JOURNAL OF APPROXIMATION THEORY 49, 110–114 (1987)

Notes on Lacunary Interpolation with Splines IV. (0, 2) Interpolation with Splines of Degree 6

TH. FAWZY AND F. HOLAIL

Mathematics Department, Suez-Canal University, Ismailia, Egypt

Communicated by Paul G. Nevai

Received August 31, 1984

1. INTRODUCTION

In 1957 Turan and Balazs [1] initiated the study of "lacunary interpolation." Several authors have studied the use of splines to solve such interpolation problems (see, e.g., Meir and Sharma [7]; Swartz and Varga [10]; Varma [11]; Mishra and Mathur [8]). All of these methods are global and require the solution of a large system of equations.

Recently, Fawzy [2–5] presented several local methods for solving lacunary interpolation problems using piecewise polynomials with certain continuity properties.

In this paper we study the following (0, 2)-interpolation problem:

Problem 1: Given Δ: $\{x_i = ih\}_{i=0}^{n}$ and real numbers $\{f_i, f_i''\}_{i=0}^{n}$, find S such that

$$S(x_i) = f_i, \qquad S''(x_i) = f_i'', \qquad i = 0, 1, ..., n. \tag{1.1}$$

The purpose of this paper is to construct a spline method for solving problem 1 using piecewise polynomials of degree 6 such that for all functions $f \in C^6$, the order of approximation is the same as the best order of approximation using 6th degree splines.

2. CONSTRUCTION OF THE SPLINE INTERPOLANT

We shall construct a solution S of problem 1 in the form:

$$S_\Delta(x) = S_k(x) = \sum_{j=0}^{6} \frac{S_k^{(j)}}{j!} (x - x_k)^j, \qquad x \in [x_k, x_{k+1}], \tag{2.1}$$

where $k = 0, 1, ..., n-1$.

0021-9045/87 $3.00

We shall define each of the $S_k^{(j)}$ explicitly in terms of the data. In particular we choose

$$S_k^{(0)} = f_k, \qquad S_k^{(2)} = f_k^{(2)}, \qquad k = 0, 1, ..., n-1. \tag{2.2}$$

For $k = 1, 2, ..., n-3$, we take

$$S_k^{(6)} = \frac{1}{h^4}\{f''_{k+3} - 4f''_{k+2} + 6f''_{k+1} - 4f''_k + f''_{k-1}\} \tag{2.3}$$

$$S_k^{(5)} = \frac{1}{h^3}\{f''_{k+2} - 3f''_{k+1} + 3f''_k - f''_{k-1}\} - \frac{h}{2} S_k^{(6)} \tag{2.4}$$

$$S_k^{(4)} = \frac{1}{h^2}\{f''_{k+1} - 2f''_k + f''_{k-1}\} - \frac{h^2}{12} S_k^{(6)} \tag{2.5}$$

$$S_k^{(3)} = \frac{1}{h}\left\{f''_{k+1} - f''_k - \sum_{r=2}^{4} \frac{h^r}{r!} S_k^{(r+2)}\right\} \tag{2.6}$$

and

$$S_k^{(1)} = \frac{1}{h}\left\{f_{k+1} - f_k - \frac{h^2}{2} f''_k - \sum_{r=3}^{6} \frac{h^r}{r!} S_k^{(r)}\right\}. \tag{2.7}$$

For $k = 0$, we choose

$$S_0^{(6)} = S_1^{(6)} \tag{2.8}$$

$$S_0^{(5)} = S_1^{(5)} - hS_1^{(6)} \tag{2.9}$$

$$S_0^{(4)} = S_1^{(4)} - hS_1^{(5)} - \frac{h^2}{2} S_1^{(6)} \tag{2.10}$$

$$S_0^{(3)} = \frac{1}{h}\left\{f''_1 - f''_0 - \sum_{r=2}^{4} \frac{h^r}{r!} S_0^{(r+2)}\right\} \tag{2.11}$$

and

$$S_0^{(1)} = \frac{1}{h}\left\{f_1 - f_0 - \frac{h^2}{2} f''_0 - \sum_{r=3}^{6} \frac{h^r}{r!} S_0^{(r)}\right\}. \tag{2.12}$$

Finally, for $k = n-2$ and $n-1$, we take

$$S_k^{(j)} = S_{k-1}^{(j)}(x_k), \qquad j = 1, 3, 4, 5, \text{ and } 6. \tag{2.13}$$

Clearly, the function S defined in (2.1)–(2.13) solves the (0, 2)-interpolation problem 1. Moreover, by construction it is clear that S is a piecewise polynomial of degree 6.

The $S_k^{(3)}$ have been chosen to make S''_Δ right continuous, i.e.,

$$D_L^2 S_k(x_{k+1}) = D_R^2 S_{k+1}(x_{k+1}),$$

while the $S_k^{(1)}$ have been chosen to make S continuous. Thus,

$$S \in C^{(0,2)}[x_0, x_n] = \{f \in C[x_0, x_n] : D_R^2 f \in C[x_0, x_n]\}, \tag{2.14}$$

where D_R is the right derivative.

Indeed, S is the unique piecewise polynomial of degree 6 in

$$C^{(0,2)}[x_0, x_n] \cap C^6[x_{n-3}, x_n],$$

satisfying the interpolation conditions (1.1).

S is a special kind of g-spline; we refer to it as a lacunary g-spline.

3. Error Bounds

Suppose $f \in C^6[x_0, x_n]$. Then, using the Taylor and dual Taylor expansions it is easy to establish the following lemma estimating how well the $S_k^{(j)}$ approximate $f^{(j)}(x_k)$ in terms of the modulus of continuity $\omega(D^6 f; h)$ of $f^{(6)}(x)$.

Lemma 3.1. *For $j = 1, 3, 4, 5, 6$, we have*

$$|S_k^{(j)} - f^{(j)}(x_k)| \leqslant c_{kj} h^{6-j} \omega(D^6 f; h), \tag{3.1}$$

where the constants c_{kj} are given in the following table:

	c_{k1}	c_{k3}	c_{k4}	c_{k5}	c_{k6}
$k = 0$	$\frac{3365}{4320}$	$\frac{301}{36}$	$\frac{467}{36}$	$\frac{59}{6}$	$\frac{17}{3}$
$1 \leqslant k \leqslant n-3$	$\frac{221}{1440}$	$\frac{31}{24}$	$\frac{29}{36}$	$\frac{25}{6}$	$\frac{14}{3}$
$k = n-2$	$\frac{4951}{4320}$	$\frac{119}{24}$	$\frac{263}{36}$	$\frac{53}{6}$	$\frac{17}{3}$
$k = n-1$	$\frac{4543}{864}$	$\frac{141}{8}$	$\frac{683}{36}$	$\frac{29}{2}$	$\frac{20}{3}$

Theorem 3.1. *Let $f \in C^6[x_0, x_n]$ and let S_Δ be the unique lacunary g-spline constructed in* (2.1)–(2.13). *Then, for all* $0 \leqslant j \leqslant 6$,

$$\|D^j(f - S_\Delta)\|_{L_\infty[x_k, x_{k+1}]} \leqslant c_{kj}^* h^{6-j} \omega(D^6 f; h), \tag{3.2}$$

where the constants c^*_{kj} *are given in the following table*:

	c^*_{k0}	c^*_{k1}	c^*_{k2}	c^*_{k3}	c^*_{k4}	c^*_{k5}	c^*_{k6}
$k=0$	$\frac{1{,}009}{360}$	$\frac{32{,}569}{4{,}320}$	$\frac{301}{18}$	$\frac{979}{36}$	$\frac{923}{36}$	$\frac{31}{2}$	$\frac{17}{3}$
$1 \leqslant k \leqslant n-3$	$\frac{479}{1{,}080}$	$\frac{4{,}951}{4{,}320}$	$\frac{31}{12}$	$\frac{119}{24}$	$\frac{263}{36}$	$\frac{53}{6}$	$\frac{14}{3}$
$k=n-2$	$\frac{1{,}261}{540}$	$\frac{23{,}033}{4{,}320}$	$\frac{743}{72}$	$\frac{141}{8}$	$\frac{683}{36}$	$\frac{29}{2}$	$\frac{17}{3}$
$k=n-1$	$\frac{19{,}691}{2{,}160}$	$\frac{4{,}871}{288}$	$\frac{1073}{36}$	$\frac{1079}{24}$	$\frac{1325}{36}$	$\frac{127}{6}$	$\frac{20}{3}$

Proof. We sketch the proof of the theorem for $1 \leqslant k \leqslant n-3$, while for $k=0$, $n-2$ and $n-1$, similar procedures lead to the required results.

Suppose $1 \leqslant k \leqslant n-3$, and that $x_k \leqslant x \leqslant x_{k+1}$. Then, using the Taylor expansion of f, we have

$$\begin{aligned} |f(x)-S_\Delta(x)| &= |f(x)-S_k(x)| \\ &\leqslant \sum_{j=0}^{5} \frac{|f^{(j)}(x_k)-S_k^{(j)}|}{j!} h^j + \frac{|f^{(6)}(\xi_k)-S_k^{(6)}|}{6!} h^6 \\ &\leqslant \sum_{j=0}^{6} \frac{|f^{(j)}(x_k)-S_k^{(j)}|}{j!} h^j + \frac{|f^{(6)}(\xi_k)-f^{(6)}(x_k)|}{6!} h^6, \end{aligned}$$

where $x_k < \xi_k < x_{k+1}$. Using Lemma 3.1 and the definition of the modulus of continuity of $f^{(6)}(x)$, we easily obtain the required result. The other results for derivatives can be easily obtained by following the same technique.

4. Numerical Results

The method is tested by the following example:

$$f(x) = 1 + xe^x, \qquad x_0 = 0,\ x_n = 1.0,\ h = 0.1.$$

The following results are obtained for $x = 0{,}55$:

	Exact value	Numerical value	Error
f	1.953289160	1.953289159	$1(10)^{-9}$
f'	2.686542178	2.686542177	$1(10)^{-9}$
f''	4.419795196	4.419796802	$1.606(10)^{-6}$
$f^{(3)}$	6.153048214	6.153042055	$6.159(10)^{-6}$
$f^{(4)}$	7.886301232	7.884871125	$1.430107(10)^{-3}$
$f^{(5)}$	9.61955425	9.635925000	$1.637075(10)^{-2}$
$f^{(6)}$	11.35280727	12.0521	$6.9929273(10)^{-1}$

5. Remarks

(i) The method defined here, in contrast to the methods of [7], [10], and [11], does not require any end conditions.

(ii) Similar lacunary methods were constructed for degrees $r=2, 3, 4, 5$ in [3] and [4]. These earlier methods give optimal order of approximation for functions in $C^r[x_0, x_n]$, but not for $r=6$.

(iii) Other lacunary interpolation problems, e.g., (0, 3), (0, 4), (0, 1, 3), (0, 2, 3), and (0, 2, 4), with similar constructions and optimal approximation results are submitted for publication in other journals.

(iv) Details for the computations of the constants c_{kj} and c_{kj}^* can be found in [2], [3], and [4]. The constants presented here are not guaranteed to be the best.

Acknowledgment

The authors are very grateful to Prof. Larry Schumaker for many valuable suggestions during the preparation of this work.

References

1. J. Balazs and P. Turan, Notes on interpolation. II, III, IV, *Acta Math. Acad. Sci. Hungar*, **8** (1957), 201–215; **9** (1958), 195–214, 243–258.
2. Th. Fawzy, Note on lacunary interpolation with splines. I. (0, 3)-Lacunary interpolation, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **28** (1985).
3. Th. Fawzy, Note on lacunary interpolation by splines. II, *Ann. Univ. Sci. Budapest. Sect. Comput.* **5**, in press.
4. Th. Fawzy, Notes on lacunary interpolation by splines. III. (0, 2) Case, *Acta Math. Acad. Sci. Hungar.*, in press.
5. Th. Fawzy, Lacunary interpolation by g-splines: (0, 2, 3) Case, *Studia Sci. Math. Hungar.*, in press.
6. Z. R. Guo, Note on lacunary interpolation by splines, *Math. Num. Sinica* **3** (2) (1981), 175–178.
7. A. Meir and A. Sharma, Lacunary interpolation by splines, *Siam J. Num. Anal.* **10** (3) (1973).
8. R. S. Mishra and K. K. Mathur, Lacunary interpolation by splines: (0, 2, 3) and (0, 2, 4) Cases, *Acta Math. Acad. Sci. Hungar.* **36** (1980), 251–260.
9. L. Schumaker, "Spline Functions: Basic Theory," Wiley, New York/Toronto, 1981.
10. B. K. Swartz and R. S. Varga, A note on lacunary interpolation by splines, *Siam J. Num. Anal.* **10** (1) (1973), 443–447.
11. A. K. Varma, Lacunary interpolation by splines, *Acta Math. Acad. Sci. Hungar.* **31** (3–4) (1978), 183–192.