STABILITY OF ALMOST COMMUTATIVE INVERSE SYSTEMS OF COMPACTA

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Recently, S. Mardešić and L.R. Rubin have defined approximate inverse systems of compacta. In these systems the bonding maps commute only up to certain controlled values. In this paper it is shown that such systems are stable in the sense that small perturbations of bonding maps yield again an approximate system and do not affect the limit space. In particular, an inverse system of near homeomorphisms can be replaced by an approximate inverse system of homeomorphisms having the same limit space. In such a system projections from the limit space need not be (near) homeomorphisms, but are always refinable maps.

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1. Approximate and almost commutative systems

We quote from [5] the two basic definitions.

Definition 1. An approximate inverse system of metric compacta \( X = (X_a, \varepsilon_a, p_{aa'}, A) \) consists of the following: A directed ordered set \( (A, \leq) \); for each \( a \in A \), a compact metric space \( X_a \), with metric \( d \) and a real number \( \varepsilon_a > 0 \); for each pair \( a \leq a' \) from \( A \), a mapping \( p_{aa'} : X_a \rightarrow X_{a'} \), satisfying the following conditions:

\[
\begin{align*}
(A1) \quad & d(p_{a_1a_2}p_{a_2a_3}, p_{a_1a_3}) \leq \varepsilon_{a_1}, \quad a_1 \leq a_2 \leq a_3; \quad p_{aa} = \text{id}. \\
(A2) \quad & (\forall a \in A)(\forall \eta > 0)(\exists a' \geq a)(\forall a_2 \geq a_1 \geq a') \quad d(p_{aa}, p_{a_2a_2}, p_{aa_2}) \leq \eta.
\end{align*}
\]

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(A3) \((\forall a \in A)(\forall \eta > 0)(\exists a' \geq a)(\forall a'' \geq a')(\forall x, x' \in X_{a'}) \)
\[d(x, x') \leq \varepsilon_a \Rightarrow d(p_{aa}(x), p_{aa}(x')) \leq \eta.\]

We refer to the numbers \(\varepsilon_a\) as the meshes of the approximate system \(X\).

Often we will assume that \(A\) is cofinite, i.e., every element \(a' \in A\) has only a finite number of predecessors \(a \preceq a'\). This number is denoted by \(|a'| \geq 1\).

If \(\pi_a : \prod_{a \in A} X_a \to X_{a'}, a \in A,\) denote projections, we define the limit space \(X = \lim X\) and the natural projections \(p_a : X \to X_{a}\) as follows.

**Definition 2.** A point \(x = (x_a) \in \prod_{a \in A} X_a\) belongs to \(X = \lim X, X = (X_a, \varepsilon_a, p_{aa}, A),\) provided, for every \(a \in A,\)
\[(L) \quad x_a = \lim_{u \downarrow} p_{aa}(x_u).\]

The projections \(p_a : X \to X_{a}\) are given by \(p_a = \pi_a|X.\)

Note that the numbers \(\varepsilon_a\) appear only in conditions (A1) and (A3). Their choice is delicate, because (A1) prevents them from being too small and (A3) (related to uniform continuity) prevents them from being too large. The next definition introduces a related new concept, where the meshes \(\varepsilon_a\) do not make up a part of the structure.

**Definition 3.** An almost commutative system \(X = (X_a, p_{aa}, A)\) consists of a directed set \((A, \preceq)\), metric compacta \(X_a\) and maps \(p_{aa}\) such that it is possible to associate with each \(a \in A\) an \(\varepsilon_a > 0\) so as to obtain an approximate system as in Definition 1.

Since Definition 2 does not depend on the meshes \(\varepsilon_a,\) it applies with no changes also to almost commutative systems.

Almost commutative systems retain many nice properties of usual (commutative) inverse systems, but are general enough so as to allow expansion theorems, which are false in the case of usual systems. We now quote (as propositions) several results from [5] and [7]. We state them for almost commutative systems.

**Proposition 1.** Every (commutative) inverse system \(X = (X_a, p_{aa}, A)\) of metric compacta over a cofinite set \(A\) is also an almost commutative system and the limit \(X,\) in the sense of Definition 2, coincides with the inverse limit in the usual sense (see [5, Remark 2]).

**Proposition 2.** If \(X = (X_a, p_{aa}, A)\) is an almost commutative inverse system and \(B \subseteq A\) is a cofinal subset, then the restriction \(Y = (X_{b}, p_{bb}, B)\) is also an almost commutative system and the projection \(\prod_{a \in A} X_a \to \prod_{b \in B} X_b\) induces a homeomorphism of the limit spaces (see [7, Proposition 2]).

In particular, if \(A\) is finite and \(a_0 \in A\) is its maximal element, then the limit \(X\) of \(X\) coincides with \(X_{a_0}\). This is why we often tacitly assume that \(A\) is infinite.
Proposition 3. If \( X = (X_a, p_{aa}, A) \) is an almost commutative system of non-empty metric compacta, then the limit \( X = \lim X \) is a non-empty compact Hausdorff space (see [5, Theorems 1 and 2]).

Proposition 4. Let \( X = (X_a, p_{aa}, A) \) be an almost commutative system. Then, for any \( a \in A \), \( p_a = \lim_{a} p_{aa} p_{aa} \), i.e., \( \lim_{a} d(p_{aa} p_{aa}, p_a) = 0 \), where \( d(f, g) = \sup \{d(f(x), g(x)) : x \in X \} \) (see [5, Lemma 4]).

Proposition 5. Every almost commutative system \( X = (X_a, p_{aa}, A) \) has the following two properties:

(B1) Let \( a \in A \) and let \( U \subseteq X_a \) be an open set such that \( p_a(X) \subseteq U \). Then there exists an \( a' \geq a \) such that \( p_{aa}(X_{a'}) \subseteq U \), for any \( a'' \geq a' \).

(B2) For every open covering \( \mathcal{U} \) of \( X \) there exists an \( a \in A \) such that for any \( a_1 \geq a \) there exists an open covering \( \mathcal{V} \) of \( X_{a_1} \) for which \( (p_{a_1})^{-1}(\mathcal{V}) \) refines \( \mathcal{U} \).

Proposition 5 was proved in [5, Theorem 3] and [7, Theorem 1]. Note that properties (B1) and (B2) characterize limits in the case of commutative inverse systems (see [6, I, § 3.2, Remark 5]).

Proposition 6. If in an almost commutative system \( X = (X_a, p_{aa}, A) \) all bonding maps \( p_{aa} \) are surjective, then all projections \( p_a \) are also surjective (see [7, Corollary 1]).

It was shown in [5, Theorem 5] that every compact Hausdorff space \( X \) with \( \dim X \leq n \) is the limit of an almost commutative system \( X = (X_a, p_{aa}, A) \), where all \( X_a \) are polyhedra of dimension \( \dim X_a \leq n \). An analogous result for \( \mathcal{P} \)-like continua was obtained in [7, Theorem 3]. Analogous results for commutative systems are false.

In this paper we establish another property of almost commutative systems, not possessed by commutative systems. This is their stability, i.e., the property that sufficiently small perturbations of the bonding maps of an almost commutative system yield again an almost commutative system with the same limit space (see Section 4, Theorem 2). We also consider almost commutative systems whose bonding maps are near homeomorphisms. The limit space \( X \) of such a system \( X \) is also obtainable as the limit of an almost commutative system whose bonding maps are homeomorphisms (Section 5, Theorem 4). This does not imply that \( X \) is homeomorphic with the members \( X_a \) of \( X \). However, we show (Section 5, Theorem 5) that the natural projections \( p_a : X \to X_a \) are refinable maps.

2. Contiguous approximate systems

In this section we show that nearby approximate systems have homeomorphic limits.
Theorem 1. Let $X = (X_a, e_a, p_{aa}, A)$, $X' = (X_a, e_a, p'_{aa}, A)$ be two approximate systems, which differ only in their bonding maps. If
\begin{equation}
 d(p_{aa}, p'_{aa}) \leq \varepsilon_a, \quad a_1 \leq a_2,
\end{equation}
then the limits $X = \lim X$ and $X' = \lim X'$ are homeomorphic. More precisely, if $p_a : X \to X_a$, $p'_a : X' \to X_a$ are the natural projections, then there is a homeomorphism $f : X' \to X$ such that
\begin{equation}
 d(p_a, f(p'_a)) \leq \varepsilon_a.
\end{equation}

We refer to approximate systems $X$ and $X'$ satisfying (2.1) as contiguous systems.

Proof. We will first show that, for any $a \in A$, $(p_{aa}, p'_a, a_1 \geq a)$ is a Cauchy net.

For a given $\eta > 0$ choose $a' \geq a$ such that (A2) and (A3) hold for $X$ and $i_\eta$. We claim that $a, a, a_1', a_2$ implies
\begin{equation}
 d(p_{aa}, p'_{a_1}, p_{aa}, p'_{a_2}) \leq \eta.
\end{equation}

Indeed, choose $a_1, a_2$ so large that
\begin{align}
 d(p'_{a_1}, p'_{a_2}, p_{a_1}) &\leq \varepsilon_{a_1}, \\
 d(p'_{a_2}, p'_{a_1}, p_{a_2}) &\leq \varepsilon_{a_2},
\end{align}

This is possible by Proposition 4, applied to $X'$. Note that, by (A3), (2.4) implies
\begin{equation}
 d(p_{aa}, p'_{a_1}, p_{aa}, p'_{a_2}) \leq \frac{1}{3} \eta.
\end{equation}

Furthermore, by (A2),
\begin{equation}
 d(p_{aa}, p_{a_1}, p_{aa}) \leq \frac{1}{5} \eta,
\end{equation}

and therefore
\begin{equation}
 d(p_{aa}, p_{a_1, a_2}, p_{aa}, p'_{a_2}) \leq \frac{1}{5} \eta.
\end{equation}

By (2.1),
\begin{equation}
 d(p_{a_1}, p'_{a_1, a_2}) \leq \varepsilon_{a_1},
\end{equation}

and therefore,
\begin{equation}
 d(p_{a_1}, p'_{a_1, a_2}, p_{a_1, a_2, p'_{a_2}}) \leq \varepsilon_{a_1}.
\end{equation}

By (A3), (2.10) implies
\begin{equation}
 d(p_{aa}, p_{a_1, a_2}, p'_{a_1, a_2, p'_{a_2}}) \leq \frac{1}{5} \eta.
\end{equation}

Now, (2.6), (2.11) and (2.8) imply
\begin{equation}
 d(p_{aa}, p'_{a_1}, p_{aa}, p'_{a_2}) \leq \frac{1}{3} \eta.
\end{equation}

In the same way we see that
\begin{equation}
 d(p_{aa}, p'_{a_1}, p_{aa}, p'_{a_2}) \leq \frac{1}{3} \eta.
\end{equation}
(2.12) and (2.13) yield the desired formula (2.3), which shows that \((p_{a_0}, p'_{a_1}, a_1 \geq a)\) is indeed a Cauchy net.

This enables us to define maps \(f_a : X' \to X_a, a \in A\), by
\[
f_a = \lim_{a_1} p_{a_0, p'_{a_1}}.
\]

(2.14)

We now claim that for every \(a \in A\) and \(\eta > 0\) there is an \(a' \geq a\) such that for any \(a_1 \geq a'\)
\[
d(\lim_{a_1} p_{a_0, p'_{a_1}}, f_a) \leq \eta.
\]

(2.15)

Indeed, choose \(a' \geq a\), by (A2) applied to \(X\). Then, for \(a_1 \geq a, 2 a'\), one has
\[
d(p_{a_0, p'_{a_1}, p_{a_0, p'_{a_1}}}, p_{a_0, p'_{a_1}}) \leq \eta.
\]

(2.16)

(2.15) is obtained from (2.16) by passing to the limit with \(a_2\). By Definition 2, (2.15) insures that, for any \(x' \in X', (f_a(x')) \in X\). Therefore, there is a mapping \(f : X' \to X\) such that
\[
p_a f = f_a, \quad a \in A.
\]

(2.17)

We now show that \(f\) satisfies (2.2). Indeed, (2.1) yields
\[
d(\lim_{a_1} p_{a_0, p'_{a_1}}, \lim_{a_1} p_{a_0, p'_{a_1}}) \leq \varepsilon_a, \quad a \leq a_1.
\]

(2.18)

Passing to the limit with \(a_1\) and taking into account (2.14) and Proposition 4 (applied to \(X'\)), we obtain
\[
d(f_a, p_{a_0, p'_{a_1}}) \leq \varepsilon_a, \quad a \in A,
\]

(2.19)

(2.17) and (2.19) yield (2.2).

In order to show that \(f : X' \to X\) is a homeomorphism, we also define mappings \(g_a : X \to X_a, a \in A\), by
\[
g_a = \lim_{a_1} p_{a_0, p_{a_1}}, \quad a \in A,
\]

(2.20)

and show that they induce a mapping \(g : X \to X'\), which satisfies
\[
p'_{a} g = g_a, \quad a \in A.
\]

(2.21)

This is done just as in the case of the mappings \(f_a\) and \(f\). To complete the proof it suffices to show that \(g f = 1_X\) and \(f g = 1_X\).

We first show that
\[
d(g_a f, p_{a_0, p'_{a_1}}) \leq \varepsilon_a, \quad a \in A.
\]

(2.22)

We apply (A3) to \(X'\) and \(\eta = \varepsilon_a\) to obtain \(a' \geq a\). Then (2.19), applied to any \(a_1 \geq a'\), yields
\[
d(p_{a_0, p'_{a_1}}, p_{a_0, p'_{a_1}}) \leq \varepsilon_a, \quad a_1 \geq a'.
\]

(2.23)

However, by (2.17) and (2.20),
\[
\lim_{a_1} p_{a_0, p'_{a_1}} = \lim_{a} p_{a_0, p_{a_1}} = g_a f,
\]

(2.24)
and by Proposition 4 (applied to $X'$),
\[ \lim_{a_1} p'_{a_1} p'_{a_1} = p'_a. \] (2.25)

Therefore, (2.22) follows from (2.23) by passing to the limit with $a_1$. By (2.21), (2.22) can be rewritten as
\[ d(p'_a g_f, p'_a) \leq \varepsilon_a, \quad a \in A \] (2.26)

We will now show that, for any $\eta > 0$ and any $a \in A$,
\[ d(p'_a g_f, p'_a) \leq \eta, \] (2.27)

so that $p'_a g_f = p'_a$, for all $a \in A$, and therefore $g_f = 1_{X'}$.

For a given $a \in A$, choose $a' \geq a$, by (A3) applied to $X'$. Then, for any $a_1 \geq a'$, (2.26) for $a_1$ implies
\[ d(p'_{a_1} g_f, p'_{a_1}) \leq \eta, \quad a_1 \geq a'. \] (2.28)

Passing to the limit with $a_1$, taking into account Proposition 4 (applied to $X'$), we see that (2.28) yields (2.27).

The proof that $f g = 1_X$ is analogous. \( \square \)

**Corollary 1.** Let $X = (X_i, p_{i+1})$, $X' = (X_i, p'_{i+1})$ be two inverse sequences of metric compacta and let $p_{ij} = p_{i+1} \cdots p_{j-1}, p'_{ij} = p'_{i+1} \cdots p'_{j-1}, i < j$. If $\varepsilon_i > 0$ are numbers such that
\[ d(x, x') \leq \varepsilon_j \Rightarrow d(p_{ij}(x), p_{ij}(x')) \leq \varepsilon_i / 2^j, \quad i < j, \] (2.29)
\[ d(x, x') \leq \varepsilon_j \Rightarrow d(p'_{ij}(x), p'_{ij}(x')) \leq \varepsilon_i / 2^j, \quad i < j, \] (2.30)
\[ d(p_{i+1}, p'_{i+1}) \leq \frac{1}{2} \varepsilon_i, \] (2.31)

then the inverse limits $X = \lim X$, $X' = \lim X'$ are homeomorphic.

**Proof.** $X = (X_i, \varepsilon_i, p_{ij}, \mathbb{N})$ is an approximate inverse system. (A1) and (A2) are fulfilled because $X$ is commutative. For a given $i \in \mathbb{N}$ and $\eta > 0$, by (2.29),
\[ d(x, x') \leq \varepsilon_j \Rightarrow d(p_{ij}(x), p_{ij}(x')) \leq \eta, \] (2.32)

for any $j \geq i \geq i$, provided $\varepsilon_i / 2^j \leq \eta$. This establishes (A3). The same argument shows that $X' = (X_i, \varepsilon_i, p'_{ij}, \mathbb{N})$ is an approximate system.

The assertion of Corollary 1 follows from Theorem 1 if we show that
\[ d(p'_{ij}, p_{ij}) \leq \varepsilon_i, \quad i < j. \] (2.33)

We prove (2.33) by induction on $j - i$. In the case $j - i = 1$, (2.33) follows from (2.31). For $j - i > 1$,
\[ d(p'_{ij}, p_{ij}) \leq d(p'_{i+1} p'_{i+1+j}, p_{i+1} p'_{i+1+j}) + d(p_{i+1} p'_{i+1+j}, p_{i+1} p_{i+1+j}). \] (2.34)
By (2.31), the first term on the right-hand side of (2.34) is \( \leq \frac{1}{2} \varepsilon_i \). By the induction hypothesis,
\[
d(p_{i+1,j}, p_{i+1,j}) \leq \varepsilon_{i+1}.
\]  
This and (2.29) imply that the second term on the right-hand side of (2.34) is also \( \leq \frac{1}{2} \varepsilon_i \).  

**Remark 1.** Corollary 1 is related to Theorem 2 of [2].

### 3. Increasing the meshes in an approximate system

We will now establish a technical lemma to the effect that in an approximate system \( X = (X_a, \varepsilon_a, p_{aa'}, A) \) one can slightly increase the meshes \( \varepsilon_a \) and still have an approximate system.

**Lemma 1.** Let \( X = (X_a, \varepsilon_a, p_{aa'}, A) \) be an approximate system of metric compacta over a cofinite set \( A \). Then there exist numbers \( \varepsilon'_a > \varepsilon_a, a \in A \), such that \( X' = (X_a, \varepsilon'_a, p_{aa'}, A) \) is also an approximate system. Its limit \( X' = X = \lim X \) and the natural projections \( p'_a : X' \to X_a \) coincide with the projections \( p_a : X \to X_a \).

**Proof.** For any pair \( (a, a_1) \in A \times A, a \preceq a_1 \), consider the mapping \( d_{aa'} : X_{a_1} \times X_{a_1} \to \mathbb{R} \), given by
\[
d_{aa'}(x, x') = d(p_{aa'}(x), p_{aa'}(x')).
\]  
For any \( a_1 \in A \) put
\[
E_{a_1} = \{(x, x') \in X_{a_1} \times X_{a_1} : d(x, x') \leq \varepsilon_{a_1}\}.
\]  
Clearly, \( E_{a_1} \) is compact and, therefore, the set \( d_{aa'}(E_{a_1}) \) has a maximum \( m_{aa_1} = \max(d_{aa'}(E_{a_1})) \).

If \( |a_1| \geq 1 \) denotes the number of predecessors of \( a_1 \), then \( m_{aa_1} + 1/2 |a_1| > m_{aa_1} \), and therefore, there is a neighborhood \( U \) of \( E_{a_1} \) in \( X_a \times X_a \) such that
\[
(x, x') \in U \Rightarrow d_{aa'}(x, x') < m_{aa_1} + 1/2 |a_1|.
\]  
We claim that there exists a number \( \varepsilon'_{aa_1} > \varepsilon_{a_1} \) such that
\[
x, x' \in X_{a_1}, \ d(x, x') \leq \varepsilon'_{aa_1} \Rightarrow (x, x') \in U.
\]  
Indeed, if one assumes that this is not the case, then one can find two convergent sequences \( x_n, x'_n \in X_{a_1} \) such that \( d(x_n, x'_n) \leq \varepsilon_{a_1} + 1/n \), but \( (x_n, x'_n) \in (X_{a_1} \times X_{a_1}) \setminus U \). Clearly, the limits \( x = \lim x_n, x' = \lim x'_n \) satisfy \( d(x, x') \leq \varepsilon_{a_1} \), i.e., \( (x, x') \in E_{a_1} \subseteq U \) and \( (x, x') \in (X_a \times X_a) \setminus U \), which is a contradiction.

Using the fact that \( A \) is cofinite, we now define \( \varepsilon'_{a_1} \) by
\[
\varepsilon'_{a_1} = \min\{\varepsilon'_{aa_1} : a \preceq a_1\}, \quad a_1 \in A.
\]  

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It follows from (3.5), (3.4), (3.3) and (3.1) that \( \epsilon'_a > \epsilon_a \) and for any \( a < a_1 \),
\[
x, x' \in X_{a_1}, \; d(x, x') \leq \epsilon'_a \implies d_{aa}(x, x') < m_{aa} + 1/2^{[a]}, \quad a < a_1.
\] (3.6)

We will now show that \( X' = (X_a, \epsilon'_a, p_{aa}, A) \) is also an approximate system. Property (A1) holds because \( \epsilon_a < \epsilon'_a \). Property (A2) holds because it does not involve any meshes.

In order to establish (A3), for a given \( a \in A \) and \( \eta > 0 \) choose \( a' \geq a \) so as to satisfy (A3) for \( X \) and \( \frac{1}{2} \eta \). Moreover, let \( a' \) be so large that
\[
1/2^{[a]} \leq \frac{1}{2} \eta.
\] (3.7)

Then, for any \( a'' \geq a' \),
\[
1/2^{[a'']} \leq 1/2^{[a']} \leq \frac{1}{2} \eta.
\] (3.8)

Moreover, by (A3),
\[
(v, v') \in E_{a''} \Rightarrow d_{aa}(v, v') \leq \eta/2, \quad a'' \geq a',
\] (3.9)

and therefore,
\[
m_{aa} \leq \eta/2, \quad a'' \geq a'.
\] (3.10)

Now (3.6), (3.8) and (3.10) imply
\[
d(x, x') \leq \epsilon'_a \implies d(p_{aa}(x), p_{aa}(x')) = d_{aa}(x, x')
\]
\[
< m_{aa} + 1/2^{[a]} \leq \eta, \quad a'' \geq a'.
\]

Finally, note that the definition of the limit and the natural projections do not depend on the meshes.

4. The stability theorem

The following is the main result of this paper.

**Theorem 2.** Let \( X = (X_a, p_{aa}, A) \) be an almost commutative system of metric compacta over a cofinite directed set \( A \). Then there exist numbers \( \delta_{a_1, a_2} > 0, \; a_1 < a_2, \) such that for any choice of maps \( p_{a_1, a_2} : X_{a_2} \to X_{a_1} \), such that
\[
d(p_{a_1, a_2}, p_{a_2, a_1}) \leq \delta_{a_1, a_2}, \quad a_1 < a_2,
\] (4.1)

\( X' = (X_a, p'_{aa}, A) \) is also an almost commutative system. Moreover, the limits \( X = \lim X \) and \( X' = \lim X' \) coincide, \( X' = X \), and so do the natural projections \( p'_a = p_a : X \to X_a, \; a \in A \).

**Proof.** Since \( X \) is an almost commutative system, there exist numbers \( \epsilon_a > 0, \; a \in A \), such that \( X = (X_a, \epsilon_a, p_{aa}, A) \) is an approximate system. By Lemma 1, there exist numbers \( \epsilon'_a > \epsilon_a, \; a \in A \), such that \( X' = (X_a, \epsilon'_a, p_{aa}, A) \) is also an approximate system with limit \( X \) and projections \( p_a : X \to X_a \).
For every pair \( a_1 < a_2 \) we define a number \( \delta_{a_1a_2} > 0 \) satisfying the following three conditions:

\[
\delta_{a_1a_2} \leq \frac{1}{3}(\varepsilon'_{a_1} - \varepsilon_{a_1}), \tag{4.2}
\]

\[\forall x, x' \in X_{a_1}, \quad d(x, x') \leq \delta_{a_1a_2}
\Rightarrow d\left(p_{aa_1}(x), p_{aa_1}(x')\right) \leq \frac{1}{3}(\varepsilon'_{a_1} - \varepsilon_{a_1}) \quad \text{for all } a \leq a_1, \tag{4.3}\]

\[
\delta_{a_1a_2} \leq 1/2^{\lfloor a_2 \rfloor}. \tag{4.4}
\]

To satisfy (4.3), one uses uniform continuity of the maps \( p_{aa_1} \) and the fact that \( a_1 \) has only finitely many predecessors \( a \).

We will now show that, for any choice of maps \( p'_{a_1a_2} : X_{a_2} \to X_{a_1} \) satisfying (4.1), \( X' = (X_a, \varepsilon'_a, p'_{aa_1}, A) \) is an approximate system.

We first prove (A1) for \( X' \), i.e., we show that, for any \( a_1 < a_2 < a_3 \), one has

\[
d\left(p'_{a_1a_2}, p'_{a_2a_3}, p'_{a_1a_3}\right) \leq \varepsilon'_{a_1}. \tag{4.5}\]

To this effect, notice that (4.1) and (4.2) imply

\[
d\left(p'_{a_1a_2}, p'_{a_2a_3}, p_{a_1a_2}, p_{a_2a_3}\right) \leq \frac{1}{3}(\varepsilon'_{a_1} - \varepsilon_{a_1}). \tag{4.6}\]

Formula (4.1) (for \( a_2 < a_3 \)) and formula (4.3) (applied to \( a_1 < a_2 < a_3 \)) yield

\[
d\left(p_{a_1a_2}, p'_{a_2a_3}, p_{a_1a_2}p_{a_2a_3}\right) \leq \frac{1}{3}(\varepsilon'_{a_1} - \varepsilon_{a_1}). \tag{4.7}\]

By (A1) for \( X \) one has

\[
d\left(p_{a_1a_2}, p_{a_2a_3}, p_{a_1a_3}\right) \leq \varepsilon_{a_1}. \tag{4.8}\]

Finally, by (4.1) and (4.2)

\[
d\left(p_{a_1a_2}, p'_{a_1a_2}, p'_{a_1a_3}\right) \leq \frac{1}{3}(\varepsilon'_{a_1} - \varepsilon_{a_1}). \tag{4.9}\]

(4.6), (4.7), (4.8) and (4.9) yield the desired formula (4.5).

We will now prove (A2) for \( X' \). For a given \( a \in A, \eta > 0 \), choose \( a' \approx a \) so that (A2) and (A3) hold for \( \eta/4 \) and the approximate system \( X^* = (X_a, \varepsilon'_{a}, p'_{aa_1}, A) \). Moreover, let

\[
1/2^{\lfloor a' \rfloor} \leq \frac{1}{4}\eta. \tag{4.10}\]

We claim that for any \( a_2 > a_1 > a' \) one has

\[
d\left(p_{aa_1}, p'_{aa_1}, p'_{a_2a_3}\right) \leq \eta. \tag{4.11}\]

By (4.1), (4.4) and (4.10),

\[
d\left(p_{aa_1}, p'_{aa_1}, p_{aa_1}p'_{a_2a_3}\right) \leq \frac{1}{4}\eta. \tag{4.12}\]

By (4.1) and (4.2),

\[
d\left(p'_{a_2a_3}, p_{a_1a_3}\right) \leq \varepsilon'_{a_1}. \tag{4.13}\]
Hence, by the choice of \( a' \) (property (A3)),

\[
d(p_{a_1}, p'_{a_1,a_2}, p_{a_1}, p_{a_1,a_2}) \leq \frac{1}{4} \eta. \quad (4.14)
\]

Also, by the choice of \( a' \) (property (A2)),

\[
d(p_{a_1}, p_{a_1,a_2}, p_{a_1}) \leq \frac{1}{4} \eta. \quad (4.15)
\]

Finally, by (4.1), (4.4) and (4.10),

\[
d(p_{a_2}, p'_{a_2}) \leq \frac{1}{4} \eta. \quad (4.16)
\]

Now, (4.12), (4.14), (4.15) and (4.16) yield the desired formula (4.11).

In order to prove (A3) for \( X' \), for given \( a \in A \) and \( n > 0 \) choose \( a'' \geq a' \) such that

\[
\frac{1}{2} < a'' \leq \frac{1}{4} \eta. \quad (4.17)
\]

We claim that \( a'' \geq a' \) and \( d(x, x') \leq \varepsilon_{a'} \) implies

\[
d(p'_{a}^{a''}(x), p'_{a}^{a''}(x')) \leq \eta. \quad (4.18)
\]

By the choice of \( a' \),

\[
d(x, x') \leq \varepsilon_{a'} \Rightarrow d(p'_{a}^{a''}(x), p'_{a}^{a''}(x')) \leq \frac{1}{4} \eta. \quad (4.19)
\]

Moreover, by (4.1), (4.4) and (4.17),

\[
d(p'_{a}^{a''}(x), p_{a}^{a''}(x)) \leq \frac{1}{4} \eta, \quad (4.20)
\]

\[
d(p'_{a}^{a''}(x'), p_{a}^{a''}(x')) \leq \frac{1}{4} \eta. \quad (4.21)
\]

Formulae (4.20), (4.19) and (4.21) yield the desired formula (4.18).

Now note that, for any pair \( a, a' \), (4.1) and (4.2) imply

\[
d(p'_{a}^{a''}(x), p_{a}^{a''}(x')) \leq \varepsilon_{a}. \quad (4.22)
\]

We are now able to apply Theorem 1 to \( X' \) and \( X' \). We obtain a homeomorphism

\[
f: X' \to X \text{ satisfying}
\]

\[
d(p_{a}^{a''}, p_{a}^{a''}) \leq \varepsilon_{a}, \quad a \in A. \quad (4.23)
\]

We will show that, actually

\[
p_{a}^{a''} = p_{a}^{a}, \quad a \in A. \quad (4.24)
\]

It suffices to show that for any \( \eta > 0 \) one has

\[
d(p_{a}^{a''}, p_{a}^{a''}) \leq \eta. \quad (4.25)
\]

For this purpose choose \( a' \geq a \) so large that

\[
\frac{1}{2} < a' \leq \frac{1}{4} \eta, \quad (4.26)
\]

that \( a' \) satisfies (A3) for \( \frac{1}{4} \eta \) and \( X' \), and that

\[
d(p_{a}^{a''}, p_{a}^{a''}, f(x)) \leq \frac{1}{4} \eta, \quad a \geq a', \quad (4.27)
\]

\[
d(p_{a}^{a''}, p_{a}^{a''}) \leq \frac{1}{4} \eta, \quad a \geq a'. \quad (4.28)
\]
Note that (4.27) and (4.28) are consequences of Proposition 4, applied to $X^*$ and $X'$. By (4.23) (applied to $a_1$), we obtain
\begin{equation}
    d(p_{aa}^t p_{a} f, p_{aa}^t p_{a}^t) \leq \frac{1}{4} \eta.
\end{equation}
Moreover, by (4.1), (4.4) and (4.26), we have
\begin{equation}
    d(p_{aa} p_{a} f, p_{aa}^t p_{a} f) \leq 1/2 |a| \leq \frac{1}{4} \eta.
\end{equation}
Now (4.27), (4.30), (4.29) and (4.28) yield (4.25).

Recall that $p_a = p A X$ and $p_a = p A X'$. Therefore, (4.24) becomes
\begin{equation}
    \pi_a f(x') = \pi_a(x'), \quad x' \in X', \quad a \in A.
\end{equation}
Since (4.31) holds for all $a \in A$, we conclude that $f(x') = x'$, for all $x' \in X'$. However, $f(x') \in X$, which proves that $x' \subset X'$ implies $x' \subset X$, i.e., $X' \subset X$. In the same way, using the inverse $g = f^{-1}: X \rightarrow X'$, one can show that $X \subset X'$. Consequently, $X' = X$ and $p_a = p_a = \pi_a |X$, $a \in A$. \hfill \Box

**Remark 2.** Theorem 2 applies also to commutative systems $X$. However, the obtained system $X'$, in general, is only almost commutative.

**Remark 3.** One is tempted to conjecture that every inverse sequence of metric compacta $X = (X_i, p_{i+1})$ admits a sequence of numbers $\omega_i > 0$ such that for any choice of mappings $p_{i+1}: X_i \rightarrow X_{i-1}$, satisfying
\begin{equation}
    d(p_{i+1}, p_{i+1}) \leq \omega_i,
\end{equation}
the limit $X'$ of the sequence $X' = (X_i, p_{i+1})$ coincides with or is homeomorphic to $X = \lim X$. However, this is false as demonstrated by the following simple example.

**Example 1.** Let $S^1 = \{z \in \mathbb{C}: |z| = 1\}$ be the unit sphere having the point 1 as its base-point. Let $Z = \bigvee_{j=1}^\infty S_j$ be the Hawaiian earring, i.e., the wedge of a sequence of copies $S_j$ of $S^1$. We view $Z$ as embedded in $\bigcup_{j=1}^\infty S_j$ and endow it with the usual product metric.

Let $X_i = Z$ and let each $p_{i+1}: X_i \rightarrow X_i$ be the identity mapping $1_Z$. Then $X = (X_i, p_{i+1})$ is an inverse sequence of metric compacta with limit $X = Z$.

We will now show that for any sequence of numbers $\omega_i > 0$ one can choose maps $p_{i+1}: X_i \rightarrow X_i$, such that (4.32) holds, but $X' = \lim X', X' = (X_i, p_{i+1})$, is not homeomorphic to $X$.

It suffices to prove the assertion under the additional assumption that $\omega_1 \geq \omega_2 \geq \cdots$. For each $i$ choose an integer $n_i$ so large that
\begin{equation}
    \text{diam } S_{n_i} \leq \frac{1}{2} \omega_i.
\end{equation}
One can assume that $n_1 > n_2 > \cdots$. We now define maps $p_{i+1}': Z \rightarrow Z$ by putting
\begin{equation}
    p_{i+1}'(x) = \begin{cases} x & \text{if } x \in S_j, \quad j \neq n_{i+1}, \\ \phi_j(x) & \text{if } x \in S_{n_{i+1}}. \end{cases}
\end{equation}
here $\phi_i$ maps $S_{n_i}$ to $S_n$ and is given by $\phi_i(t) = t^2$. Note that (4.32) holds. Indeed, if $x \in S_j$, $j \neq n_{i+1}$, then $d(p_{i+1}(x), p_{i+1}(x)) = d(x, x) = 0$. If $x \in S_{n_{i+1}}$, then

$$d(p_{i+1}(x), p_{i+1}(x)) = d(\phi_i(x), x) \leq \text{diam } S_n + \text{diam } S_{n_{i+1}}$$

$$\leq \frac{1}{2} \omega_i + \frac{1}{2} \omega_{i+1} \leq \omega_i.$$  

(4.35)

Now put $Y_i = S_{n_i} \subseteq Z$ and note that $Y = (Y_i, p_{i+1})$ is an inverse sequence whose limit $Y$ is the dyadic solenoid, because $\phi_i: S^1 \to S^1$, $\phi_i(t) = t^2$. It is well known that the solenoid $Y$ does not embed in the plane (and is not locally connected). Since $X' = \lim X'$ contains $Y$, it does not embed in the plane (and is not hereditarily locally connected). On the other hand, the Hawaiian earring is a planar continuum (and is hereditarily locally connected). Therefore, $X$ and $X'$ cannot be homeomorphic.

5. Approximate systems of near homeomorphisms

For compact metric spaces $X'$, $X''$ let $\text{Map}(X', X'')$ denote the space of continuous maps $f: X' \to X''$ with the metric $d(f, g) = \sup(d(f(x), g(x)))$. The following result is a consequence of Theorem 2.

**Theorem 3.** Let $X = (X_a, p_{aa'}, A)$ be a cofinite almost commutative system of metric compacta. For each pair $a < a'$ let $M_{aa'} \subseteq \text{Map}(X_{a'}, X_a)$ be a set, such that $p_{aa'} \in \text{Cl}(M_{aa'})$; also let the identity map $1_{X_a} \in M_{aa'}$. Then there exists an almost commutative system $X' = (X_a, p_{aa'}, A)$, such that $p_{aa'} \in M_{aa'}$, the limits $X' = \lim X'$, $X = \lim X$ coincide, $X' = X$, and the natural projections coincide, $p_a = p_a$, $a \in A$.

**Proof.** By Definition 3, there exist numbers $\varepsilon_a > 0$ such that $X = (X_a, \varepsilon_a, p_{aa'}, A)$ is an approximate inverse system. Choose numbers $\delta_{a_1 a_2} > 0$, $a_1 < a_2$, as in Theorem 2. Since $p_{aa'} \in \text{Cl}(M_{aa'})$, it is possible to choose maps $p_{aa'} \in M_{aa'}$, $a < a'$, such that $d(p_{aa'}, p_{aa'}) \leq \delta_{aa'}$. Then $X' = (X_a, p_{aa'}, A)$ is an almost commutative system, $X' = X$ and $p_a = p_a$, $a \in A$. $\square$

**Remark 4.** This result is analogous to Theorem 3 of [2], which we state as follows.

**Corollary 2.** Let $X = (X_i, p_{i+1})$ be an inverse sequence of metric compacta and let $M_i \subseteq \text{Map}(X_{i+1}, X_i)$ be sets such that $p_{i+1} \in \text{Cl} M_i$. Then there is an inverse sequence $X' = (X_i, p_{i+1}')$ such that $p_{i+1}' \in M_i$ and the limits $X' = \lim X'$ and $X = \lim X$ are homeomorphic.

**Proof.** By Corollary 1, it suffices to define numbers $\varepsilon_i > 0$ and maps $p_{i+1}' \in M_i$ such that (2.29)-(2.31) hold. This is done by induction on $i$. Assume that we have already defined $\varepsilon_1, \ldots, \varepsilon_k$ and $p_{i+1}'$, $i = k - 1$. Since $p_{k+1} \in \text{Cl}(M_k)$, there exists a mapping
$p_{k+1} \in M_k$ such that $d(p_{k+1}, p_{k+1}) = \frac{1}{2} \varepsilon_k$. We now consider the maps $p_{k+1} = p_{i+1} \cdots p_{k+1}, p_{i+1} = p_{i+1} \cdots p_{k+1}, i \leq k$. By their uniform continuity, there is a number $\varepsilon_{k+1} > 0$ so small that $d(x, x') = \varepsilon_{k+1}$ implies

\[ d(p_{i+1}(x), p_{i+1}(x')) = \frac{\varepsilon_i}{2^{k+1}}, \]

\[ d(p_{i+1}(x), p_{i+1}(x')) = \frac{\varepsilon_i}{2^{k+1}}. \]

**Definition 4.** A mapping $f : X \to Y$ between metric compacta is called a near homeomorphism provided $f \in \text{Cl}(H(X, Y))$, where $H(X, Y) \subseteq \text{Map}(X, Y)$ is the set of all homeomorphisms of $X$ onto $Y$. Clearly, the existence of a near homeomorphism $f : X \to Y$ implies that $X$ and $Y$ are homeomorphic.

**Theorem 4.** Let $X = (X_0, p_{aa'}, A)$ be a cofinite almost commutative system of metric compact. If all $p_{aa'}$ are near homeomorphisms, then there exists an almost commutative system $X' = (X_0, p_{aa'}, A)$ such that all the bonding maps $p_{aa'}$ are homeomorphisms, the limit spaces $X' = \lim X'$ and $X = \lim X$ coincide, $X' = X$, and so do the natural projections, $p_a = p_{a'}$.

**Proof.** Apply Theorem 3 to the case $M_{aa'} = H(X_0, X_0)$. □

Theorem 4 is an analogue of the well-known M. Brown theorem [2, Theorem 4], which we now state and prove.

**Corollary 3.** Let $X = (X_i, p_{i+1})$ be an inverse sequence of metric compacta all of whose bonding maps are near homeomorphisms. Then $X = \lim X$ is homeomorphic to any of the spaces $X_i$.

**Proof.** Corollary 2 yields an inverse sequence $X' = (X_i, p_{i+1})$, where all the bonding maps are homeomorphisms and $X' = \lim X'$ is homeomorphic with $X = \lim X$. Since $X'$ is homeomorphic with every term $X_i$ of $X'$, so is $X$. □

**Remark 5.** Recently, F. D. Ancel has given a new proof of Brown’s theorem [1].

**Remark 6.** In contradistinction to the case of a sequence (see Corollary 3), in Theorem 4 one cannot assert that the limit space $X$ is homeomorphic to the spaces $X_i$. Namely, it is easy to define cofinite inverse systems of metric compacta $X_o$, whose bonding maps are near homeomorphisms, but $X = \lim X$ is a nonmetrizable space and therefore, it cannot be homeomorphic to any of the terms $X_o$. We now describe such an example.
Example 2. Let $A$ be the set of all finite subsets of the unit interval $I = [0, 1]$. We order $A$ by inclusion $\subseteq$. For $a = \{t_1, \ldots, t_k\} \in A$, we put

$$X_a = I \times 0 \cup \bigcup_{i=1}^{k} (t_i \times I) \subseteq I \times I. \quad (5.1)$$

Ordering $X_a$ lexicographically, we obtain an ordered space homeomorphic to $I$. If $a \subseteq a' = \{t_1, \ldots, t_k\}$, we define $p_{aa} : X_a \to X_a$ by $p_{aa}(t, 0) = t, t \in I$; $p_{aa}(t_i, s) = t_i, i \in \{1, \ldots, k\}, s \in I$. Clearly, each $p_{aa}$ is a near homeomorphism and $X = (X_a, p_{aa}, A)$ is a cofinite inverse system. The limit $X$ of $X$ is a nonmetrizable ordered continuum, because it admits uncountable collections of disjoint nonempty open sets. Such a collection is given by the sets $(p_{t(t)})^{-1}((\text{Int}(t \times I))$, where $t$ ranges over $I$.

Remark 7. Example 2 shows that Theorems 3 and 4 are false if one replaces almost commutative systems by commutative systems.

Remark 8. Concerning Theorem 2 and commutative systems $X = (X_a, p_{aa}, A)$, Example 2 shows that it is not possible to find numbers $\delta_{aa} > 0, a < a'$, with the following property: If $X' = (X_a, p_{aa}', A)$ is a commutative system and $d(p_{aa}', p_{aa}) \leq \delta_{aa}$, then the limits $X' = \lim X'$ and $X = \lim X$ are homeomorphic spaces.

Remark 9. If $X = (X_a, p_{aa}, A)$ is an almost commutative system all of whose bonding maps are homeomorphisms, one cannot conclude that the projections $p_a : X \to X_a$ are homeomorphisms or near homeomorphisms. Indeed, assume that this is the case and consider $X$ from Example 2. Then Theorem 4 yields an approximate system $X'$ with homeomorphisms as bonding mappings and $X' = X$. The assumption would imply that $X \approx X_a$, which is not the case.

To obtain a positive result on approximate systems having homeomorphisms as bonding maps, we need an extension of the definition of a refinable map, originally defined for maps between metric compacta [3] (see [4]).

Definition 5. Let $f : X \to Y$ be a mapping between compact Hausdorff spaces. We say that $f$ is refinable provided it is onto and for arbitrary open coverings $\mathcal{U}$ of $X$ and $\mathcal{V}$ of $Y$ there is an onto mapping $g : X \to Y$ such that $g$ is a $\mathcal{U}$-mapping (i.e., sets $g^{-1}(y), y \in Y$, refine $\mathcal{U}$) and the maps $f$ and $g$ are $\mathcal{V}$-near (i.e., sets $\{f(x), g(x)\}, x \in X$, refine $\mathcal{V}$).

Our notion of refinable map applies also to maps between a non-metric and a metric compact space. Note that maps, which are near homeomorphisms, are refinable.

Theorem 5. Let $X = (X_a, p_{aa}, A)$ be an almost commutative system of metric compacta with limit $X$. If all $p_{aa}$ are near homeomorphisms (homeomorphisms), then all the projections $p_a : X \to X_a$ are refinable mappings.
Proof. Since the bonding mappings $p_{aa'}$ are surjective, so are the projections $p_a : X \to X_a$ (see Proposition 6). Given an open covering $\mathcal{U}$ of $X$ and an $\eta > 0$, we must produce a surjective map $g : X \to X_a$ such that $d(p_a, g) \leq \eta$ and $(g^{-1}(y), y \in X_a)$ refines $\mathcal{U}$. By Proposition 5 (property (B2)), there is an index $a' \geq a$ and an open covering $\mathcal{V}$ of $X_{a'}$ such that $p_{a'}^{-1}(\mathcal{V})$ refines $\mathcal{U}$. By Proposition 4, one can achieve that $d(p_a, p_{aa'} p_{a'}) \leq \frac{1}{2} \eta$. Since $p_{aa'}$ is a near homeomorphism, there exists a homeomorphism $p'_{aa'} : X_{aa'} \to X_a$ such that $d(p'_{aa'}, p_{aa'}) \leq \frac{1}{2} \eta$ and, therefore, $d(p_a, p'_{aa'} p_a) \leq \eta$. We now put $g = p'_{aa'} p_a$. Since $p'_{aa'}$ and $p_a$ are surjections, so is $g : X \to X_a$. Moreover, for any $y \in X_a$, $g^{-1}(y) = (p_a)^{-1}(z)$, where $z = (p'_{aa'})^{-1}(y)$. Clearly, $z \in V$ and $(p_a)^{-1}(z) \subseteq (p_a)^{-1}(V) \subseteq U$, for some $V \in \mathcal{V}$ and some $U \in \mathcal{U}$. This proves that $p_a$ is indeed a refinable mapping. 

Remark 10. If the compacta $X_a$ in Theorem 5 are calm (e.g., if the $X_a$ are FANR’s or polyhedra), then the projections $p_a : X \to X_a$ induce shape equivalences. This is an immediate consequence of Theorem 5 and of [4, Corollary 2].

References


