendable, then $\kappa(G) \geq n + 1$.*

3. *M*-alternating paths in defect 1-extendable bipartite graphs

Theorem 3.1. *Let $G = (U, W)$ be a bipartite graph with $|W| = |U| + 1$. Let M be a near perfect matching in G and w be the M-unsaturated vertex. Then G is defect 1-extendable if and only if for any $u \in U$, there is an M-alternating wu-path in G.*

Proof. Firstly, we prove the necessity. Assume $x_0, x_1, \dots, x_r$ are all the vertices in U that can be reached by an M-alternating path beginning with w. Let $A = \{x_i : 0 \le i \le r\}$. It suffices to prove that $A = U$.

Suppose to the contrary $A \ne U$. Since M is a near perfect matching and w is the only M-unsaturated vertex, there is a vertex y_i such that $x_i y_i \in M$ for any $0 \le i \le r$. Let $B = U \setminus A$, $C = \{y_i : 0 \le i \le r\}$ and $D = W \setminus (C \cup \{w\})$. Then $|A| = |C|$ and $|B| = |D| \ne 0$ since $A \ne U$.

Note that no vertex in B joins to any vertex in C. Otherwise, suppose there is a vertex $v \in B$ that joins to a vertex y_k, $0 \le k \le r$. Then $x_k \in A$ and hence there is an M-alternating wx_k-path P in G. Suppose $v \in V(P)$. Then wPv is an M-alternating wv-path and hence $v \in A$, a contradiction to $v \in B$. So $v \notin V(P)$. Clearly, both end edges in P are in $E(G) \setminus M$, $x_k y_k \in M$ and $y_k v \in E(G) \setminus M$. Thus $wPx_k y_k v$ is an M-alternating wv-path and hence $v \in A$, a contradiction to $v \in B$.

Analogously, we can prove that no vertex in B joins to w. So vertices in B can only join to vertices in $W \setminus (C \cup \{w\}) = D$. Then there is at least an edge between D and $U \setminus B = A$ as $\kappa(G) \ge 1$. Assume ux_j is such an edge where $u \in D$ and $x_j \in A$, $0 \le j \le r$. Since G is defect 1-extendable, ux_j is contained in a near perfect matching of G which matches vertices in $B \cup \{x_j\}$ to vertices in D. Hence $|D| \ge |B \cup \{x_j\}| = |B| + 1$, a contradiction to $|D| = |B|$. Hence $A = U$.

Now we prove sufficiency. Since G has a near perfect matching M and there is an M-alternating wv-path for all $v \in U$, G is connected. Choose any edge e in G. Assume $e = xy$ where $x \in U$ and $y \in W$. It is enough to prove that there is a near perfect matching in G containing e.

Since M is a near perfect matching and $x \in U$, there is a vertex y' in G such that $xy' \in M$.

Suppose $y = y'$. Then $e \in M$ and hence M is a near perfect matching in G containing e. Suppose $y = w$. Then $(M \cup \{e\}) \setminus \{xy'\}$ is a near perfect matching in G containing e. Suppose $y \in W \setminus \{y', w\}$. Since w is the only M-unsaturated vertex, there is a vertex x' in U such that $yx' \in M$. Then there is an M-alternating wx'-path P' in G. Note that P' begins and ends with an edge in $E(G) \setminus M$, thus $y \notin V(P')$.

If $x \in V(P')$, then $C = xP'x'yx$ is an M-alternating cycle and hence $M \Delta E(C)$ is a near perfect matching in G containing e.

If $x \notin V(P')$, then $y' \notin V(P')$ and hence $P = wP'x'yxy'$ is an M-alternating wy'-path beginning with w and ending with an edge in M. Clearly, $M \Delta E(P)$ is a near perfect matching in G containing e. □

Remark 3.2. Given a bipartite graph $G = (U, W)$ where $|W| = |U| + 1$. If G has no near perfect matching, then G is not defect 1-extendable. If G has a near perfect matching M where w is the M-unsaturated vertex, we can construct a directed graph $\vec{G}$ from G by giving orientation to all edges in M from U to W and orientation to the other edges of G from W to U. To identify if G is defect 1-extendable, Theorem 3.1 shows that we only need to check if for any vertex $u \in U$, there is an M-alternating wu-path in G, which is equal to check if in $\vec{G}$, w can reach all the vertices in U and can be done by doing a BFS (Breadth-First Search) of $\vec{G}$ beginning from vertex w. Since finding a near perfect matching in G needs $O(p^{1/2}q)$ time [6] and doing a BFS on $\vec{G}$ costs $O(p + q)$ time, verifying if G is defect 1-extendable can be done in $O(p^{1/2}q)$ time by Theorem 3.1.

4. *M*-alternating paths in defect *n*-extendable bipartite graph *G* with $\kappa(G) \ge 2$

Let $G = (U, W)$ be a bipartite graph with $|W| = |U| + 1$ and M be a near perfect matching of G, $u \in U$ and $v \in W$ be a pair of vertices.

The **predecessor of a vertex** in an M-alternating path or M-alternating cycle in G is defined as follows:

If $P = a_1 a_2 \dots a_k$ is an M-alternating path in G such that $a_1 \in W$, we define the predecessor of vertex a_i $(i \ne q1)$ in P, denoted by a_i^{-P}, to be $a_i^{-P} = a_{i-1}$;

If C is an M-alternating cycle in G, each vertex a has exactly two neighbors a' and a'' in C with $aa' \in M$ and $aa'' \in E(G) \setminus M$. Then we define the predecessor of vertex a in C, denoted by a^{-C}, to be $a^{-C} = a'$ if $a \in W$ and $a^{-C} = a''$ if $a \in U$.

Let k be a positive integer. If $P_1, P_2, \dots, P_{k-1}$, are $k - 1$ M-alternating vu-paths in G, Q is an M-alternating path starting from v and ending at some vertex $y \in W$ (possibly $y = v$) and Γ is a set of M-alternating cycles in G, then $S = (P_1, P_2, \dots, P_{k-1}, Q, \Gamma)$ is said to be **a *k*-system of *G* with respect to *v* and *u*** if it satisfies:

(1) $P_1, P_2, \dots, P_{k-1}$ are internally disjoint;
(2) $V(P_i) \cap V(Q) = \{v\}$ for each $1 \le i \le k - 1$;
(3) the cycles in Γ are mutually vertex disjoint;
(4) $(\bigcup_{i=1}^{k-1} V(P_i) \cup V(Q)) \cap V(C_i) \subseteq \{v\}$ for all $C_i \in \Gamma$.

Given a k-system $S = (P_1, P_2, \dots, P_{k-1}, Q, \Gamma)$ with respect to v and u. We define $\boldsymbol{V(S)}$ and $\boldsymbol{E(S)}$ by $V(S) = \bigcup_{i=1}^{k-1} V(P_i) \cup V(Q) \cup \bigcup_{c \in \Gamma} V(C)$ and $E(S) = \bigcup_{i=1}^{k-1} E(P_i) \cup E(Q) \cup \bigcup_{c \in \Gamma} E(C)$. And for each $x \in U \setminus \{u\}$, **the predecessor of *x* with respect to *S***, denoted by $\boldsymbol{x^{-S}}$, is defined as:

(1) If $x \in V(P_i)$, then $x^{-S} = x^{-P_i}$;
(2) If $x \in V(Q)$, then $x^{-S} = x^{-Q}$;
(3) If $x \in V(C)$ for some $C \in \Gamma$, then $x^{-S} = x^{-C}$;
(4) If $x \notin V(S)$, then $x^{-S} = y$ where $xy \in M$ for some $y \in W$.

Furthermore, for any $V \subseteq U \setminus \{u\}$, we define $\boldsymbol{V^{-S}}$ by $V^{-S} = \{x^{-S} | x \in V\}$.

The functions $\boldsymbol{A(i, S)} : \{(i, S) | i$ is an integer and S is a k-system of G with respect to v and $u\} \to W$ and $\boldsymbol{B(i, S)} : \{(i, S) | i$ is an integer and S is a k-system of G with respect to v and $u\} \to U$ are inductively defined as follows:

$$B(i, S) = \begin{cases} \phi & \text{if } i = 0 \\ \Gamma_{G-u}(A(i-1, S)) & \text{if } i \geq 1 \end{cases}$$

$$A(i, S) = \begin{cases} v & \text{if } i = 0 \\ A(i-1, S) \cup (B(i, S))^{-S} & \text{if } i \geq 1. \end{cases}$$

It is easily seen that $A(i, S) \subseteq W$, $B(i, S) \subseteq U$, $B(i, S) \subseteq B(j, S)$ and $A(i, S) \subseteq A(j, S)$ when $i < j$. Let S be a k-system in G, we define $\boldsymbol{A(S)}$ and $\boldsymbol{B(S)}$ by $A(S) = \bigcup_{i=0}^{\infty} A(i, S)$ and $B(S) = \bigcup_{i=0}^{\infty} B(i, S)$. Moreover, for each $z \in A(S) \cup B(S)$, **the height of z with respect to S**, denoted by $\boldsymbol{h(z, S)}$, is defined by

$$h(z, S) = \begin{cases} \min\{i | z \in A(i, S)\} & \text{if } z \in A(S) \\ \min\{i | z \in B(i, S)\} & \text{if } z \in B(S). \end{cases}$$

Lemma 4.1. *Let $G = (U, W)$ be a defect n-extendable bipartite graph with $|W| = |U| + 1$ and $\kappa(G) \geq 2$. Then for any $S \subseteq W$ and $|S| \geq 2$, if $\Gamma_G(S) \neq U$, then $|\Gamma_G(S)| \geq |S| + n - 1$.*

Proof. Let G be as defined in the statement, $S \subseteq W$, $|S| \geq 2$ and $\Gamma_G(S) \neq U$.

Suppose $|S| \geq |W| - n + 1$. Then there is a set $S' \subseteq S$ and $|S'| = |W| - n$. Since G is a defect n-extendable bipartite graph with $\kappa(G) \geq 2$, Theorem 1.2 implies that $|\Gamma_G(S')| \geq |S'| + n - 1 = |W| - n + n - 1 = |U|$. However $\Gamma_G(S') \subseteq U$ as $S' \subseteq W$. So $\Gamma_G(S') = U$. Since $S' \subseteq S$, we have $U = \Gamma_G(S') \subseteq \Gamma_G(S)$. Note that $\Gamma_G(S) \subseteq U$ as $S \subseteq W$. Thus $\Gamma_G(S) = U$, a contradiction to the hypothesis of $\Gamma_G(S) \neq U$.

So $|S| \leq |W| - n$ and hence Theorem 1.2 implies that $|\Gamma_G(S)| \geq |S| + n - 1$. □

Lemma 4.2. *Let n be a positive integer, $G = (U, W)$ be a bipartite graph with $|W| = |U| + 1$ and M be a near perfect matching of G. Let $u \in U$ and $v \in W$ be a pair of vertices in G and $S^0 = (P_1^0, P_2^0, \ldots, P_{n-1}^0, Q^0, \Gamma^0)$ be an n-system of G with respect to v and u such that $Q^0 = v$. Then for each $x \in A(S^0)$, there is an n-system $S = (P_1, P_2, \ldots, P_{n-1}, Q, \Gamma)$ with respect to v and u such that*

(1) x is the end vertex of Q; and

(2) for each $y \in U \setminus \{u\}$, if $h(y, S^0) > h(x, S^0)$, then $y^{-S} = y^{-S^0}$.

Proof. For simplicity, we use A_i, B_i, A, B and $h(z)$ to denote $A(i, S^0)$, $B(i, S^0)$, $A(S^0)$, $B(S^0)$ and $h(z, S^0)$ respectively for any integer i and vertex $z \in A(S^0) \cup B(S^0)$. We prove the lemma by induction on $h(x)$.

If $h(x) = 0$, then $x = v$ and S^0 is the required n-system. Assume $h(x) > 0$ and the lemma holds for $h(x) \leq m - 1$, $m \geq 1$. Now we consider the case of $h(x) = m$.

Note that $x \in A_m - A_{m-1} \subseteq B_m^{-S^0}$. Thus $x = y_0^{-S^0}$ for some $y_0 \in B_m$ and hence $h(y_0) \leq m$. Suppose $h(y_0) = t < m$, then $x \in B_t^{-S^0} \subseteq A_t$, and hence $h(x) \leq t < m$, a contradiction to $h(x) = m$. So $h(y_0) = m$. Then there is a vertex x_0 in A_{m-1} such that $y_0 \in \Gamma_{G-u}(x_0)$. Obviously, $h(x_0) \leq m - 1$. Suppose $h(x_0) = r < m - 1$, then $y_0 \in \Gamma_{G-u}(A_r) = B_{r+1}$, and hence $h(y_0) \leq r + 1 \leq m - 1$, a contradiction to $h(y_0) = m$. Thus $h(x_0) = m - 1$.

By the induction hypothesis, there is an n-system $S' = (P'_1, P'_2, \ldots, P'_{n-1}, Q', \Gamma')$ with respect to v and u such that:

(1) x_0 is the end vertex of Q'; and

(2) for any $y \in U \setminus \{u\}$, if $h(y) > h(x_0)$, then $y^{-S'} = y^{-S^0}$.

Since $h(y_0) = m > m - 1 = h(x_0)$, $y_0^{-S'} = y_0^{-S^0} = x$. We consider two cases:

Case 1. $x_0 y_0 \notin M$.

(1) If $y_0 \in V(P'_i)$, $1 \leq i \leq n-1$, then $x = y_0^{-S'} = y_0^{-P'_i}$. Let $P_i = vQ'x_0y_0P'_iu$, $P_j = P'_j$ for all $j \neq i$, $Q = vP'_ix$ and $\Gamma = \Gamma'$. Note that $y_0 \neq qu$ as $y_0 \in \Gamma_{G-u}(x_0)$. So $P_i \neq vu$ and hence $P_1, P_2, \ldots, P_{n-1}$ are $n - 1$ internally disjoint M-alternating vu-paths.

(2) If $y_0 \in V(Q')$, then $x = y_0^{-S'} = y_0^{-Q'}$. Since $x_0y_0 \notin M$, we have $y_0 \neq x_0^{-Q'}$, and hence $C = x_0y_0Q'x_0$ is an M-alternating cycle. Let $P_i = P'_i$ for all $1 \leq i \leq n-1$, $Q = vQ'x$ and $\Gamma = \Gamma' \cup \{C\}$.

(3) If $y_0 \in V(C')$ for some $C' \in \Gamma'$, then $x = y_0^{-S'} = y_0^{-C'}$. Let $C' = a_0a_1 \ldots a_ra_0$ where $a_ia_{i+1} \in M$ and $a_i \in U$ if i is odd. Without loss of generality, assume $y_0 = a_j$. Then $1 \leq j \leq r$ and $x = a_{j-1}$.

(3.1) If $v \notin V(C')$, then let $P_i = P'_i$ for $1 \leq i \leq n-1$, $Q = vQ'x_0y_0a_{j+1}a_{j+2} \ldots x$ and $\Gamma = \Gamma' \setminus \{C'\}$.

(3.2) If $v \in V(C')$, without loss of generality, assume $v = a_0$, then let $P_i = P_i'$ for $1 \le i \le n-1$, $Q = va_1a_2 \ldots x$, $C'' = x_0y_0a_{j+1}a_{j+2} \ldots a_0Q'x_0$ and $\Gamma = (\Gamma' \setminus \{C'\}) \cup \{C''\}$.
(4) If $y_0 \notin V(S')$, then $y_0x_1 \in M$ for some $x_1 \in W$. By the definition of predecessor, $x = y_0^{-S'} = x_1$. Note that $x \ne v$. Otherwise, $h(x) = h(v) = 0$, a contradiction to the assumption hypothesis that $h(x) > 0$. Thus $x \notin V(S')$. Let $P_i = P_i'$ for $1 \le i \le n-1$, $Q = vQ'x_0y_0x$ and $\Gamma = \Gamma'$.
Case 2. $x_0y_0 \in M$.

Suppose $x_0 = v$. Then $vy_0 \in M$ and hence $x = y_0^{-S^0} = v$. Therefore, $h(x) = h(v) = 0$, a contradiction to $h(x) > 0$. So $x_0 \ne v$ and hence $x_0y_0 \in E(Q')$. Therefore, $x = y_0^{-S^0} = y_0^{-S'} = y_0^{-Q'}$. Let $P_i = P_i'$ for $1 \le i \le n-1$, $Q = vQ'x$ and $\Gamma = \Gamma'$.

In both cases, let $S = (P_1, P_2, \ldots, P_{n-1}, Q, \Gamma)$. Then S is an n-system in G with respect to v and u such that x is the end vertex of Q. Select any $y \in U \setminus \{u\}$. Clearly, $y^{-S} = y^{-S'}$ if $y \ne y_0$. If $h(y) > h(x)$, since $h(x) = h(y_0)$, then $y \ne y_0$ and hence $y^{-S} = y^{-S'}$. Moreover, $h(y) > h(x) > h(x_0)$, so by induction hypothesis, $y^{-S} = y^{-S'} = y^{-S^0}$. Thus in both cases, S is the required n-system and the lemma holds. □

Lemma 4.3. *Let n be a positive integer, $G = (U, W)$ be a bipartite graph with $|W| = |U| + 1$ and M be a near perfect matching in G. Let $u \in U$, $v \in W$ and S^0 be an n-system of G with respect to v and u. Then*

(1) $\Gamma_{G-u}(A(S^0)) = B(S^0)$
(2) $(B(S^0))^{-S^0} \subseteq A(S^0)$.

Proof. For simplicity, we use A_i, B_i, A, B and $h(x)$ to denote $A(i, S^0), B(i, S^0), A(S^0), B(S^0)$ and $h(x, S^0)$ respectively for any integer i and vertex $x \in A(S^0) \cup B(S^0)$.
(1) Choose any $y \in \Gamma_{G-u}(A)$. Then there is a vertex $x \in A$ such that $y \in \Gamma_{G-u}(x)$. Let $t = h(x)$. Then $x \in A_t$ and hence $y \in \Gamma_{G-u}(A_t) = B_{t+1} \subseteq B$. So $\Gamma_{G-u}(A) \subseteq B$.

Suppose $B = \phi$. Then $\Gamma_{G-u}(A) = \phi = B$ as $\Gamma_{G-u}(A) \subseteq B$.

Suppose $B \ne q\phi$. Select any $y' \in B$. Let $m = h(y')$. Note that $m \ge 1$ as $B_0 = \phi$. Then $y' \in B_m = \Gamma_{G-u}(A_{m-1}) \subseteq \Gamma_{G-u}(A)$. So $B \subseteq \Gamma_{G-u}(A)$ and hence $\Gamma_{G-u}(A) = B$.
(2) Choose any $x \in B^{-S^0}$. Then $x = y^{-S^0}$ for some $y \in B_t$ where $t \ge 1$ since $B_0 = \phi$. Therefore, $x = y^{-S^0} \in B_t^{-S^0} \subseteq A_t \subseteq A$. Thus $B^{-S^0} \subseteq A$. □

Lemma 4.4. *Let n be a positive integer, $G = (U, W)$ be a bipartite graph with $|W| = |U| + 1$ and M be a near perfect matching of G. Let $u \in U$ and $v \in W$ be a pair of vertices in G and $S^0 = (P_1^0, P_2^0, \ldots, P_{n-1}^0, Q^0, \Gamma^0)$ be an n-system of G with respect to v and u such that $Q^0 = v$. If there are not n internally disjoint M-alternating vu-paths in G, then*

(1) *$u \notin \Gamma_G(A(S^0) \setminus \{v\})$ and*
(2) *$\Gamma_G(A(S^0) \setminus \{v\}) \subseteq B(S^0)$.*

Proof. For simplicity, we use A and B to denote $A(S^0)$ and $B(S^0)$ respectively.
(1) Suppose to the contrary $u \in \Gamma_G(A \setminus \{v\})$. Then there is a vertex $x \in A \setminus \{v\}$ such that $xu \in E(G)$. Lemma 4.2 implies that there is an n-system $S = (P_1, P_2, \ldots, P_{n-1}, Q, \Gamma)$ of G with respect to v and u such that x is the end vertex of Q. By the definition of n-system with respect to v and u, $u \notin V(Q)$. Thus $xu \notin M$ and hence $P_1, P_2, \ldots, P_{n-1}, vQxu$ are n internally disjoint M-alternating vu-paths, a contradiction to the hypothesis. Therefore, $u \notin \Gamma_G(A \setminus \{v\})$.
(2) By statement (1) and Lemma 4.3 (1), we have $\Gamma_G(A \setminus \{v\}) = \Gamma_{G-u}(A \setminus \{v\}) \subseteq \Gamma_{G-u}(A) = B$. □

Lemma 4.5. *Let n be a positive integer, $G = (U, W)$ be a defect n-extendable bipartite graph with $|W| = |U| + 1$ and $\kappa(G) \ge 2$ and M be a near perfect matching of G. Let $u \in U$ and $v \in W$ be a pair of vertices in G such that $d_G(v) \ge n$. Then the following statements hold:*

(1) *if $v = w$, there is an n-system $S^0 = (P_1^0, P_2^0, \ldots, P_{n-1}^0, Q^0, \phi)$ of G with respect to v and u such that $Q^0 = v$ and there are not n internally disjoint M-alternating vu-paths in G, then $|A(S^0)| \ge 3$.*
(2) *if $v \ne qw$, there is an $(n-1)$-system $S^0 = (P_1^0, P_2^0, \ldots, P_{n-2}^0, Q^0, \phi)$ of G with respect to v and u such that $Q^0 = v$ and there are not $n-1$ internally disjoint M-alternating vu-paths in G, then $|A(S^0)| \ge 3$.*

Proof. For simplicity, we use A_i, B_i, A and B to denote $A(i, S^0), B(i, S^0), A(S^0)$ and $B(S^0)$ respectively for any integer i.
(1) Suppose $v = w$, there is an n-system $S^0 = (P_1^0, P_2^0, \ldots, P_{n-1}^0, Q^0, \phi)$ with respect to v and u in G such that $Q^0 = v$ and there are not n internally disjoint M-alternating vu-paths.

Assume $vy_i \in P_i^0$ for all $1 \le i \le n-1$. Since $d_G(v) \ge n$, there is a vertex $z_1 \notin \{y_1, y_2, \ldots, y_{n-1}\}$ such that $vz_1 \in E(G)$. Note that $z_1 \ne u$, otherwise $P_1^0, P_2^0, \ldots, P_{n-1}^0, vu$ are n internally disjoint M-alternating vu-paths, a contradiction to the assumption hypothesis. So $z_1 \in \Gamma_{G-u}(\{v\}) = \Gamma_{G-u}(A_0) = B_1$. Let $x_1 = z_1^{-S^0}$. Then $x_1 \in B_1^{-S^0} \subseteq A_1 \subseteq A$.

Since $z_1 \notin \{y_1, y_2, \ldots, y_{n-1}\} \cup \{u\}$ and $v = w$, we have $x_1 \ne qv$ and hence $|\Gamma_G(\{v, x_1\})| \ge 2 + n - 1 = n + 1$ by Theorem 1.2. So there is a vertex $z_2 \in \Gamma_G(\{v, x_1\})$ such that $z_2 \notin \{z_1\} \cup \{y_1, y_2, \ldots, y_{n-1}\}$. Then $vz_2 \in E(G)$ or $x_1z_2 \in E(G)$.

Suppose $z_2 = u$. Note that $vz_2 \notin E(G)$. Otherwise, $P_1^0, P_2^0, \ldots, P_{n-1}^0, vu$ are n internally disjoint M-alternating vu-paths, a contradiction to the assumption hypothesis. Thus $x_1z_2 \in E(G)$. Since $x_1 \in A$, Lemma 4.2 implies that there is an n-system

$S = (P_1, P_2, \ldots, P_{n-1}, Q, \Gamma)$ with respect to v and u such that x_1 is the end vertex of Q. Clearly, $P_1, P_2, \ldots, P_{n-1}, vQx_1u$ are n internally disjoint M-alternating vu-paths as $x_1 \neq qv$, a contradiction to the assumption hypothesis.

So $z_2 \neq u$ and hence $z_2 \in \Gamma_{G-u}(\{v, x_1\}) \subseteq \Gamma_{G-u}(A_1) = B_2$. Let $x_2 = z_2^{-S^0}$. Then $x_2 \in B_2^{-S^0} \subseteq A$ and hence $\{v, x_1, x_2\} \subseteq A$. Moreover, since $z_2 \notin \{z_1\} \cup \{y_1, y_2, \ldots, y_{n-1}\}$, we have $x_2 \notin \{v, x_1\}$. Therefore, $|A| \geq |\{v, x_1, x_2\}| \geq 3$ and statement (1) holds.

(2) Suppose $v \neq qw$. Then there is a vertex u_0 in G such that $vu_0 \in M$. The proof of statement (2) is similar with that of statement (1) by replacing n and y_{n-1} with $n-1$ and u_0 respectively. □

Theorem 4.6. *Let n be a positive integer and $G = (U, W)$ be a defect n-extendable bipartite graph with $|W| = |U| + 1$ and $\kappa(G) \geq 2$. Let M be a near perfect matching in G and w be the M-unsaturated vertex. Let $u \in U$ and $v \in W$. Then the following statements hold:*

(1) *If $v = w$, then there are $\min\{d_G(u) - 1, d_G(v), n\}$ internally disjoint M-alternating vu-paths in G.*
(2) *If $v \neq w$, then there are $\min\{d_G(u) - 1, d_G(v) - 1, n - 1\}$ internally disjoint M-alternating vu-paths in G.*

Proof. (1) Assume $v = w$. We prove statement (1) by induction on n.

Suppose $n = 1$. Since $\kappa(G) \geq 2$, we have $d_G(u) \geq 2$ and $d_G(v) \geq 2$. Then $\min\{d_G(u) - 1, d_G(v), n\} = 1$. Note that G is a defect 1-extendable bipartite graph, thus Theorem 3.1 implies that there is an M-alternating vu-path in G. Therefore, statement (1) holds when $n = 1$.

Assume statement (1) holds when G is defect k-extendable for any integer $k \leq n - 1$. Now we consider the case when G is defect n-extendable where $n \geq 2$.

Suppose $\min\{d_G(u) - 1, d_G(v)\} \leq n - 1$, then $\min\{d_G(u) - 1, d_G(v), n\} = \min\{d_G(u) - 1, d_G(v), n - 1\}$. Since G is defect $(n-1)$-extendable, by induction hypothesis, there are $\min\{d_G(u) - 1, d_G(v), n-1\}$ internally disjoint M-alternating vu-paths in G and hence statement (1) holds.

Assume $\min\{d_G(u) - 1, d_G(v)\} \geq n$, then $\min\{d_G(u) - 1, d_G(v), n\} = n$. It suffices to prove that there are n internally disjoint M-alternating vu-paths in G. Suppose to the contrary, G has no such n vu-paths.

Since G is defect $(n-1)$-extendable and $\min\{d_G(u) - 1, d_G(v), n - 1\} = n - 1$ as $\min\{d_G(u) - 1, d_G(v)\} \geq n$, by induction hypothesis, there are $(n-1)$ internally disjoint M-alternating vu-paths, say $P_1^0, P_2^0, \ldots, P_{n-1}^0$, in G. Let $Q^0 = v$. Then $S^0 = (P_1^0, P_2^0, \ldots, P_{n-1}^0, Q^0, \phi)$ is an n-system of G with respect to v and u. For simplicity, we use the abbreviation $A = A(S^0)$, $B = B(S^0)$, $y^- = y^{-S^0}$ for any $y \in U \setminus \{u\}$ and $V^- = V^{-S^0}$ for any $V \subseteq U \setminus \{u\}$.

Let $vy_i \in E(P_i^0)$, $1 \leq i \leq n - 1$. We discuss two cases:

Case 1. Suppose $u \notin \{y_1, y_2, \ldots, y_{n-1}\}$.

Suppose $vu \in E(G)$. Then $vu \notin M$ as v is an M-unsaturated vertex. So $P_1^0, P_2^0, \ldots, P_{n-1}^0, vu$ are n internally disjoint M-alternating vu-paths, a contradiction to the assumption that such paths do not exist. So $vu \notin E(G)$. Since $u \notin \Gamma_G(A \setminus \{v\})$ by Lemma 4.4(1), we have $u \notin \Gamma_G(A)$. Furthermore, Lemma 4.3(1) implies that $\Gamma_G(A) = \Gamma_{G-u}(A) = B$. Note that for distinct vertex v_1 and v_2 in $U \setminus \{u\}$, $v_1^- = v_2^-$ occurs only if $\{v_1, v_2\} \subseteq \{y_1, y_2, \ldots, y_{n-1}\}$. Clearly, $\{y_1, y_2, \ldots, y_{n-1}\}^- = \{v\}$, then $|B^-| \geq |B| - (n-1) + 1$ and hence $|B| \leq |B^-| + n - 2$. Therefore, by $\Gamma_G(A) = B$ and Lemma 4.3(2), we have $|\Gamma_G(A)| = |B| \leq |B^-| + n - 2 \leq |A| + n - 2$.

On the other hand, note that G is defect n-extendable with $\kappa(G) \geq 2$, $|A| \geq 3$ by Lemma 4.5 and $u \notin \Gamma_G(A)$. Therefore, Lemma 4.1 implies that $|\Gamma_G(A)| \geq |A| + n - 1$, a contradiction to $|\Gamma_G(A)| \leq |A| + n - 2$.

Case 2. Suppose $u \in \{y_1, y_2, \ldots, y_{n-1}\}$

Note that for distinct vertex v_1 and v_2 in $U \setminus \{u\}$, $v_1^- = v_2^-$ occurs only if $\{v_1, v_2\} \subseteq \{y_1, y_2, \ldots, y_{n-1}\} \setminus \{u\}$. So $|B^-| \geq |B| - (n-2) + 1$ and hence $|B| \leq |B^-| + n - 3$. Moreover, by Lemmas 4.4(2) and 4.3(2), we have $|\Gamma_G(A \setminus \{v\})| \leq |B| \leq |B^-| + n - 3 \leq |A| + n - 3$.

On the other hand, note that $|A \setminus \{v\}| \geq 2$ by Lemma 4.5(1) and $u \notin \Gamma_G(A \setminus \{v\})$ by Lemma 4.4(1). Moreover, Lemma 4.1 implies that $|\Gamma_G(A \setminus \{v\})| \geq |A \setminus \{v\}| + n - 1 = |A| + n - 2$, a contradiction to $|\Gamma_G(A)| \leq |A| + n - 3$.

In both cases, we can find a contradiction, so G has n internally disjoint M-alternating vu-paths when $\min\{d_G(u) - 1, d_G(v)\} \geq n$ and hence statement (1) holds.

(2) Assume $v \neq w$. Now we prove statement (2) by induction on n.

If $n = 1$, then $\min\{d_G(u) - 1, d_G(v) - 1, n - 1\} = 0$ and statement (2) is obviously true. Assume statement (2) is true when G is defect k-extendable for all integer $k \leq n - 1$. Now we consider the case when G is defect n-extendable.

By Lemma 2.4, G is defect $(n-1)$-extendable, thus by induction hypothesis, G has $\min\{d_G(u) - 1, d_G(v) - 1, n - 2\}$ internally disjoint M-alternating vu-paths.

If $\min\{d_G(u) - 1, d_G(v) - 1\} \leq n - 2$, then $\min\{d_G(u) - 1, d_G(v) - 1, n - 1\} = \min\{d_G(u) - 1, d_G(v) - 1, n - 2\}$ and hence G has $\min\{d_G(u) - 1, d_G(v) - 1, n - 1\}$ internally disjoint M-alternating vu-paths by induction hypothesis.

In the following, we consider the case of $\min\{d_G(u) - 1, d_G(v) - 1\} \geq n - 1$.

Note that $\min\{d_G(u) - 1, d_G(v) - 1, n - 1\} = n - 1$ and $\min\{d_G(u) - 1, d_G(v) - 1, n - 2\} = n - 2$. Therefore, G has $n - 2$ internally disjoint M-alternating vu-paths, say $P_1^0, P_2^0, \ldots, P_{n-2}^0$. It is enough to prove that G has $n - 1$ internally disjoint M-alternating vu-paths. Suppose G has no such $n - 1$ vu-paths.

Let $Q^0 = v$. Then $S^0 = (P_1^0, P_2^0, \ldots, P_{n-2}^0, Q^0, \phi)$ is an $(n-1)$-system of G with respect to v and u. Assume $vu_0 \in M$ and $vy_i \in E(P_i^0)$, $1 \leq i \leq n - 2$. we discuss two cases: $u \notin \{u_0, y_1, y_2, \ldots, y_{n-2}\}$ and $u \in \{u_0, y_1, y_2, \ldots, y_{n-2}\}$. The proof of the two cases are similar to that in statement (1) by replacing y_{n-1} with u_0. □

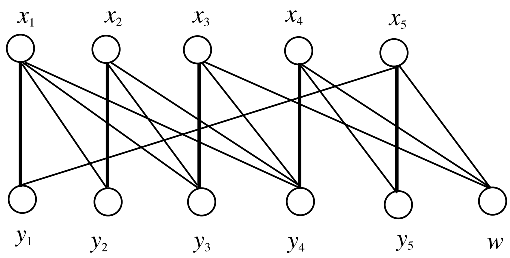

Fig. 1. A defect 2-extendable graph G.

Remark 4.7. (1) It is easy to check that the bounds in Theorem 4.6 are sharp (cf. Fig. 1). Clearly, $U = \{x_i : 1 \le i \le 5\}$ and $W = \{y_i : 1 \le i \le 5\} \cup \{w\}$ are the bipartitions of G with $|W| = |U| + 1$. It is easy to check that $\kappa(G) \ge 2$ and for any $S \subseteq U$ and $1 \le |S| \le |U| - 2$, $|\Gamma_G(S)| \ge |S| + 2$. So Lemma 2.1 implies that G is defect 2-extendable. Note that $M = \{x_iy_i : 1 \le i \le 5\}$ is a near perfect matching of G, w is the M-unsaturated vertex and $\min\{d_G(x_1) - 1, d_G(w), 2\} = 2$. Clearly, there are exactly two internally disjoint M-alternating wx_1-paths in G because each M-alternating wx_1-path must contain y_3 or y_4. So the bound in Theorem 4.6(1) is sharp. Note that $\min\{d_G(x_4) - 1, d_G(y_5) - 1, 2 - 1\} = 1$ and x_4y_5 is the only M-alternating x_4y_5-path in G. Thus the bound in Theorem 4.6(2) is sharp, too.

(2) Given a defect n-extendable bipartite graph $G = (U, W)$ with $\kappa(G) \ge 2$ and $|W| = |U| + 1$, and a near perfect matching M of G. Select any $u \in U$ and $v \in W$. Since Wen and Yang [12] prove that $2 \le \delta(G) \le n + 1$, Theorem 4.6 implies that the number of internally disjoint M-alternating vu-paths may be much less than n. But by Lemma 2.1, $d(u) \ge n + 1$ and by Theorem 1.2, $|\Gamma_G(\{v, y\})| \ge n + 1$ for any $y \in W$ and $y \ne v$. Then it is reasonable to guess that the number of internally disjoint M-alternating vu-paths may be not less than n if we add some new edges to G such that v is adjacent to all vertices in $\Gamma_G(v, y)$. So we introduce the operation $*$ defined as follows. Let G be a graph, $v, y \in V(G)$ and $S \subseteq V(G)$. Let $E(v, y) = \{$ edges joining vertex v to all vertices in G which are adjacent to y but not adjacent to $v\}$. We use $\boldsymbol{G*(S, y)}$ to denote a graph constructed from G by adding all edges in $\bigcup_{v\in S} E(v, y)$ to G. Then we get the following theorem.

Theorem 4.8. *Let n be a positive integer, $G = (U, W)$ be a bipartite graph such that $|W| = |U| + 1$ and $\kappa(G) \ge 2$. Then G is defect n-extendable if and only if for any $w \in W$ the following two statements hold:*

(1) *there is a near perfect matching M in G such that w is the M-unsaturate vertex;*
(2) *for any near perfect matching M in G such that w is the M-unsaturated vertex and any vertex $u \in U$ and $v \in W \setminus \{w\}$, there are n internally disjoint M-alternating wu-paths in $H = G * (\{w\}, v)$.*

Proof. First we prove the necessity. Let $G = (U, W)$ be a defect n-extendable bipartite graph with $|W| = |U| + 1$ and $\kappa(G) \ge 2$. Select any vertex w in G.
(1) Since $n \ge 1$ and $\kappa(G) \ge 2$, we have $\min\{\kappa(G) - 1, n - 1\} \ge 0$. Thus Lemma 2.5 implies that $G - w$ has a perfect matching M. Clearly, M is a near perfect matching in G such that w is the M-unsaturated vertex. Then statement (1) holds.
(2) Select any $v \in W\setminus\{w\}$. Let $H = G*(\{w\}, v)$. Then by Lemma 2.1, H is a defect n-extendable bipartite graph with $\kappa(G) \ge 2$. Note that $d_H(w) = |\Gamma_G(\{v, w\})| \ge 2 + n - 1 = n + 1$ by Theorem 1.2, and $d_H(u) \ge |\Gamma_G(u)| \ge n + 1$ by Lemma 2.1. So $\min\{d_H(u) - 1, d_H(w), n\} = n$ and hence Theorem 4.6(1) implies that there are n internally disjoint M-alternating wu-paths in H. Thus statement (2) holds.

Now we prove the sufficiency. Suppose to the contrary G is not defect n-extendable. Then there is a matching S of size n in G which is not contained in any near perfect matching of G. By statement (1), we may assume that M is a near perfect matching of G that contains as many edges in S as possible and w is the M-unsaturated vertex. Clearly, there is an edge $e \in S$ and $e \notin M$. Assume $e = ab$ where $a \in U$ and $b \in W$. Then there is a vertex v in W such that $av \in M$. If $b = w$, then $M' = (M \setminus \{av\}) \cup \{ab\}$ is a near perfect matching in G and $|M' \cap S| > |M \cap S|$, a contradiction to the choice of M.

So $b \ne w$. Then there is a vertex u in U such that $ub \in M$. Let $H = G * (\{w\}, v)$. It is not difficult to see that M is a near perfect matching in H. Statement (2) implies that there are n internally disjoint M-alternating wu-paths, say $P_1, P_2, \ldots, P_n$, in H. Clearly, for all $1 \le i \le n$, $ub \notin E(P_i)$ as $ub \in M$. Therefore, $e \notin E(P_i)$ for all $1 \le i \le n$. Since $|S \setminus \{e\}| = n - 1$, there is at least a path P_j among $P_1, P_2, \ldots, P_n$ satisfies that $E(P_j) \cap S = \phi$.

Suppose $v \in V(P_j)$. By the definition of H, we have $C = vP_jubav$ is an M-alternating cycle in G. Let $M' = M\Delta E(C)$. Then M' is also a near perfect matching in G such that $|M' \cap S| > |M \cap S|$, a contradiction to the choice of M.

So $v \notin V(P_j)$ and hence $a \notin V(P_j)$. Assume $P_j = wa_0a_1 \ldots a_ku$. Then $a_0 \in \Gamma_G(w) \cup \Gamma_G(v)$ as $H = G * (\{w\}, v)$.

Suppose $a_0 \in \Gamma_G(w)$. Then P_j is an M-alternating wu-path in G. Since $v \notin V(P_j)$, we have that $Q = wP_jubav$ is an M-alternating wv-path in G and $E(Q) \cap S = \{e\}$. Therefore, $M'' = M\Delta E(Q)$ is a near perfect matching in G such that $|M'' \cap S| > |M \cap S|$, a contradiction to the choice of M.

So $a_0 \notin \Gamma_G(w)$. Then $a_0 \in \Gamma_G(v)$. Since $v \notin V(P_j)$ and $H = G * (\{w\}, v)$, we have $R = va_0a_1 \ldots a_ku$ is an M-alternating vu-path in G. Moreover, since $a \notin V(P_j)$, we have $C' = vRubav$ is an M-alternating cycle in G and $E(C') \cap S = \{e\}$. Let $T = M\Delta E(C')$. Then T is a near perfect matching in G such that $|T \cap S| > |M \cap S|$, a contradiction to the choice of M.

So every matching of size n is contained in a near perfect matching of G and we complete the proof of sufficiency. □

To identify if a bipartite graph G with $\kappa(G) \geq 2$ is defect n-extendable, Theorem 4.8 requires us to check every near perfect matching in G. However, the following theorem greatly reduces the number.

Theorem 4.9. *Let n be a positive integer and $G = (U, W)$ be a bipartite graph with $|W| = |U| + 1$, which has a near perfect matching. Let M and M_0 be two near perfect matchings of G, w be the M-unsaturated vertex and w_0 be the M_0-unsaturated vertex. Let $u \in U$ and $v \in W$. If $w = w_0$ and G has k internally disjoint M_0-alternating uv-paths, then G also has k internally disjoint M-alternating uv-paths.*

Proof. Assume $w = w_0$ and G has k internally disjoint M_0-alternating uv-paths, say $P_1, P_2, \ldots, P_k$, in G, where $u \in U$ and $v \in W$. Let H be a graph constructed from G by adding a new vertex x and joining x to w only. Let $M' = M \cup \{xw\}$ and $M'_0 = M_0 \cup \{xw\}$. Obviously, M' and M'_0 are perfect matchings in H and $P_1, P_2, \ldots, P_k$ are k internally disjoint M'_0-alternating uv-paths in H. So Lemma 2.3 implies that there are also k internally disjoint M'-alternating uv-paths, say $Q_1, Q_2, \ldots, Q_k$, in H. Since x only joins to w in H, we have $x \notin V(Q_i)$ for all $1 \leq i \leq k$. Then $Q_1, Q_2, \ldots, Q_k$ are also M-alternating uv-paths in G and this complete the proof of the theorem. □

By Theorems 4.8 and 4.9, we can get the following theorem immediately.

Theorem 4.10. *Let n be a positive integer, $G = (U, W)$ be a bipartite graph such that $|W| = |U| + 1$ and $\kappa(G) \geq 2$. Then G is defect n-extendable if and only if for any $w \in W$ the following two statements hold:*

(1) *There is a near perfect matching M in G such that w is the M-unsaturate vertex;*
(2) *Let M be a near perfect matching in G such that w is the M-unsaturated vertex. Then for any vertex $u \in U$ and $v \in W \setminus \{w\}$, there are n internally disjoint M-alternating wu-paths in $H = G * (\{w\}, v)$.*

Proof. It follows from Theorems 4.8 and 4.9. □

Remark 4.11. Given a bipartite graph $G = (U, W)$ with $\kappa(G) \geq 2$ and $|W| = |U| + 1$, to identify if G is defect n-extendable, Theorem 4.10 shows that for any $w \in W$, it is enough to check that if G contains a near perfect matching M such that w is the M-unsaturated vertex and if it does, then we continue to check that for any $u \in U$ and $v \in W \setminus \{w\}$, if there are n internally disjoint M-alternating uw-paths in $H = G * (\{w\}, v)$. Note that to find a near perfect matching in G is equal to find a maximum matching in G, which needs $O(p^{1/2}q)$ time [6]. If we construct a directed graph $\overrightarrow{H}$ from H by giving orientation to all edges in M from U to W and orientation to the other edges of H from W to U, then the maximum number of internally disjoint M-alternating wu-paths in H is equivalent to the maximum number of internally disjoint directed path from w to u in $\overrightarrow{H}$. Such paths can be found in $O(p^3)$ time by using the algorithm of finding the maximum flow between w and u in $\overrightarrow{H}$ each edge of which is assigned with unit capacity [9]. Since w, v and u are arbitrary, we have to compute the maximum number of internally disjoint alternating paths between two vertices $O(p^3)$ times. Furthermore, we have to find maximum matching $|W| = O(p)$ times. So determining if a bipartite graph G is defect n-extendable can be done in $O(p^3 * p^3 + p * p^{1/2}q) = O(p^6)$ time. Especially, when $\kappa(G) \geq n$, we can greatly decrease the time complexity by the following theorem.

Theorem 4.12. *Let n be a positive integer and $G = (U, W)$ be a bipartite graph such that $|W| = |U| + 1$ and $\kappa(G) \geq n$. Let G' be a graph constructed from G by adding a vertex $x \notin V(G)$ and joining x to all vertices in W. Then G is defect n-extendable if and only if G' is n-extendable.*

Proof. Let G' be as defined in the statement.

First we prove the necessity. Assume G is defect n-extendable. We consider two cases.

Case 1. $n = 1$. Select any $e \in E(G')$. It suffices to prove there is a perfect matching in G' that contains e.

Suppose $e \in E(G)$. Since G is defect 1-extendable, there is a near perfect matching M in G containing e. Assume w is the M-unsaturated vertex in G. Obviously, $M \cup \{xw\}$ is a perfect matching in G' containing e.

Suppose $e \notin E(G)$. Then e is incident with x. Assume $e = xv$. Then $v \in W$ and Lemma 2.5 implies that $G - v$ is $\min\{\kappa(G) - 1, n - 1\}$-extendable. Note that $\min\{\kappa(G) - 1, n - 1\} \geq 0$ as $\kappa(G) \geq n \geq 1$. Then $G - v$ has a perfect matching M'. Clearly $M' \cup \{e\}$ is a perfect matching in G' containing e.

Case 2. $n \geq 2$. Select any $S \subseteq W$ such that $1 \leq |S| \leq |W| - n$. By the definition of G', we have $\Gamma_{G'}(S) = \Gamma_G(S) \cup \{x\}$, thus $|\Gamma_{G'}(S)| = |\Gamma_G(S)| + 1$.

Suppose $|S| = 1$. Assume $S = \{t\}$. Then $\Gamma_G(t) = d_G(t) \geq \kappa(G) \geq n$ and hence $|\Gamma_{G'}(S)| = |\Gamma_G(t)| + 1 \geq n + 1 = |S| + n$.

Suppose $|S| \geq 2$. Since $\kappa(G) \geq n \geq 2$, Theorem 1.2 implies that $|\Gamma_{G'}(S)| = |\Gamma_G(S)| + 1 \geq |S| + n - 1 + 1 = |S| + n$.

So $|\Gamma_{G'}(S)| \geq |S| + n$ for any $1 \leq |S| \leq |W| - n$, and hence Lemma 2.7 implies that G' is n-extendable.

Now we prove the sufficiency. Assume G' is n-extendable. Select any matching F of size n in G. Then F is also a matching in G'. So there is a perfect matching M in G' containing F and $xw \in M$ for some $w \in W$. Note that $xw \notin F$ as $x \notin V(F)$. Thus $M \setminus \{xw\}$ is a near perfect matching in G containing F. Therefore, G is defect n-extendable. □

Remark 4.13. Theorem 4.12 shows that verifying if a graph G with p vertices, q edges and $\kappa(G) \geq n$ is defect n-extendable is equal to verify if a bipartite graph G' with $p + 1$ vertices and $(p + 1)/2 + q$ edges is n-extendable, which was proved in [13] to be done in $O((p + 1)((p + 1)/2 + q)) = O(pq)$ time.

Theorem 4.14. *Let k and n be positive integer and $G = (U, W)$ be a defect n-extendable bipartite graph with $\kappa(G) \geq 2$. Let $e = xy$ be an edge such that $x, y \notin V(G)$ and H be a graph constructed from G by joining x to at least $k + 1$ vertices in W and joining y to at least k vertices in U. Then H is defect $\min\{n, k\}$-extendable.*

Proof. Let $U' = U \cup \{x\}$ and $W' = W \cup \{y\}$. Then (U', W') be the bipartitions of H. Let $m = \min\{n, k\}$. Since $\kappa(G) \geq 2$, by the construction of H, we have $\kappa(H) \geq 2$. Select any $S' \subseteq U'$ such that $1 \leq |S'| \leq |U'| - m$. Lemma 2.1 implies that it is enough to prove that $|\Gamma_H(S')| \geq |S'| + m$. We consider the following cases:
Case 1. $x \notin S'$.
Case 1.1. $|S'| \leq |U| - n$. Since G is defect n-extendable with $\kappa(G) \geq 2$, Lemma 2.1 implies that $|\Gamma_H(S')| \geq |\Gamma_G(S')| \geq |S'| + n \geq |S'| + \min\{n, k\} = |S'| + m$.
Case 1.2. $|S'| > |U| - n$. Note that $|\Gamma_G(S')| \geq |W| - 1$ by Lemma 2.2.
Case 1.2.1. $k \geq n$. Then $|S'| > |U| - n \geq |U| - k$. Since y is adjacent to k vertices in U, y joins to at least a vertex in S'. Therefore $|\Gamma_H(S')| = |\Gamma_G(S') \cup \{y\}| = |\Gamma_G(S')| + 1 \geq |W| - 1 + 1 = |W| = |U'| \geq |S'| + m$ since $|S'| \leq |U'| - m$.
Case 1.2.2. $k < n$. If $|S'| \leq |U| - k$, then $m = \min\{n, k\} = k$. So $|\Gamma_H(S')| \geq |\Gamma_G(S')| \geq |W| - 1 = |U| \geq |S'| + k = |S'| + m$. If $|S'| > |U| - k$, then similarly to the proof in Case 1.2.1, we have $|\Gamma_H(S')| \geq |S'| + m$.
Case 2. $x \in S'$.

Let $S = S' \setminus \{x\}$. Then $S \subseteq U$.

If $S = \emptyset$, then $S' = \{x\}$ and hence $|\Gamma_H(S')| = k + 1 \geq 1 + \min\{n, k\} = |S'| + m$.

If $1 \leq |S| \leq |U| - n$, then Lemma 2.1 implies that $|\Gamma_G(S)| \geq |S| + n$. Since y is adjacent to x, we have $|\Gamma_H(S')| \geq |\Gamma_G(S) \cup \{y\}| \geq |S| + n + 1 = |S'| + n \geq |S'| + \min\{n, k\} = |S'| + m$.

If $|S| > |U| - n$, then Lemma 2.2 implies that $|\Gamma_G(S)| \geq |W| - 1$. Moreover, y is adjacent to x and $|S'| \leq |U'| - m$, then we have $|\Gamma_H(S')| \geq |\Gamma_G(S) \cup \{y\}| \geq |W| - 1 + 1 = |W| = |U'| \geq |S'| + m$.

Thus in all cases, we have $|\Gamma_H(S')| \geq |S'| + m$ and the proof is completed. □

5. Verify defect n-extendable bipartite graph G with $n \geq 2$ and $\kappa(G) = 1$

Using M-alternating paths, Sections 3 and 4 present the methods to decide if a bipartite graph G is defect n-extendable in polynomial time for the case of $n = 1$ or $\kappa(G) \geq 2$. In this section, we will solve the case of $n \geq 2$ and $\kappa(G) = 1$. Firstly, we define two types of bipartite graphs G with $\kappa(G) = 1$. Let $G = (U, W)$ be a bipartite graph with $\kappa(G) = 1$ and $|W| = |U| + 1$. If G contains no cut vertex in W, we called it a **Type-A** graph, otherwise we call it a **Type-B** graph. Clearly any defect n-extendable bipartite graph G with $\kappa(G) = 1$ and $n \geq 2$ belongs to either Type-A or Type-B graph. We will characterize the two types of defect n-extendable bipartite graph respectively.

Theorem 5.1. *Let $G = (U, W)$ with $\kappa(G) = 1$ and $|W| = |U| + 1$ be a Type-A bipartite graph. Let n be a positive integer with $2 \leq n \leq |U| - 1$, x be a cut vertex in G and $H = (X, Y)$ be a component in $G - x$. Then G is defect n-extendable if and only if the following statements hold:*

(1) *There are exactly two components in $G - x$.*
(2) $||X| - |Y|| = 1$.
(3) *If $|X| = |Y| + 1 = m + 1$, then*
   (3.1) *$Y \subseteq U$ and $X \subseteq W$.*
   (3.2) *H is isomorphic to $K_1, K_{2,1}$ or a defect s-extendable graph with $\kappa(H) \geq 2$ where $s = \min\{n, m - 1\}$.*
   (3.3) *For any $w \in V(H)$ such that $wx \in E(G)$, each component $H' = (X', Y')$ in $H - w$ with $|X'| = m'$ is t-extendable where $t = \min\{n - 1, m' - 1\}$.*
   (3.4) *If $|\Gamma_G(x) \cap V(H)| < |X|$ holds for each component $H = (X, Y)$ in $G - x$ with $|X| = |Y| + 1$, then $d_G(x) \geq n + 1$.*

Proof. Sufficiency is immediate from Theorem 1.1. Now we prove the necessity.

Since G is a Type-A defect n-extendable graph, Theorem 1.1 implies that we only need to prove statement (3.2). Assume $|X| = |Y| + 1 = m + 1$. We discuss three cases.
Case 1. $m = 0$. Then $|X| = 1$ and $|Y| = 0$. So H is isomorphic to K_1.
Case 2. $m = 1$. Then $|X| = 2$ and $|Y| = 1$. Since H is connected, we have H is isomorphic to $K_{2,1}$.
Case 3. $m \geq 2$. Let $s = \min\{n, m - 1\}$. Since $n \geq 2$ and $m \geq 2$, we have $s \geq 1$. Theorem 1.1(4.2) implies that H is defect s-extendable. Thus it suffices to prove that $\kappa(H) \geq 2$. Suppose to the contrary $\kappa(H) = 1$. We consider two cases.
Case 3.1. Suppose H is a Type-B graph.

Then there is a cut vertex w of H in X. Clearly, $w \in W$. Let $H_1 = (X_1, Y_1)$ and $H_2 = (X_2, Y_2)$ be two components of $H - w$ where $X_i \in X$ and $Y_i \in Y$, $i = 1, 2$. Since H is defect s-extendable and $w \in X$, Theorem 1.1(1) and (2) imply that $|X_i| = |Y_i| \geq 1$, $i = 1, 2$.

Note that $xv \in E(G)$ for some $v \in X_1$. Otherwise H_1 is a component of $G - w$ and hence w is a cut vertex of G in W, a contradiction, since G is a Type-A graph. Obviously, $wy_2 \in E(G)$ for some $y_2 \in Y_2$ as H_2 is a component of $H - w$. Since $|Y_1| > |X_1 \setminus \{v\}|$ and vertices in Y_1 can only join to vertices in $X_1 \setminus \{v\}$ in $G - \{v, x, w, y_2\}$, there is no near perfect matching in G containing $\{vx, wy_2\}$, a contradiction to the hypothesis that G is defect n-extendable where $n \geq 2$.
Case 3.2. Suppose H is a Type-A graph.

Then there is no cut vertex of H in X. Since $\kappa(H) = 1$, there is a cut vertex, say y, in Y and Theorem 1.1 (1) and (2) imply that there are exactly two components $C_1 = (U_1, W_1)$ and $C_2 = (U_2, W_2)$ in $H - y$ where $W_i \subseteq X$ and $U_i \subseteq Y$, $i = 1, 2$. Then $xz \in E(G)$ for some $z \in W_1 \cup W_2$. Without loss of generality, assume $z \in W_1$. Since H is defect s-extendable and $y \in Y$, Theorem 1.1(1) and (4) imply that $|W_i| = |U_i| + 1$, $i = 1, 2$. Let $k_i = |U_i|$, $i = 1, 2$. Then $k_i \geq 0$.

Suppose $U_1 = \phi$. Then $W_1 = \{z\}$. Note that $xa \in E(G)$ for some $a \in W_2$, otherwise, $G - z$ is disconnected, while $z \in W$, a contradiction, since G is a Type-A graph. Since $U_1 = \phi$ and $|U_1 \cup U_2 \cup \{y\}| = m \geq 2$, we have $U_2 \neq q\phi$. Note that y joins to a vertex b in W_2 such that $a \neq b$. Otherwise, y only joins vertex a in W_2 and hence $H_2 - a$ and H_1 are different components in $G - a$, a contradiction to the assumption hypothesis that G is Type-A graph. It is not difficult to see that $|U_2| > |W_2 \setminus \{a, b\}|$ and vertices in U_2 can only join to vertices in $W_2 \setminus \{a, b\}$ in $G - \{a, x, b, y\}$, so there is no near perfect matching in G containing $\{ax, by\}$, a contradiction to the hypothesis that G is defect n-extendable where $n \geq 2$.

So $U_1 \neq \phi$. Analogous to the proof above, we can prove that y joins to a vertex c in W_1 such that $c \neq z$ and there is no near perfect matching in G containing matching $\{zx, cy\}$, a contradiction to the hypothesis that G is defect n-extendable where $n \geq 2$.

In both Cases 3.1 and 3.2, we can find a contradiction. Therefore, $\kappa(H) \geq 2$ and statement (3.2) follows. □

Theorem 5.2. *Let $G = (U, W)$ with $\kappa(G) = 1$ and $|W| = |U| + 1$ be a Type-B bipartite graph. Let n be a positive integer such that $2 \leq n \leq |U| - 1$ and v be a cut vertex of G in W. Then G is defect n-extendable if and only if for any component $H = (X, Y)$ in $G - v$, the following statements hold:*

(1) $|X| = |Y|$.
(2) *If* $|X| = |Y| = m$, *then*
  (2.1) *H is s-extendable where $s = \min\{n - 1, m - 1\}$;*
  (2.2) *$H' = G[V(H) \cup \{v\}]$ is defect t-extendable where $t = \min\{n, m - 1\}$ and if $\kappa(H') = 1$, then H' is a Type-A graph.*

Proof. Sufficiency is immediate by Theorem 1.1(1)–(3). To prove the necessity, by Theorem 1.1, we only need to prove that $H' = G[V(H) \cup \{v\}]$ is a Type-A graph when $\kappa(H') = 1$. Assume $\kappa(H') = 1$. Suppose to the contrary H' is not Type-A graph.

Without loss of generality, assume $X \subseteq U$ and $Y \subseteq W$, then $X' = X$ and $Y' = Y \cup \{v\}$ are the two bipartitions of H' with $|Y'| = |X'| + 1$.

Since H' is not Type-A graph, then H' is a Type-B graph and hence H' has a cut vertex y in Y'. Let $H_1 = (U_1, W_1)$ be the component in $H' - y$ that contains v where $U_1 \subseteq U$ and $W_1 \subseteq W$. Let H_2 be another component in $H' - y$. Note that H_1 and H_2 are in different components of $G - y$. Therefore, y is also a cut vertex in G.

Let $H'' = (X'', Y'')$ be a component in $G - v$ such that $H'' \neq H$, $X'' \subseteq U$ and $Y'' \subseteq W$ and $Q = (U', W')$ be the component in $G - y$ that contains vertex v, $U' \subseteq U$ and $W' \subseteq W$.

Note that $y \notin V(H_1) \cup V(H'')$, $v \in V(Q)$, $v \in V(H_1)$ and v joins to at least a vertex in H''. So $V(H_1) \cup V(H'') \subseteq V(Q)$ and hence $U_1 \cup X'' \subseteq U'$. Since H' is defect $\min\{n, m - 1\}$ extendable and H_1 is a component of $H' - y$, by Theorem 1.1(1) and (2), we have $|U_1| = |W_1| \geq 1$. Analogously, we have $|X''| = |Y''| \geq 1$. Thus $|U'| \geq |U_1 \cup X''| \geq 2$. Furthermore, since Q is a component of $G - y$, by Theorem 1.1(3) again, we have Q is $\min\{n - 1, |U'| - 1\} \geq 1$ extendable as $n \geq 2$. So Lemma 2.8 implies that $\kappa(Q) \geq 2$. However, note that $H_1 - v$ and H'' are in different components of $Q - v$. Hence $\kappa(Q) = 1$, a contradiction to $\kappa(Q) \geq 2$. Thus H' is a Type-A graph. □

Remark 5.3. A bipartite graph $G = (U, W)$ with $\kappa(G) = 1$ and $|W| = |U| + 1$ is a Type-A graph can be determined in $O(p^2)$ time. This together with Theorem 5.1 imply that verifying if G is Type-A defect n-extendable ($n \geq 2$) can be done in polynomial time as statements (1), (2), (3.1) and (3.4) can be checked in $O(p^2)$ time, Section 4 shows that verifying statement (3.2) needs $O(p^6)$ time and verifying statement (3.3) needs $O(p^2q)$ time as it only needs to test if a bipartite graph is t-extendable p times at most and each can be done in $O(pq)$ time [13]. So the total time complexity to determine if G is Type-A defect n-extendable using Theorem 5.1 is $O(p^6)$. Moreover, by Theorem 5.2, to determine if G is Type-B defect n-extendable can also be done in polynomial time.

Acknowledgements

The authors are grateful to the anonymous referees for his/her careful reading and many valuable suggestions. Financial support from the Natural Science Foundation of Guangdong Province (No. 9451009001002740) and the Natural Science Foundation of Guangdong Province (No. 9451030007003340) are also gratefully acknowledged.

References

[1] R.E.L Aldred, D.A. Holton, Dingjun Lou, Akira Saito, M-alternating paths in n-extendable bipartite graphs, Discrete Math. 269 (2003) 1–11.
[2] J.A. Bondy, U.S.R Murty, Graph Theory, Springer, New York, 2007.
[3] C.H.C. Little, D.D. Grant, D.A. Holton, On defect d-matching in graphs, Discrete Math. 13 (1975) 41–54.
[4] G.Z. Liu, Q.L. Yu, Generalization of matching extensions in graphs, Discrete Math. 231 (2001) 311–320.
[5] Dingjun Lou, Akira Saito, Lihua Teng, A note on internally disjoint alternating paths in bipartite graphs, Discrete Math. 290 (2005) 105–108.
[6] S. Micali, V.V. Vazirani, An $O(|V|^{1/2}|E|)$ algorithm for finding maximum matching in general graphs, in: The 21st Annual Symposium on Foundations of Computer Science, Syracuse, NY, 1980, pp. 17–27.
[7] M.D. Plummer, On n-extendable graphs, Discrete Math. 31 (1980) 201–210.
[8] M.D. Plummer, Matching extension in bipartite graphs, Congr. Numer. 54 (1986) 245–258.
[9] R.E. Tarjan, A simple version of Katzanov's blocking flow algorithm, Oper. Res. Lett. 2 (1984) 265–268.
[10] Xuelian Wen, Dingjun Lou, Characterizing defect n-extendable bipartite graphs with different connectivities, Discrete Math. 307 (2007) 1898–1908.
[11] Xuelian Wen, Dingjun Lou, Path decomposition of defect 1-extendable bipartite graphs, Australas. J. Combin. 39 (2007) 175–182.
[12] Xuelian Wen, Zihui Yang, Minimum degree of minimal defect n-extendable bipartite graphs, Discrete Math. 309 (2009) 6255–6264.
[13] Fuji Zhang, Heping Zhang, Construction for bicritical graphs and k-extendable bipartite graphs, Discrete Math. 306 (2006) 1415–1423.