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TOPOLOGY ON $S^{-1}S$ FOR BANACH ALGEBRAS

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IN THIS note we show that Quillen's $S^{-1}S$-construction on the category of finitely generated projective modules over a Banach algebra Λ with identity has a topological enrichment $S^{-1}S^{\text{top}}(\Lambda)$ and $B(S^{-1}S^{\text{top}}(\Lambda))$ has the homotopy type of $K_0\Lambda \times \mathrm{BGL}^{\text{top}}\Lambda$. By applying the singular complex functor, we obtain a first quadrant spectral $E^1_{p,q}(\Lambda) = K^{\text{alg}}_q\Lambda(\Delta^p)$, where $\Lambda(\Delta^p)$ is the ring of continuous Λ-valued functions on Δ^p, that converges to $K^{\text{top}}_{p+q}\Lambda := \pi_{p+q}(K_0\Lambda \times \mathrm{BGL}^{\text{top}}\Lambda)$ for $p + q \geqslant 0$. Copyright © 1996 Elsevier Science Ltd

0. INTRODUCTION

Let Λ be a Banach algebra with identity. For $p > 0$ the topological K-theory of Λ is defined to be $K^{\text{top}}_p\Lambda := \pi_p\mathrm{BGL}^{\text{top}}\Lambda$, where $\mathrm{GL}^{\text{top}}\Lambda$ is the colimit of finite-dimensional invertible matrices over Λ and $\mathrm{BGL}^{\text{top}}\Lambda$ is its classifying space. Since the Grothendieck group $K_0\Lambda$ of Λ only depends on the algebraic structure of Λ, $K^{\text{top}}_*\Lambda = \pi_*(K_0\Lambda \times \mathrm{BGL}^{\text{top}}\Lambda)$ where $K_0\Lambda$ is given the discrete topology. On the other hand, the algebraic K-theory of Λ is defined to be $K^{\text{alg}}_*\Lambda := \pi_*(K_0\Lambda \times \mathrm{BGL}^{\delta}\Lambda^+)$, where $\mathrm{GL}^{\delta}\Lambda$ is the colimit of the discrete groups of finite-dimensional invertible matrices over Λ and $(-)^+$ is Quillen's plus construction [2]. The map $\mathrm{GL}^{\delta}\Lambda \to \mathrm{GL}^{\text{top}}\Lambda$ induces a map

$$K_0\Lambda \times \mathrm{BGL}^{\delta}\Lambda^+ \to K_0\Lambda \times \mathrm{BGL}^{\text{top}}\Lambda \tag{1}$$

of topological spaces and a map

$$K^{\text{alg}}_*\Lambda \to K^{\text{top}}_*\Lambda, \quad * \geqslant 0 \tag{2}$$

from the algebraic K-theory to the topological K-groups of Λ.

In this note we approach the topological and algebraic K-theory of Λ through Quillen's $S^{-1}S$-construction [7]. Recall that if $\mathscr{P}$ is the category of finitely generated (left) projective Λ-modules, then the homotopy groups of the classifying space of the category $S^{-1}S(\mathscr{P})$ are the algebraic K-theory of Λ—we shall assume that most *all* categories are small. Like the topological enrichment $\mathrm{GL}^{\text{top}}\Lambda$ of $\mathrm{GL}^{\delta}\Lambda$, there is a topological enrichment $S^{-1}S^{\text{top}}(\mathscr{P})$ of the category $S^{-1}S(\mathscr{P})$. The central result of this note, Theorem 3.1, asserts $K^{\text{top}}_p\Lambda \cong \pi_pS^{-1}S^{\text{top}}(\mathscr{P})$ and the forgetful functor

$$S^{-1}S(\mathscr{P}) \to S^{-1}S^{\text{top}}(\mathscr{P}) \tag{3}$$

induces the map in (2). This result is part of the folklore of group completions [1]; however, the proof given does not deal with localization of homology groups. Instead we show that there is a continuous extension of the setting considered in [7] and the fibres of the continuous map $\mathrm{BGL}^{\text{top}}\Lambda \to \mathrm{B}(S^{-1}S^{\text{top}}(\mathscr{P}))$ are contractible; in fact these fibres can be identified with a realization of a simplicial model of frames in Λ^{∞}. The discerning feature of the functor in (3) is that it is the "identity," whereas the map in (1) is induced by the "identity" and involves universal properties of H-spaces and the + -construction.

In Section 1 we extend Quillen's Theorem A [19], which says that a functor with contractible fibres induces a homotopy equivalence of classifying spaces, to topological categories, and mention that Thomason's homotopy colimit theorem [24] also has a topological extension. We topologize Quillen's $S^{-1}S$-construction in Section 3 after giving a Banach space analog of the Stiefel manifold of k-frames in $\mathbb{R}^n$.

In Section 4 we replace the topological category with the simplicial category

$$[p] \mapsto S^{-1}S^{\mathrm{top}}(\mathscr{P})(\Delta^p).$$

Using Karoubi's extension [11] of a theorem of Swan [23], which states that there is equivalence between the category of $\mathscr{P}$-fibre bundles over a compact space X and the category of finitely generated projective modules over the ring of continuous functions from X to Λ, we are able to identify the homotopy groups of the classifying space of $S^{-1}S^{\mathrm{top}}(\mathscr{P})(\Delta^p)$ with the algebraic K-theory of $\Lambda(\Delta^p)$, where $\Lambda(\Delta^p)$ is the ring of continuous Λ-valued functions on the geometric p-simplex Δ^p. We then obtain a spectral sequence

$$K_q^{\mathrm{alg}}\Lambda(\Delta^p) = E^1_{p,q}(\Lambda) \;\Rightarrow\; K^{\mathrm{top}}_{p+q}\Lambda$$

whose edge homomorphism is the map in (2). If $\Lambda = \mathbb{C}(X)$ is the commutative Banach algebra of continuous $\mathbb{C}$-valued functions on X, then

$$K_1^{\mathrm{alg}}\Lambda(\Delta^p) \cong \mathbb{C}^*(X \times \Delta^p) \oplus [X, \mathrm{SU}]$$

$$E^2_{p,1}(\Lambda) \cong \begin{cases} H^1(X,\mathbb{Z}) \oplus [X, \mathrm{SU}] & \text{if } p = 0 \\ H^{1-p}(X,\mathbb{Z}) & \text{if } p > 0. \end{cases}$$

Motivated by this observation we define the homotopy groups of the algebraic K-theory of Λ to be $\pi_p K_q^{\mathrm{alg}}\Lambda := E^2_{p,q}(\Lambda)$.

1. TOPOLOGICAL CATEGORIES

The term *topological category* should be taken to mean a category in which each hom set is endowed with the structure of a compactly generated topological space and the usual structure maps are continuous. Let $\mathscr{C}$ and $\mathscr{D}$ be topological categories. A functor $F: \mathscr{C} \to \mathscr{D}$ is called *continuous* if for each pair (A, B) of objects of $\mathscr{C}$ the map $F: \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ is continuous. Let Y be an object in $\mathscr{D}$. The comma category Y/F is not, in general, a topological category, however, it does have the structure of a category object (see [5, Section 7]) in **Top**, the cartesian closed category of compactly generated topological spaces. To be precise, the space of objects of Y/F is

$$\mathrm{obj}\; Y/F = \coprod_X \mathrm{Hom}_{\mathscr{D}}(Y, F(X))$$

and the space of arrows is

$$\mathrm{arr}\; Y/F = \coprod_{X_0, X_1} \mathrm{Hom}_{\mathscr{D}}(Y, F(X_0)) \times \mathrm{Hom}_{\mathscr{C}}(X_0, X_1).$$

Similarly one can define the category object F/Y. Note that the product of two topological spaces is given the compactly generated topology [21].

Given a category object $\mathscr{C}$ in **Top**, let $\mathrm{N}\mathscr{C}$ denote the nerve of $\mathscr{C}$. Recall that $\mathrm{N}\mathscr{C}$ is a simplicial object in **Top**, and the classifying space of $\mathscr{C}$ is the geometric realization $\mathrm{B}\mathscr{C} = |\mathrm{N}\mathscr{C}|$ of $\mathrm{N}\mathscr{C}$ obtained from $\coprod_n \Delta^n \times \mathrm{N}\mathscr{C}_n$ by the relation $(t, \alpha^* x) \sim (\alpha_* t, x)$ for all $\alpha: [m] \to [n]$, $t \in \Delta^m$ and $x \in \mathrm{N}\mathscr{C}_n$.

Let $\mathscr{C}$ be a topological category and X an object of $\mathscr{C}$. Denote by $X/\mathscr{C}$ the comma category X/F with F equal to the identity functor. Let $\rho: X/\mathscr{C} \to X/\mathscr{C}$ be the continuous functors given by $\rho(f) = id_X$.

LEMMA 1.1. *Let $\mathscr{C}$ be a topological category, and let X be an object in $\mathscr{C}$. There is a continuous natural transformation $\eta: \rho \xrightarrow{\bullet} 1$. In particular the classifying space $\mathrm{B}(X/\mathscr{C})$ is contractible.*

Proof. Let η: obj $X/\mathscr{C} \to$ arr $X/\mathscr{C}$ be the map given by

$$(f: X \to X_0) \mapsto (id: X \to X, f: X \to X_0).$$

Clearly η is continuous. Let $(f: X \to X_0, f_0: X_0 \to X_1)$ be an arrow in $X/\mathscr{C}$. Since the following diagram is commutative.

$$\begin{array}{ccc} (id: X \to X) & \to & (f: X \to X_0) \\ \| & & \downarrow \\ (id: X \to X) & \to & (f_0 \circ f: X \to X_1) \end{array}$$

η is a natural transformation. □

Remark. A similar proof that the space $\mathrm{B}(\mathscr{C}/X)$ is also contractible for every X in $\mathscr{C}$.

Quillen's Theorem A. Recall that a bisimplicial space X is a functor $([p], [q]) \mapsto X_{p,q} \in \mathsf{Top}$. One may view X as a family of spaces in the first quadrant of the plane together with horizontal and vertical face and degeneracy operators that commute and satisfy the familiar identities [13]. Given a bisimplicial space X, there are natural homeomorphisms $|[p] \mapsto |X_{p,*}|| \cong |\mathrm{diag}\, X| \cong |[q] \mapsto |X_{*,q}||$, where $\mathrm{diag}\, X$ is the simplicial space $[p] \mapsto X_{p,p}$. Following Segal [20] we call the simplicial space X *good* if all the degeneracy operators $s_i: X_n \to X_{n+1}$ are closed cofibrations. Good simplicial spaces have the property that a simplicial map which is termwise a homotopy equivalence determines a homotopy equivalence of the corresponding geometric realizations.

The following theorem is a version of Quillen's Theorem A [19] for topological categories. We shall follow his proof with little change.

THEOREM 1.2. *Let $\mathscr{C}$ and $\mathscr{D}$ be topological categories, with the property that the inclusion of the point $*_X \mapsto 1_X \in \mathrm{Hom}(X, X)$ are cofibrations for all objects of $\mathscr{C}$ and $\mathscr{D}$, and let $F: \mathscr{C} \to \mathscr{D}$ be a continuous functor. If the classifying space $\mathrm{B}(Y/F)$ has the homotopy type of a point for each object Y in $\mathscr{D}$, then*

$$\mathrm{B}F: \mathrm{B}\mathscr{C} \to \mathrm{B}\mathscr{D}$$

is a homotopy equivalence.

Proof. Let $S(F)$ be the category object in Top whose space of objects is

$$\mathrm{obj}\, S(F) = \coprod_{X,Y} \mathrm{Hom}_{\mathscr{D}}(Y, F(X))$$

and whose space of arrows arr $S(F)$ is

$$\coprod_{Y_i, X_i} \mathrm{Hom}_{\mathscr{D}}(Y_1, Y_0) \times \mathrm{Hom}_{\mathscr{D}}(Y_0, F(X_0)) \times \mathrm{Hom}_{\mathscr{C}}(X_0, X_1).$$

A triple

$$(g_0: Y_1 \to Y_0, g: Y_0 \to F(X_0), f_0: X_0 \to X_1)$$

is an arrow from $g: Y_0 \to F(X_0)$ to $F(f_0) \circ g \circ g_0: Y_1 \to F(X_1)$. Define $\pi_1: S(F) \to \mathscr{D}^{\mathrm{op}}$ by sending $(g: Y \to F(X); X)$ to Y, and define $\pi_2: S(F) \to \mathscr{C}$ by sending the same object to X. Let $T(F)$ be the bisimplicial space whose (p, q)-simplices are of the form

$$(Y_q \to \cdots \to Y_0 \to F(X_0), X_0 \to \cdots \to X_p).$$

$T(F)_{p,q}$ is topologized as the disjoint union

$$\coprod_{X_i, Y_j} \mathrm{Hom}_{\mathscr{D}}(Y_q, Y_{q-1}) \times \cdots \times \mathrm{Hom}_{\mathscr{C}}(X_{p-1}, X_p).$$

Let $\mathrm{N}\mathscr{C}^v$ be the bisimplicial space equal to the nerve of $\mathscr{C}$ in the horizontal direction and constant in the vertical direction. The obvious projections induce a map

$$T(F) \to \mathrm{B}\mathscr{C}^v \tag{4}$$

of bisimplicial spaces. Since the nerve of $S(F)$ is equal to diagonal of $T(F)$, the realization of the map in (4) is equal to $\mathrm{B}\pi_2$. By first realizing (4) with respect to the vertical direction we get a map

$$\coprod_{X_i} \mathrm{B}(\mathscr{D}/F(X_p)) \times \mathrm{Hom}(X_p, X_{p-1}) \times \cdots \times \mathrm{Hom}(X_1, X_0) \to \mathrm{N}\mathscr{C}_p$$

of good simplicial spaces. By the previous lemma, $\mathrm{B}(\mathscr{D}/F(X_p))$ is contractible. It follows that $\mathrm{B}\pi_2$ is a homotopy equivalence. Similarly there is a map

$$T(F) \to \mathrm{N}\mathscr{D}^{\mathrm{op}h} \tag{5}$$

of bisimplicial spaces, where $\mathrm{N}\mathscr{D}^{\mathrm{op}h}$ is constant in the horizontal direction; furthermore, the realization of this map is $\mathrm{B}\pi_1$. By first realizing (5) with respect to the horizontal direction we get a map

$$\coprod_{Y_i} \mathrm{Hom}(Y_0, Y_1) \times \cdots \times \mathrm{Hom}(Y_{q-1}, Y_q) \times \mathrm{B}(Y_q/F) \to \mathrm{B}\mathscr{D}^{\mathrm{op}}$$

of good simplicial spaces. Since $\mathrm{B}(Y_q/F)$ is contractible, $\mathrm{B}\pi_1$ is a homotopy equivalence. Piecing these maps together we obtain the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{B}\mathscr{D}^{\mathrm{op}} & \xleftarrow{\mathrm{B}\pi_1} & \mathrm{B}S(F) & \xrightarrow{\mathrm{B}\pi_2} & \mathrm{B}\mathscr{C} \\
\downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle B(F)} \\
\mathrm{B}\mathscr{D}^{\mathrm{op}} & \xleftarrow{\mathrm{B}\pi_1} & \mathrm{B}S(id) & \xrightarrow{\mathrm{B}\pi_2} & \mathrm{B}\mathscr{D}
\end{array}$$

where the middle map is induced by the continuous functor $S(F) \to S(1_{\mathscr{D}})$

$$(Y, X; g: Y \to F(X)) \mapsto (Y, F(X); g: Y \to F(X)).$$

Since all the horizontal maps are homotopy equivalences, it follows that $\mathrm{B}F$ is a homotopy equivalence. □

Grothendieck construction. Let **Cat** be the category of small categories and functors. Suppose $F: J \to \mathbf{Cat}$ is a small diagram. By composing F with the nerve functor N, we obtain a small diagram $\mathrm{N}F$ of simplicial sets. Thomason [24] has shown that the homotopy colimit [3] of $\mathrm{N}F$ is, up to weak equivalence, the nerve of $J \int F$, the Grothendieck construction on F. $J\int F$ is the category whose objects are the pairs (j, X), where j is an object in J and X is an object in $F(J)$. An arrow from (j, X) to (k, Y) consists of a pair (α, ϕ), where $\alpha: j \to k$ is an arrow in J and $\phi: F(\alpha)(X) \to Y$ is an arrow in $F(k)$.

If F is a small diagram of topological categories, composition with the classifying space functor B gives a small diagram $j \mapsto \mathrm{B}F(j)$ of topological spaces. The simplicial replacement functor $\amalg_*$ associates to the J-indexed diagram $j \mapsto \mathrm{B}F(j)$ the simplicial space $[p] \mapsto \amalg_p \mathrm{B}F$, where

$$\amalg_p \mathrm{B}F = \coprod_{j_0 \to \cdots \to j_p} \mathrm{B}F(j_0).$$

The face and degeneracy maps d_j, s_j for $j > 0$ are induced by the identity $\mathrm{B}F(j_0) \to \mathrm{B}F(j_0)$ and $d_0: \mathrm{B}F(j_0) \to \mathrm{B}F(j_1)$ is the map associated to $j_0 \to j_1$. The homotopy colimit of the diagram $j \mapsto \mathrm{B}F(j)$ is the geometric realization of $\amalg_* \mathrm{B}F$.

THEOREM 1.3. *Let $F: J \to \mathbf{Cat}^{\mathrm{top}}$ be a small diagram of topological categories. Suppose the inclusion of the point corresponding to the identity map, $1_X \in \mathrm{Hom}_{F(j)}(X, X)$ is a cofibration for each object X in each $F(j)$, $j \in J$. Then there is a natural homotopy equivalence*

$$\eta: \underrightarrow{\mathrm{holim}}\, \mathrm{B}F \to \mathrm{B}(J\int F).$$

Proof. One can mimic Thomason's proof [24, Theorem 1.2]. □

2. FRAMES IN BANACH SPACES

Let k denote either the field of real or complex numbers. Recall that a Banach space is a complete normed k-vector space. Given two Banach spaces E and F the product norm on $E \times F$ is given by $|(e,f)| = \max\{|e|, |f|\}$. Recall that a k-linear map $T: E \to F$ is continuous if and only if there is a positive constant C such that $|T(e)| \leqslant C|e|$ for all $e \in E$. Let $L(E, F)$ be the k-vector space of continuous k-linear maps from E to F. The operator norm on $L(E, F)$ is given by

$$|T| = \sup\{|T(e)|: e \in E, |e| \leqslant 1\}.$$

It is well known (see [9]) that $L(E, F)$ is a Banach space and $|T \circ S| \leqslant |T||S|$ for all $T, S \in L(E, F)$.

A k-algebra Λ with unit I is a Banach algebra if it is a Banach space, $|I| = 1$ and $|\lambda \cdot \mu| \leqslant |\lambda| \cdot |\mu|$ for all $\lambda, \mu \in \Lambda$. If E is a Banach space then $L(E, E)$ is a Banach algebra, and the k-algebra $k(X)$ of continuous k-valued functions defined on a compact space X with norm $|f| = \sup\{|f(x)|: x \in X\}$ is a Banach algebra.

Let Λ be a Banach algebra, and let $\mathscr{P}(\Lambda)$ be the category of finitely generated projective (left) Λ-modules and Λ-linear maps. Given $P \in \mathscr{P}(\Lambda)$ choose a surjective map $\varphi: \Lambda^n \to P$, and give P the quotient topology.

LEMMA 2.1. *The topology on P is independent of the choice of surjective map φ above.*

Proof. If $\psi: \Lambda^m \to P$ is a surjective map let P_ψ be the space with the quotient topology induced by ψ. Given ϕ and ψ as above there exists a Λ-linear map $T: \Lambda^m \to \Lambda^n$ such that $\psi = \phi \circ T$. In particular the identity map $P_\psi \to P_\phi$ is continuous. Since there also exists a Λ-linear map $S: \Lambda^n \to \Lambda^m$ such that $\phi = \psi \circ S$, the two spaces P_ϕ and P_ψ have the same topology. □

Corollary 2.2. *Let P, Q be modules in $\mathscr{P}(\Lambda)$. Then P and Q are Banach spaces, and* $\mathrm{Hom}(P, Q)$ *is a Banach space with the operator norm.*

Proof. To prove that P is a Banach space it is enough to show that P is a closed linear subspace of some Banach space. Choose a surjective map $\phi: \Lambda^n \to P$, and let $\rho: P \to \Lambda^n$ be a section of ϕ. Then P is isomorphic to the image of ρ which is also equal to the kernel of the map $(I - \rho \circ \phi): \Lambda^n \to \Lambda^n$. Since a Λ-linear map is continuous, the result follows.

We shall prove that $\mathrm{Hom}(P, Q)$ is a closed linear subspace of $L(P, Q)$. Let $\{\phi_n\}$ be a Cauchy sequence in $\mathrm{Hom}(P, Q)$. Since $L(P, Q)$ is a Banach algebra $\lim \phi_n$ exists in $L(P, Q)$, and it is given by mapping $p \in P$ to $\lim \phi_n(p)$. Hence it is enough to show that $\lim \phi_n(\lambda \cdot p) = \lambda \cdot \lim \phi_n(p)$, $\lambda \in \Lambda$. This follows from $|(\phi_m - \phi_n)(\lambda \cdot p)| \leqslant |\lambda| \cdot |\phi_m - \phi_n| \cdot |p|$. □

Denote the Banach space of Λ-continuous maps from P to Q by $\mathrm{Hom}(P, Q)$, and the topological subspace of automorphisms of P by $\mathrm{Aut}(P)$.

Corollary 2.3. *The set* $\mathrm{Aut}(P)$ *is open in* $\mathrm{Hom}(P, P)$.

Proof. Let I be the identity map of P, and put $B_1(0)$ equal to the open subset $\{\varphi: P \to P: |\varphi| < 1\}$. Notice that if $\varphi \in B_1(0)$ then the sequence

$$I + \varphi + \varphi^2 + \cdots$$

converges to $(I - \varphi)^{-1}$. Hence the open ball $B_1(I) = \{I - \varphi: \varphi \in B_1(0)\} \subseteq \mathrm{Aut}(P)$ is a neighborhood of I. Since composition is continuous the open ball $B_1(I)$ can be translated to an open neighborhood of any $\psi \in \mathrm{Aut}(P)$. □

Banach manifolds. The natural inclusion $P \to P \oplus Q$ induces an injective map $\mathrm{Aut}(P) \to \mathrm{Aut}(P \oplus Q)$. In order to understand the coset space

$$\mathrm{Aut}(P \oplus Q)/\mathrm{Aut}(P)$$

we shall introduce the concept of a Banach manifold. Roughly speaking a Banach manifold is a topological space that is locally a Banach space. To be more specific we give the following (cf. [12])

Definition 2.4. Let X be a topological Hausdorff space.

(1) A chart for X is a pair (U, φ) where U is an open subset of X and φ is a homeomorphism of U onto an open subset $\varphi(U)$ of some Banach space.

(2) An atlas for X is a collection (U_i, φ_i), $i \in I$ of charts of X such that $X = \bigcup U_i$ and $\varphi_i(U_i \cap U_j)$ is an open subset of $\varphi_i(U_i)$ and

$$\varphi_j \varphi_i^{-1}: \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is an isomorphism for each pair of indices i, j.

A topological space that admits an atlas is called a manifold. In order to rule out pathological examples of manifolds we shall assume that all manifolds are paracompact with a countable base; in particular all manifolds are metrizable and have the homotopy type of CW-complexes [17]. By a Lie group we mean a manifold with a group structure.

LEMMA 2.5. *Let H be a closed subgroup of the Lie group G. Then the natural projection $\pi: G \to G/H$ is a Serre fibration.*

Proof. This follows from [14, 7.2 and 8.4]. □

Stiefel manifolds. Fix a Banach algebra Λ. Let $\mathscr{P} = \mathscr{P}(\Lambda)$, and for $P \in \mathscr{P}$ let $\mathscr{P}_n(P)$ be the full subcategory of the groupoid Iso $\mathscr{P}$ of isomorphisms in $\mathscr{P}$ whose objects are $A \in \mathscr{P}$ such that $P \oplus A \cong \Lambda^n$. Consider the space

$$V_n(P) = \varinjlim_{A \in \mathscr{P}_n(P)^{\mathrm{op}}} \mathrm{Iso}_{\mathscr{P}}(P \oplus A, \Lambda^n).$$

If $\phi \in \mathrm{Iso}_{\mathscr{P}}(P \oplus A, \Lambda^n)$ let $F(\phi): \mathrm{Iso}_{\mathscr{P}}(P \oplus \Lambda, \Lambda^n) \to \mathrm{Aut}(P \oplus A)$ be $F(\phi)(\psi) = \phi^{-1} \circ \psi$. If $\pi_0 \mathscr{P}_n(P)$ is a skeletal subcategory of $\mathscr{P}_n(P)$ then $V_n(P)$ is isomorphic to

$$\coprod_{A \in \pi_0 \mathscr{P}_n(P)} \mathrm{Aut}(P \oplus A)/\mathrm{Aut}(0 \oplus A).$$

The topological space $V_n(P)$ is an infinite-dimensional analog of the finite-dimensional Stiefel manifold of p-frames in k^n.

The map $\mathrm{Iso}_{\mathscr{P}}(P \oplus A, \Lambda^n) \to \mathrm{Iso}_{\mathscr{P}}(P \oplus A \oplus \Lambda, \Lambda^{n+1})$ given by $\phi \mapsto \phi \oplus I$ induces a map $V_n(P) \xrightarrow{\imath} V_{n+1}(P)$. Let $V_\infty(P)$ be the colimit of the sequence

$$\cdots \to V_n(P) \xrightarrow{\imath} V_{n+1}(P) \to \cdots. \tag{6}$$

PROPOSITION 2.6. *The space $V_\infty(P)$ has the homotopy of a point.*

Proof. First observe that $\imath: V_n(P) \to V_{n+1}(P)$ is a closed cofibration [17]. Thus $V_\infty(P)$ has the homotopy of a CW-complex (cf. [15, Appendix]), and it suffices to show that $V_\infty(P)$ has the weak homotopy of a point.

Suppose $\phi_0, \phi_1 \in V_\infty(P)$, then there is n such that ϕ_0, $\phi_1 \in V_n(P)$. We may view an element $\phi \in V_n(P)$ as monomorphisms from P to Λ^n. Indeed $\mathrm{Iso}_{\mathscr{P}}(P \oplus A, \Lambda^n)/\mathrm{Aut}(A)$ is a torsor under the contractible Banach Lie group $\mathrm{Hom}(A, P)$, and so $V_n(P) \to \mathrm{Mono}(P, \Lambda^n)$ is Serre fibration and a homotopy equivalence. Define a family of monomorphisms $\Phi_t: P \to \Lambda^n \oplus \Lambda^n$ for $0 \leqslant t \leqslant 1$ by

$$\Phi_t(p) = (t \cdot \phi_0(p) + (1 - t) \cdot \phi_1(p), t(- t) \cdot \phi_0(p)).$$

The monomorphism Φ determines a path from ϕ_0 and ϕ_1 in $V_{2n}(P)$ and shows that $V_\infty(P)$ is path connected.

Now consider an element $x \in \pi_q(V_\infty(P), *)$, where $* \in \mathrm{Iso}_{\mathscr{P}}(P \oplus A, \Lambda^n)/\mathrm{Aut}(A)$ is some fixed base point and $q > 0$. We may represent x by a map $f: S^q \to V_n(P)$ for any continuous map $S^q \to V_\infty(P)$ factors as $S^q \to V_n(P) \to V_\infty(P)$ by [21, Section 9] for some n. The fibration $\mathrm{Iso}_{\mathscr{P}}(P \oplus A, \Lambda^n) \xrightarrow{p} \mathrm{Iso}_{\mathscr{P}}(P \oplus A, \Lambda^n)/\mathrm{Aut}(A)$ induces an isomorphism (cf. [26, Ch. IV, Theorem 8.5])

$$p_*: \pi_p(\mathrm{Iso}_{\mathscr{P}}(P \oplus \mathrm{A}, \Lambda^n), \mathrm{Aut}(A)) \to \pi_p(\mathrm{Iso}_{\mathscr{P}}(P \oplus A, \Lambda^n)/\mathrm{Aut}(A))$$

and by fixing an isomorphism $P \oplus A \cong \Lambda^n$, we may identify $\mathrm{Iso}_{\mathscr{P}}(P \oplus A, \Lambda^n)$ with $\mathrm{Aut}(P \oplus A)$.

Given $x \in \pi_q(\mathrm{Aut}(P \oplus A), \mathrm{Aut}(0 \oplus A))$ is image in

$$\pi_q(\mathrm{Aut}(P \oplus A \oplus P \oplus A), \mathrm{Aut}(0 \oplus A \oplus P \oplus A))$$

induced by the map $S \mapsto S \oplus I$, is zero since the homotopy of maps $\mathrm{Aut}(P \oplus A) \to \mathrm{Aut}(P \oplus A \oplus P \oplus A)$ given by

$$H(t, S) = \begin{pmatrix} \cos \frac{\pi}{2} t & -\sin \frac{\pi}{2} t \\ \sin \frac{\pi}{2} t & \cos \frac{\pi}{2} t \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix}$$

for $S \in \mathrm{Aut}(P \oplus A)$, carries $\mathrm{Aut}(0 \oplus A)$ to $\mathrm{Aut}(0 \oplus A \oplus 0 \oplus A)$ for all t. □

3. QUILLEN'S $S^{-1}S$-CONSTRUCTION

Recall that Quillen's $S^{-1}S$-construction on the category $\mathscr{P}$ is the category whose objects are pairs (A, B) where $A, B \in \mathscr{P}$. A morphism from (A_1, B_1) to (A_2, B_2) is an equivalence class of pairs (f, S) where

$$f:(A_1 \oplus S, B_1 \oplus S) \to (A_2, B_2)$$

is a morphism in $\mathrm{Iso}\,\mathscr{P}^2$ and $(f, S) \sim (f', S')$ if there exists an isomorphism $\alpha: S \to S'$ with $f = f'(I \oplus \alpha, I \oplus \alpha)$ (cf. [7]). In particular

$$\mathrm{Hom}_{S^{-1}S}((A_1, B_1), (A_2, B_2)) = \varinjlim_{S \in \mathscr{P}(A, B)^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{Iso}\,\mathscr{P}^2}((A_1 \oplus S, B_1 \oplus S), (A_2, B_2))$$

where $\mathscr{P}(A, B)$ is the full subcategory of the groupoid $\mathrm{Iso}\,\mathscr{P}$ whose objects are $S \in \mathscr{P}$ such that $A_1 \oplus S \cong A_2$ and $B_1 \oplus S \cong B_2$. In [7], [6] it was proved that

$$\pi_i \mathrm{B}(S^{-1}S(\mathscr{P})) = K_i^{\mathrm{alg}}(\Lambda).$$

Topological $S^{-1}S$. From the discussion of the previous section on Stiefel manifolds it follows that $S^{-1}S(\mathscr{P})$ has a topological enrichment which we denote as $S^{-1}S^{\mathrm{top}}(\mathscr{P})$. Let $*$ be the base point in $\mathrm{B}(S^{-1}S(\mathscr{P}))$ determined by the pair (0, 0). To simplify notation we denote $S^{-1}S(\mathscr{P})$ and $S^{-1}S^{\mathrm{top}}(\mathscr{P})$ by $S^{-1}S$ and $S^{-1}S^{\mathrm{top}}$, respectively.

THEOREM 3.1. *The classifying space* $\mathrm{B}(S^{-1}S^{\mathrm{top}})$ *has the homotopy type of*

$$K_0(\Lambda) \times \mathrm{BGL}^{\mathrm{top}}\Lambda$$

and $\pi_i \mathrm{B}(S^{-1}S^{\mathrm{top}}) = K_i^{\mathrm{top}}\Lambda$. *Furthermore the obvious functor*

$$S^{-1}S \to S^{-1}S^{\mathrm{top}}$$

induces the standard map $K_i^{\mathrm{alg}}\Lambda \to K_i^{\mathrm{top}}\Lambda$.

Proof. The functor

$$+ : S^{-1}S \times S^{-1}S \to S^{-1}S$$

given by sending the pair $((A_0, A_1), (B_0, B_1))$ to $(A_0 \oplus B_0, A_1 \oplus B_1)$ induces an associative H-space structure on $\mathrm{B}(S^{-1}S)$. Since $+$ is clearly continuous, the classifying space $\mathrm{B}(S^{-1}S^{\mathrm{top}})$ is also an associative H-space. Moreover,

$$\pi_0 \mathrm{B}(S^{-1}S^{\mathrm{top}}) = \pi_0 \mathrm{B}(S^{-1}S) = K_0\Lambda$$

is a group. Since the H-space $\mathrm{B}(S^{-1}S^{\mathrm{top}})$ admits a homotopy inverse [6, Lemma 3.2], there is a homotopy equivalence

$$\mathrm{B}(S^{-1}S^{\mathrm{top}}) \simeq K_0(\Lambda) \times \mathrm{B}(S^{-1}S_0^{\mathrm{top}})$$

where $\mathrm{B}(S^{-1}S_0^{\mathrm{top}})$ is the connected component of the base point and $S^{-1}S_0$ is the full subcategory of pair (A_1, A_2) for which A_1 and A_2 are stably isomorphic.

Let $S^{-1}S_{\bar{0}}^{\mathrm{top}}$ be the full subcategory of pairs (B_1, B_2) with $B_1 \cong B_2$, and let $\imath$ denote the inclusion functor. Then $\mathrm{B}(\imath)$ is a homotopy equivalence, and it is enough to show that $\mathrm{B}(S^{-1}S_{\bar{0}}^{\mathrm{top}})$ and $\mathrm{BGL}^{\mathrm{top}}\Lambda$ have the same homotopy type. To see that $\imath$ determines a homotopy equivalence it suffices by Theorem 1.2 to show that the classifying space of the comma category $(A_1, A_2)/\imath$ is contractible. If (A_1, A_2) is an object in $S^{-1}S_0^{\mathrm{top}}$, then there exists P such that $A_1 \oplus P \cong A_2 \oplus P$. Composition with the arrow $(P, 1, 1)$: $(A_1, A_2) \to (A_1 \oplus P, A_2 \oplus P)$, determines a continuous functor

$$(A_1 \oplus P, A_2 \oplus P)/\imath \to (A_1, A_2)/\imath.$$

Since $(A_1 \oplus P, A_2 \oplus P)/\imath$ has an initial object, its classifying space is contractible. There is a continuous functor from $(A_1, A_2)/\imath$ to $(A_1 \oplus P, A_2 \oplus P)/\imath$ sending (C, c_1, c_2): $(A_1, A_2) \to (B_1, B_2)$ to $(C, \tau^*c_1, \tau^*c_2): (A_1 \oplus P, A_2 \oplus P) \to (B_1 \oplus P, B_2 \oplus P)$, where τ^*c_i is the composite

$$A_i \oplus P \oplus C \xrightarrow{\tau} A_i \oplus C \oplus P \xrightarrow{c_i \oplus 1} B_i \oplus P.$$

The composition γ of the continuous functors $(A_1, A_2)/\imath \to (A_1 \oplus P, A_2 \oplus P)/\imath \to (A_1, A_2)/\imath$ sends (C, c_1, c_2) to the diagonal arrow in the following commutative diagram.

$$\begin{array}{ccc}
(A_1, A_2) & \xrightarrow{(C, c_1, c_2)} & (B_1, B_2) \\
{\scriptstyle (P,1,1)}\downarrow & \searrow & \downarrow{\scriptstyle (P,1,1)} \\
(A_1 \oplus P, A_2 \oplus P) & \xrightarrow{(C, \tau^*c_1, \tau^*c_2)} & (B_1 \oplus P, B_2 \oplus P)
\end{array}$$

The upper triangle shows that there is a continuous natural transformation from the identity functor to γ. Hence $\mathrm{B}((A_1, A_2)/\imath)$ is a homotopy retract of the contractible space $\mathrm{B}((A_1 \oplus P, A_2 \oplus P)/\imath)$. Thus $\mathrm{B}(\imath)$ is a homotopy equivalence.

Let $\mathrm{GL}_*^{\mathrm{top}}\Lambda: \mathbb{N} \to \mathbf{Cat}^{\mathrm{top}}$ be the diagram of topological categories that assigns to n the groupoid $\mathrm{GL}_n^{\mathrm{top}}\Lambda$. Denote the unique object of $\mathrm{GL}_n^{\mathrm{top}}\Lambda$ by Λ^n, and to simplify notation let $\mathbf{n}$ represent the object (n, Λ^n) in the Grothendieck construction $\mathbb{N}\int\mathrm{GL}_*^{\mathrm{top}}\Lambda$ of $\mathrm{GL}_*^{\mathrm{top}}\Lambda$. Let

$$F: \mathbb{N} \int \mathrm{GL}_*^{\mathrm{top}}\Lambda \to S^{-1}S_{\bar{0}}^{\mathrm{top}}$$

be the functor that sends $\mathbf{n}$ to the pair (Λ^n, Λ^n) and the arrow $(\mathbf{m} \leqslant \mathbf{n}, T)$ to

$$((I, T), \Lambda^{n-m}): (\Lambda^m, \Lambda^m) \to (\Lambda^n, \Lambda^n)).$$

Clearly F is well defined and continuous. Now $\mathrm{B}(\mathbb{N} \int \mathrm{GL}_*^{\mathrm{top}}\Lambda)$ has the homotopy type of the colimit

$$\cdots \to \mathrm{B}(\mathrm{GL}_n^{\mathrm{top}}\Lambda) \to \mathrm{B}(\mathrm{GL}_{n+1}^{\mathrm{top}}\Lambda) \to \cdots \tag{7}$$

by Theorem 1.3. Since each map in the sequence (3.7) is a cofibration and for every n the space $\mathrm{B}(\mathrm{GL}_n^{\mathrm{top}}\Lambda)$ has the homotopy type of a CW-complex, it follows that the classifying space of $\mathbb{N} \int \mathrm{GL}_*^{\mathrm{top}}\Lambda$ has the homotopy type of $\mathrm{BGL}^{\mathrm{top}}\Lambda$ (cf. [3, Ch. XII, Section 3]).

Let $A = (A_1, A_2)$ be an object in $S^{-1}S_0^{\text{top}}$. By the following lemma, $V_\infty(A_1)$ and $\mathrm{B}(A/F)$ have the same homotopy type. Since $V_\infty(A_1)$ is contractible by Proposition 2.6, it follows that $B(A/F)$ has the homotopy type of a point. Thus $\mathrm{B}F$ is a homotopy equivalence by Theorem 1.2.

In the discrete case the functor F is essentially the functor defined in [7, p. 224], where it was proved that $\mathrm{B}F$ is acyclic (cf. [6, Section 4]). In particular there is a commutative diagram

$$\begin{array}{ccc} \mathbb{N}\int \mathrm{GL}_*^{\delta}\Lambda & \longrightarrow & S^{-1}S_{\bar{0}} \\ \downarrow & & \downarrow \\ \mathbb{N}\int \mathrm{GL}_*^{\text{top}}\Lambda & \longrightarrow & S^{-1}S_{\bar{0}}^{\text{top}} \end{array}$$

and it follows that the map in (2) is equivalent to $\pi_i \mathrm{B}(S^{-1}S) \to \pi_i \mathrm{B}(S^{-1}S^{\text{top}})$. □

LEMMA 3.2. *$\mathrm{B}(A/F)$ and $V_\infty(A_1)$ have the same homotopy type.*

Proof. Since the sequence (6) is a sequence of closed cofibrations, $V_\infty(A_1)$ is the homotopy colimit of the diagram $n \mapsto V_n(A_1)$. Hence it is enough to show that $\mathrm{B}(A/F)$ and $|\amalg_* V(A_1)|$ have the same homotopy type.

For $[p] \in \Delta$ we have

$$\mathrm{N}(A/F)_p = \coprod_{n_0 \leqslant \cdots \leqslant n_p} \mathrm{Hom}_{S^{-1}S}(A, F(n_0)) \times \mathrm{GL}_{n_1}^{\text{top}}\Lambda \times \cdots \mathrm{GL}_{n_p}^{\text{top}}\Lambda.$$

Composition in A/F, when restricted to the first factor, is induced by the standard inclusion, so the projection map

$$\mathrm{Iso}_{\mathscr{P}\times\mathscr{P}}((A_1 \oplus P, A_2 \oplus P), (\Lambda^n, \Lambda^n)) \to \mathrm{Iso}_{\mathscr{P}}(A_1 \oplus P, \Lambda^n)$$

induces a continuous simplicial map $\pi : \mathrm{N}(A/F) \to \amalg_* V(A_1)$.

Since $A = (A_1, A_2) \in S^{-1}S_0^{\text{top}}$, there exists an isomorphism $\alpha : A_2 \to A_1$. Using α we define a continuous map $\alpha^* : \mathrm{Iso}_{\mathscr{P}}(A_1 \oplus P, \Lambda^n) \to \mathrm{Iso}_{\mathscr{P}}(A_1 \oplus P, \Lambda^n) \times \mathrm{Iso}_{\mathscr{P}}(A_2 \oplus P, \Lambda^n)$ by $\alpha^*(\varphi) = (\varphi, \varphi \circ \alpha)$. The map α^* is stable in the sense that

$$\begin{array}{ccc} \mathrm{Iso}_{\mathscr{P}}(A_1 \oplus P, \Lambda^n) & \longrightarrow & \mathrm{Iso}_{\mathscr{P}}(A_i \oplus P, \Lambda^n) \\ \downarrow & & \downarrow \\ \mathrm{Iso}_{\mathscr{P}}(A_1 \oplus P \oplus \Lambda^k, \Lambda^{n+k}) & \longrightarrow & \mathrm{Iso}_{\mathscr{P}}(A_i \oplus P \oplus \Lambda^k, \Lambda^{n+k}) \end{array}$$

commutes for non-negative k and $i = 1, 2$. Thus α^* determines a continuous simplicial map $\iota_\alpha : \amalg_* V(A_1) \to \mathrm{N}(A/F)$.

The composition $\pi \circ \iota_\alpha$ is the identity, and there is a simplicial homotopy equivalence from the identity to $\iota_\alpha \circ \pi$ defined as follows. If $(P, \varphi_1, \varphi_2) : (A_1, A_2) \to (\Lambda^n, \Lambda^n)$ is an arrow in $S^{-1}S_0^{\text{top}}$, then the isomorphism $\varphi_1(\alpha \oplus 1_P)\varphi_2^{-1} : \Lambda^n \to \Lambda^n$ is independent of the choice of representatives P, φ_1, φ_2. Now given a point $((P, \varphi_1, \varphi_2) : A \to F(n_0), T_1, \ldots, T_p)$ in $\mathrm{N}(A/F)_p$ with $T_i \in \mathrm{GL}_{n_i}^{\text{top}}\Lambda$. Let $\Theta_0 = \varphi_1(\alpha \oplus 1_P)\varphi_2^{-1}$, and define

$$\Theta_i = [\varphi_1(\alpha \oplus 1_P)\varphi_2^{-1} \oplus I]\, T_i^{-1} \in \mathrm{GL}_{n_i}^{\text{top}}\Lambda$$

for $i = 1, \dots, p$. With these isomorphisms we obtain a commutative diagram

$$\begin{array}{ccccccccc}
A & \xrightarrow{\varphi} & F(n_0) & \xrightarrow{F(T_1)} & F(n_1) & \xrightarrow{F(T_2)} \cdots \xrightarrow{F(T_p)} & F(n_p) \\
\| & & \downarrow{\scriptstyle F(\Theta_1)} & & \downarrow{\scriptstyle F(\Theta_1)} & & \downarrow{\scriptstyle F(\Theta_p)} \\
A & \xrightarrow{\iota_\alpha \circ \pi(\varphi)} & F(n_0) & \xrightarrow{F(I)} & F(n_1) & \xrightarrow{F(I)} \cdots \xrightarrow{F(I)} & F(n_p)
\end{array}$$

that determines a simplicial homotopy equivalence from the identity to $\iota_\alpha \circ \pi$. □

4. HOMOTOPY GROUPS OF ALGEBRAIC K-THEORY GROUPS

Let **s.Sets** denote the category of simplicial. In this section we utilize the adjoint functors [13, Section 16]

$$\mathbf{s.Sets} \underset{\text{Sing}}{\overset{|\cdot|}{\rightleftarrows}} \mathbf{Top}$$

to convert simplicial spaces to bisimplicial sets. Recall the singular complex functor Sing carries a topological space X to the simplicial set $p \mapsto \text{Sing}_p X = \text{Hom}(\Delta^p, X)$, and the counit $|\text{Sing}\, X| \to X$ is a weak homotopy equivalence [13, Section 16].

Let $\mathscr{P}_T(\Delta^p)$ denote the additive category of finitely generated projective (left) Λ-modules whose morphisms are Λ-linear transformations parametrized by Δ^p. The subscript T is used to suggest an identification of $\mathscr{P}_T(X)$ with the category of trivial $\mathscr{P}$-fibre bundles over a compact space X (cf. [10, p. 179]).

The underlying simplicial set of $\text{N}(S^{-1}S^{\text{top}})$ can be viewed as $\text{Sing}_0\text{N}(S^{-1}S^{\text{top}})$, similarly $\text{N}(S^{-1}S(\mathscr{P}_T(\Delta^p))) = \text{Sing}_p S^{-1}S^{\text{top}}$. Since $\text{N}(S^{-1}S^{\text{top}})_q$ has the homotopy type of CW-complex the counit

$$|\text{Sing}\,\text{N}(S^{-1}S^{\text{top}})_q| \to \text{N}(S^{-1}S^{\text{top}})_q$$

is a homotopy equivalence; moreover, $\text{N}(S^{-1}S^{\text{top}})$ and $q \mapsto |\text{Sing}\,\text{N}(S^{-1}S^{\text{top}})_q|$ are a good simplicial spaces. Thus $|\text{Sing}\,\text{N}(S^{-1}S^{\text{top}})|$ is homotopy equivalent to $\text{B}(S^{-1}S^{\text{top}})$.

LEMMA 4.1. *There is an isomorphism*

$$\pi_i \text{B}(S^{-1}S(\mathscr{P}_T(\Delta^p))) \cong K_i^{\text{alg}}\Lambda(\Delta^p).$$

Proof. Since Δ^p is contractible every locally trivial $\mathscr{P}$-fibre bundle over Δ^p is trivial (cf. [8, Ch. 4, Section 2]). Using Swan's proof [23] of the equivalence between the category of locally trivial vector bundles over a compact space and the category of finitely generated projective modules over the ring of continuous functions, one sees that $\mathscr{P}_T(\Delta^p)$ is equivalent to the category $\mathscr{P}(\Lambda(\Delta^p))$, of finitely generated projective modules over the ring of continuous Λ-valued functions on Δ^p. Since the $S^{-1}S$-construction preserves equivalences, the lemma follows. □

THEOREM 4.2. *Let Λ be a Banach algebra. There is a first quadrant spectral sequence*

$$K_q^{\text{alg}}\Lambda(\Delta^p) = E_{p,q}^1(\Lambda) \;\Rightarrow\; K_{p+q}^{\text{top}}\Lambda. \qquad (8)$$

Proof. Let $X_p = \mathrm{N}(S^{-1}S(\mathscr{P}_T(\Delta^p)))$. Since this space has the structure of an H-space and $\pi_0|X_p|$ is a group [6, Lemma 6.2] asserts that there is a spectral sequence

$$E^2_{p,q} = \pi_p([n] \mapsto \pi_q\ X_n) \ \Rightarrow\ \pi_{p+q}|X|.$$

Now $\pi_p X_n = K_q^{\mathrm{alg}}\Lambda(\Delta^n)$ by the previous lemma. Whence the E^2-term is the homology of the chain complex associated to the simplicial abelian group $[p] \mapsto K_q^{\mathrm{alg}}\Lambda(\Delta^p)$, see [13, Section 22]. □

The natural simplicial map $\mathrm{Sing}_0\mathrm{N}(S^{-1}S^{\mathrm{top}}(\mathscr{P})) \to \mathrm{diag}\,\mathrm{Sing}\,\mathrm{N}(S^{-1}S^{\mathrm{top}}(\mathscr{P}))$ induces the map in (2) and we obtain

COROLLARY 4.3. *The map $K_q^{\mathrm{alg}}\Lambda \to K_q^{\mathrm{top}}\Lambda$ is the edge homomorphism*

$$K_q^{\mathrm{alg}}\Lambda \twoheadrightarrow \cdots \twoheadrightarrow E^\infty_{0,q}\Lambda \rightarrowtail K_q^{\mathrm{top}}\Lambda$$

of the spectral sequence in (8).

Homotopy groups of algebraic K-theory. Since Δ^p is a compact topological space, each algebra $\Lambda(\Delta^p)$ is a Banach algebra. A result of Milnor [16, Corollary 7.2] states that if Λ is commutative then

$$K_1^{\mathrm{alg}}\Lambda(\Delta^p) = \Lambda^* \oplus \pi_0\mathrm{SL}^{\mathrm{top}}\Lambda(\Delta^p).$$

Since Δ^p is contractible it follows that $\pi_0\mathrm{SL}^{\mathrm{top}}\Lambda(\Delta^*) = \pi_0\mathrm{SL}^{\mathrm{top}}\Lambda$

$$E^2_{p,1}(\Lambda) = \begin{cases} \pi_0\Lambda^* \oplus \pi_0\mathrm{SL}^{\mathrm{top}}\Lambda & \text{if } p = 0 \\ \pi_p\Lambda & \text{if } p > 0. \end{cases}$$

Thinking of the simplicial abelian groups $K_q^{\mathrm{alg}}\Lambda(\Delta^*)$ as coming from the singular complex of some topological structure on $K_q^{\mathrm{alg}}\Lambda$ we propose

Definition 4.4. Let Λ be a Banach algebra. The homotopy groups of $K_*^{\mathrm{alg}}\ \Lambda$ are

$$\pi_p K_q^{\mathrm{alg}}\Lambda := E^2_{p,q}(\Lambda).$$

Let X be a compact topological space. The complex topological K-theory of X in negative degrees can be calculated as $K^{-q}X = K_q^{\mathrm{top}}\mathbb{C}(X)$. Define the higher algebraic K-theory of X to be

$$K_q^{\mathrm{alg}}X := K_q^{\mathrm{alg}}\mathbb{C}(X), \quad q \geqslant 0.$$

An immediate consequence of Swan's theory is $K_0^{\mathrm{alg}}X = K_0^{\mathrm{top}}X$, and an elementary calculation yields $\pi_0 K_1^{\mathrm{alg}}X = K_1^{\mathrm{top}}X$. Since $\mathbb{C}^*$ is an Eilengerg–Mac Lane space of type $(1, \mathbb{Z})$, we obtain for each $p > 0$ $\pi_p K_1^{\mathrm{alg}}X \cong H^{1-p}(X, \mathbb{Z})$, and the associated filtration on $K_2^{\mathrm{top}}X$ reduces to the short exact sequence

$$0 \to \pi_0 K_2^{\mathrm{alg}}X \to K_2^{\mathrm{top}}X \to \pi_1 K_1^{\mathrm{alg}}X \to 0.$$

To proceed further we consider K-theory with finite coefficients. Prasolov [18] and Fischer [4] independently have shown that the algebraic K-theory and the topological K-theory of X with finite coefficients are the same.

PROPOSITION 4.5. *For each $q \geqslant 0$ there is a natural short exact sequence*

$$0 \to K_q^{\mathrm{alg}}X \to K_q^{\mathrm{alg}}(X \times \Delta^*) \to \mathscr{Q}(q) \to 0,$$

where $\mathcal{Q}_*(q)$ is a simplicial $\mathbb{Q}$-vector space and $K_q^{\mathrm{alg}}X$ is given the structure of a discrete simplicial abelian group.

Proof. The groups $K_0^{\mathrm{alg}}(X \times \Delta^p)$ and $K_0^{\mathrm{top}}(X \times \Delta^p)$ are isomorphic for all p. By the homotopy invariance of topological K-theory $K_0^{\mathrm{alg}}\, X \to K_0^{\mathrm{alg}}(X \times \Delta^*)$ is an isomorphism and $\mathcal{Q}(0)$ is the trivial $\mathbb{Q}$-vector space.

For all p the projection map $\pi: X \times \Delta^p \to X$ induces an isomorphism

$$K_q^{\mathrm{alg}}(X, \mathbb{Z}/n) \to K_q^{\mathrm{alg}}(X \times \Delta^p, \mathbb{Z}/n)$$

for each $q > 0$, by the Prosolov–Fisher isomorphism and the homotopy invariance of topological K-theory, and a split monomorphism $K_q^{\mathrm{alg}}X \to K_q^{\mathrm{alg}}(X \times \Delta^p)$ since π admits a section. Let $\mathcal{Q}(q)$ denote the quotient $K_q^{\mathrm{alg}}(X \times \Delta^*)/K_q^{\mathrm{alg}}(X)$. The universal coefficient sequence

$$K_q^{\mathrm{alg}}(X \times \Delta^p) \oplus \mathbb{Z}/n \rightarrowtail K_q^{\mathrm{alg}}(X \times \Delta^p, \mathbb{Z}/n) \twoheadrightarrow n\text{-torsion of } K_{q-1}^{\mathrm{alg}}(X \times \Delta^p)$$

together with the previous observations proves the proposition. □

Applying the homotopy functor to the short exact sequence of the previous proposition we obtain, for each $p > 1$, a family of isomorphisms $\pi_p K_q^{\mathrm{alg}}X \cong \pi_p \mathcal{Q}(q)$, and it follows that $\pi_p K_q^{\mathrm{alg}}X$ is a $\mathbb{Q}$-vector space. For $p = 0$ and 1 there is a short exact sequence

$$0 \to \pi_1 K_q^{\mathrm{alg}}X \to \pi_1 \mathcal{Q}(q) \to K_q^{\mathrm{alg}}X \to \pi_0 K_q^{\mathrm{alg}}X \to \pi_0 \mathcal{Q}(q) \to 0. \tag{9}$$

Since $K_1^{\mathrm{alg}}(X \times \Delta^*) \cong \mathrm{Sing}\, \mathbb{C}^*(X) \oplus [X, \mathrm{SU}]$, the sequence (9) has the form

$$0 \longrightarrow H^0(X, \mathbb{Z}) \longrightarrow \mathbb{C}(X) \xrightarrow{\exp} \mathbb{C}^*(X) \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow 0$$

where $\pi_1 \mathcal{Q}(1) \xrightarrow{\sim} \mathbb{C}(X)$ is given by $[\gamma(x,t)] \mapsto \int_{\gamma(x,t)} d\log t$. And from Suslin [22] we know that

$$K_q^{\mathrm{alg}}\mathbb{C}(\Delta^q) \cong \begin{cases} \mathbb{Q}/\mathbb{Z} \oplus \text{a } \mathbb{Q}\text{-vector space} & \text{if } q > 0 \text{ is odd} \\ \text{a } \mathbb{Q}\text{-vector space} & \text{if } q > 0 \text{ is even.} \end{cases}$$

For $q > 1$, Bott periodicity along with a more detailed analysis of the spectral sequence (8) yields $(K_{2q-1}^{\mathrm{alg}}\mathbb{C})_{\mathrm{tors}} \subseteq \pi_0 K_{2q-1}^{\mathrm{alg}}\mathbb{C}(\Delta^*)$.

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