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Representation of Differential Operators in Wavelet Basis

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Abstract—Existing work on the representation of operators in one-dimensional, compactly-supported, orthonormal wavelet bases is extended to two dimensions. The nonstandard form of the representation of operators is given in separable two-dimensional, periodic, orthonormal wavelet bases. The matrix representation of the partial-differential operators ∂_x and ∂_y are constructed and a closed form formula for the matrix representation of a general partial-differential operator $g(\partial_x, \partial_y)$ is derived, where g is an analytic function.

Keywords—Wavelet, Partial-differential operator, Circulant, Block circulant.

1. INTRODUCTION

The popularity of wavelets may be attributed, to their ability to resolve phenomena within different scales of magnitude. For example, a signal may have both high- and low-frequency content, and the proper and efficient representation of such contrasting behavior is a problem to which wavelets are most amenable.

Although the theory of wavelets has been developed relatively recently (see, e.g., [1–5]), its historical origins date back to the beginning of the 20$^{\text{th}}$ century, when Haar [6] constructed the first known wavelets. The important concept of a multiresolution analysis, where the "smoothness" of a function is separated from the "details", was introduced by Meyer [7] and Mallat [2]. Another important concept is the wavelet transform, which may be continuous or discrete (see, e.g., [8,9]).

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Wavelets have found numerous applications in signal processing. To name only a few, Kronland-Martinet, Morlet and Grossmann [10] have employed wavelet transforms in the analysis of sound patterns, whereas Antoni et al. [11], Devore, Jawerth and Lucier [12], and Froment and Mallat [13] have applied wavelets to image processing. Wavelets have even permeated the area of fractals where, among other applications, fractal-wavelet transforms have been developed for image compression (see, e.g., [14]).

Wavelets have been employed in the numerical solution of various types of differential equations. For example, Engquist, Osher and Zhong [15] have employed fast, wavelet-based algorithms for the solution of linear evolution equations, Jaffard [16] has employed wavelet methods for the fast resolution of elliptic problems, and Xu and Shann [17] have employed wavelet-based Galerkin methods to solve two-point boundary-value problems.

Wavelets have also found applications in the representation of differential operators. The representation of operators in compactly-supported, one-dimensional wavelet bases, as well as the construction of their corresponding matrices, has been considered by Beylkin [18] (see also [19]). The foregoing results have been employed by Beylkin and Keiser [20] in the numerical solution of evolution equation in one temporal and one spatial dimension.

In this paper, we address the two-spatial dimension case and consider the representation of operators, in particular linear differential operators, in two-dimensional compactly supported wavelet bases. This work is an important generalization of that of Beylkin. Periodized Daubechies' wavelets are used in the construction of the matrices representing the linear differential operators. The linear differential operators are assumed to be functions of the operators ∂_x and ∂_y, i.e., $L = g(\partial_x, \partial_y)$, where g is an analytic function.

The differential operators ∂_x and ∂_y are first considered separately. Due to the periodicity of the wavelets, the matrices of such operators admit special structures, which are then exploited to derive the matrix representation of the more general differential operator $L = g(\partial_x, \partial_y)$.

This paper is organized as follows. In Section 2, we review "multiresolution analysis" of both $L^2(R)$ and $L^2(R^2)$. In Section 3, we give the definitions of the standard and nonstandard forms of the representation of operators in a wavelet multiresolution analysis. In Section 4, we discuss the construction of the matrix representation of the nonstandard form representation of a general linear operator T, while in Section 5 we construct the matrices for the cases of the differential operators ∂_x and ∂_y. Section 6 is devoted to the derivation of a closed form formula for the matrix representation of a general differential operator $g(\partial_x, \partial_y)$. Finally, in Section 7, we conclude with some remarks and future research directions.

2. MULTIRESOLUTION ANALYSIS AND WAVELET BASES

In this section, we review multiresolution analysis and wavelet bases of both $L^2(R)$ and $L^2(R^2)$. In Section 2.1, we review multiresolution analysis of $L^2(R)$ and Daubechies compactly-supported wavelets. In Section 2.2, we consider the two-dimensional case.

2.1. One-Dimensional Multiresolution Analysis

A multiresolution analysis (1-D MRA) of $L^2(R)$ is defined as an increasing sequence of closed subspaces $V_j \subset L^2(R)$, $j \in Z$, $Z = \{0, \pm 1, \pm 2, \dots\}$,

$$\{0\} \subset \cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2(R), \tag{1}$$

with the following properties:

(P1) $\bigcup_{j\in Z} V_j$ is dense in $L^2(R)$ and $\bigcap_{j\in Z} V_j = \{0\}$.

(P2) $f(x) \in V_j \iff f(2x) \in V_{j+1}$, for all $j \in Z$.

(P3) $f(x) \in V_j \iff f(x - 2^{-j}k) \in V_j$, for all $k \in Z$.

(P4) There exists a function $\phi(x) \in V_0$, with nonvanishing integral, such that the set $\{\phi_{0,k}(x) = \phi(x-k),\ k \in Z\}$ is an orthonormal basis of V_0.

We have abused notation here, for the sake of brevity. For example, (P2) means that $f \in V_j \iff g \in V_{j+1}$, where $g(x) = f(2x)$. The function $\phi(x)$ in (P4) is called the *scaling function* associated with the multiresolution analysis. It should be mentioned that the orthonormality of the basis functions in (P4) is not a strict requirement. In fact, the multiresolution analysis is also defined with the set $\{\phi_{0,k}(x) = \phi(x-k),\ k \in Z\}$ being a mere *Riesz basis* of V_0.

Let us make some observations concerning this definition. Since $\phi(x) \in V_0 \subset V_1$, there exists a sequence, $\{h_k,\ k \in Z\}$, such that

$$\phi(x) = \sum_k h_k \phi_{1,k}(x) = \sqrt{2} \sum_k h_k \phi(2x-k). \tag{2}$$

This functional equation goes by several different names: the *dilation equation*, the *two-scale difference equation*, or the *refinement equation*. We shall refer to it by the latter name. It also follows immediately that the collection of functions $\{\phi_{j,k},\ k \in Z\}$ with

$$\phi_{j,k}(x) = 2^{j/2} \phi\left(2^j x - k\right), \qquad k \in Z, \tag{3}$$

constitutes an orthonormal basis of V_j.

The scaling function ϕ is, under general conditions, uniquely defined by its refinement equation (2) and the normalization,

$$\int_{-\infty}^{\infty} \phi(x)\, dx = 1. \tag{4}$$

It is important to note that in many cases no explicit expression for ϕ is available. In many applications, we never need the scaling function itself; instead we often work directly with the coefficients h_k.

The spaces V_j are called the approximation spaces and are used to approximate general functions. This is done by defining appropriate projections onto these spaces. Associated with the spaces V_j are "detail" spaces defined as follows. Let W_j denote the orthogonal complement of V_j in V_{j+1}, i.e., a space that satisfies

$$V_{j+1} = V_j \oplus W_j.$$

The spaces W_j contain the detail information needed to go from an approximation at resolution j to an approximation at resolution $j+1$. Consequently,

$$\bigoplus_j W_j = L^2(R).$$

A wavelet is a function ψ such that the collection of functions $\{\psi(x-k),\ k \in Z\}$ constitutes an orthonormal basis of W_0. The collection of wavelet functions $\{\psi_{j,k},\ j,k \in Z\}$ is then an orthonormal basis of $L^2(R)$. The definition of $\psi_{j,k}$ is similar to the one of $\phi_{j,k}$, that is, $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$. The wavelet ψ satisfies an equation similar to that of the scaling function ϕ,

$$\psi(x) = \sqrt{2} \sum_k g_k \phi(2x-k),$$

where the coefficients g_k are given by

$$g_k = (-1)^k h_{-k+1}.$$

The wavelet bases discussed above consist of functions that are supported on the entire real line. However, in most applications, it is desirable and sometimes necessary to work with wavelets supported in a compact subset of the real line. The most popular compactly-supported wavelets are the ones constructed by Daubechies [4]. Daubechies' scaling function satisfies the finite refinement equation

$$\phi(x) = \sqrt{2} \sum_{k=0}^{L-1} h_k \phi(2x-k), \tag{5}$$

and the wavelet function satisfies

$$\psi(x) = \sqrt{2} \sum_{k=0}^{L-1} g_k \phi(2x-k), \tag{6}$$

where the coefficients g_k are given in terms of h_k by

$$g_k = (-1)^k h_{L-k-1}, \qquad k = 0, 1, 2, \dots, L-1.$$

Both the scaling function ϕ and the wavelet function ψ have support in $[0, L-1]$.

Employing the refinement equations satisfied by ϕ and ψ, namely equations (5) and (6), and the definitions of $\phi_{j,k}$ and $\psi_{j,k}$, we find that the scaling functions $\phi_{j,k}$ and the wavelet functions $\psi_{j,k}$ satisfy

$$\phi_{j,k}(x) = \sum_{m=0}^{L-1} h_m \phi_{j+1,m+2k}(x), \tag{7}$$

$$\psi_{j,k}(x) = \sum_{m=0}^{L-1} g_m \psi_{j+1,m+2k}(x). \tag{8}$$

One of the properties of Daubechies wavelets is that the scaling function satisfying (5), where $L = 2M$, has M^{th}-order approximation, in the sense that any polynomial of degree less than or equal to $M-1$ can be expressed as a linear combination of integral translates of $\phi(x)$, i.e., for any polynomial P_r of degree $r \le M-1$, there exist coefficients c_k such that

$$P_r(x) = \sum_k c_k \phi(x-k).$$

This approximation property translates into the wavelet function ψ having M vanishing moments, i.e.,

$$\int x^m \psi(x)\, ds = 0, \qquad m = 0, 1, 2, \dots, M-1. \tag{9}$$

It is important to note here that the higher the number of vanishing moments of ψ, the better the approximation is. However, larger M implies larger $L = 2M$, that is, longer *low-* and *high-*pass filters h_k and g_k, respectively.

The next section discusses the two-dimensional multiresolution analysis.

2.2. Two-Dimensional Multiresolution Analysis

There are two ways to construct two-dimensional wavelet bases. An easy and the most common method is by building an $L^2(R^2)$ multiresolution analysis which is obtained from the tensor product of a multiresolution analysis of $L^2(R)$. This leads to separable wavelet bases. A more general method is by extending the concept of multiresolution analysis to two dimensions, which leads to inseparable bidimensional wavelet bases [21,22]. In this section, we review the tensor product technique for the construction of separable wavelet bases.

Consider a one-dimensional multiresolution analysis as defined in the previous section,

$$\{0\} \subset \cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2(R), \tag{10}$$

and define the spaces $\mathbf{V}_j$, $j \in Z$, by

$$\mathbf{V}_j = V_j \otimes V_j = \{F(x,y) \mid F(x,y) = f(x)g(y),\ f,g \in V_j\}. \tag{11}$$

Clearly, the subspaces $\mathbf{V}_j$ form a "separable" multiresolution analysis of $L^2(R^2)$, that is, we have an increasing sequence of linear subspaces of $L^2(R^2)$,

$$\{0\} \subset \cdots \subset \mathbf{V}_{-2} \subset \mathbf{V}_{-1} \subset \mathbf{V}_0 \subset \mathbf{V}_1 \subset \mathbf{V}_2 \subset \cdots \subset L^2\left(R^2\right), \tag{12}$$

satisfying

(1) $\bigcap_{j\in Z} \mathbf{V}_j = \{0\}$, $\overline{\bigcup_{j\in Z} \mathbf{V}_j} = L^2(R^2)$,
(2) $f(x,y) \in \mathbf{V}_j \iff f(2x,2y) \in \mathbf{V}_{j+1}$,
(3) $f(x,y) \in \mathbf{V}_j \iff f(x - 2^{-j}k_1, y - 2^{-j}k_2) \in \mathbf{V}_j$, for all $k_1, k_2 \in Z$.

The scaling function associated with this $L^2(R^2)$ multiresolution analysis is then given by

$$\Phi(x,y) = \phi(x)\phi(y), \tag{13}$$

where $\phi(x)$ is the scaling function associated with (10). Since, for each $j \in Z$, the set $\{\phi_{j,k}(x) = 2^{j/2}\phi(2^jx - k),\ k \in Z\}$ is an orthonormal basis for V_j, it follows that the set

$$\Phi^j_{k_1,k_2}(x,y) = 2^j\phi\left(2^jx - k_1\right)\phi\left(2^jy - k_2\right), \qquad k_1, k_2 \in Z, \tag{14}$$

is an orthonormal basis for $\mathbf{V}_j$. For each $j \in Z$, denote by $\mathbf{W}_j$ the orthogonal complement of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$. Then, we have the wavelet spaces $\mathbf{W}_j$ given by

$$\mathbf{W}_j = (W_j \otimes W_j) \oplus (V_j \otimes W_j) \oplus (W_j \otimes V_j), \qquad j \in Z, \tag{15}$$

where the W_j are the wavelet spaces associated with (10). As a consequence of (15), three basic wavelets are required to define the orthogonal complement of $\mathbf{V}_0$ in $\mathbf{V}_1$, namely,

$$\Psi^h(x,y) = \phi(x)\psi(y), \tag{16}$$
$$\Psi^v(x,y) = \psi(x)\phi(y), \tag{17}$$
$$\Psi^d(x,y) = \psi(x)\psi(y), \tag{18}$$

where the superscripts h, v, and d stand for "horizontal", "vertical", and "diagonal", respectively. It then follows that an orthonormal basis for $\mathbf{W}_j$ consists of the collection of functions

$$\left\{\Psi^{h,j}_{k_1,k_2}(x,y), \Psi^{v,j}_{k_1,k_2}(x,y), \Psi^{d,j}_{k_1,k_2}(x,y),\ k_1, k_2 \in Z\right\}, \tag{19}$$

where

$$\Psi^{h,j}_{k_1,k_2}(x,y) = 2^j\Psi^h\left(2^jx - k_1, 2^jy - k_2\right) = 2^j\phi\left(2^jx - k_1\right)\psi\left(2^jy - k_2\right), \tag{20}$$
$$\Psi^{v,j}_{k_1,k_2}(x,y) = 2^j\Psi^v\left(2^jx - k_1, 2^jy - k_2\right) = 2^j\psi\left(2^jx - k_1\right)\phi\left(2^jy - k_2\right), \tag{21}$$
$$\Psi^{d,j}_{k_1,k_2}(x,y) = 2^j\Psi^d\left(2^jx - k_1, 2^jy - k_2\right) = 2^j\psi\left(2^jx - k_1\right)\psi\left(2^jy - k_2\right). \tag{22}$$

The separable orthonormal wavelet basis of $L^2(R^2)$ is the set

$$\left\{\Psi^{h,j}_{k_1,k_2}(x,y), \Psi^{v,j}_{k_1,k_2}(x,y), \Psi^{d,j}_{k_1,k_2}(x,y),\ j, k_1, k_2 \in Z\right\}. \tag{23}$$

If the original one-dimensional scaling function $\phi(x)$ and wavelet function $\psi(x)$ have compact support in $[0, L-1]$, then it is clear that $\Phi(x,y)$ and $\Psi^\lambda(x,y)$ ($\lambda = h, v$, and d) are compactly supported in $[0, L-1]^2$. In what follows, we consider compactly supported wavelets.

Since $\phi(x)$ and $\psi(x)$ satisfy the refinement equations

$$\phi(x) = \sqrt{2}\sum_{k=0}^{L-1} h_k \phi(2x-k) \qquad \text{and} \qquad \psi(x) = \sqrt{2}\sum_{k=0}^{L-1} g_k \psi(2x-k),$$

where $L = 2M$ and M is the number of vanishing moments of $\psi(x)$, the two-dimensional separable scaling and wavelet functions satisfy

$$\Phi^k_{k_1,k_2}(x,y) = 2\sum_{k_1=0}^{L-1}\sum_{k_2=0}^{L-1} h_{k_1} h_{k_2} \Phi(2x-k_1, 2y-k_2), \tag{24}$$

$$\Psi^h(x,y) = 2\sum_{k_1=0}^{L-1}\sum_{k_2=0}^{L-1} h_{k_1} g_{k_2} \Psi^h(2x-k_1, 2y-k_2), \tag{25}$$

$$\Psi^v(x,y) = 2\sum_{k_1=0}^{L-1}\sum_{k_2=0}^{L-1} g_{k_1} h_{k_2} \Psi^v(2x-k_1, 2y-k_2), \tag{26}$$

$$\Psi^d(x,y) = 2\sum_{k_1=0}^{L-1}\sum_{k_2=0}^{L-1} g_{k_1} g_{k_2} \Psi^d(2x-k_1, 2y-k_2), \tag{27}$$

and for each $j, k_1, k_2 \in Z$, $\lambda = h, v, d$, we have

$$\Phi^j_{k_1,k_2}(x,y) = \sum_{m_1,m_2=0}^{L-1} H_{m_1,m_2} \Phi^{j+1}_{m_1+2k_1,m_2+2k_2}(x,y), \tag{28}$$

$$\Psi^{\lambda,j}_{k_1,k_2}(x,y) = \sum_{m_1,m_2=0}^{L-1} G^\lambda_{m_1,m_2} \Phi^{j+1}_{m_1+2k_1,m_2+2k_2}(x,y), \tag{29}$$

where $H_{i,j} = h_i h_j$, $G^h_{i,j} = h_i g_j$, $G^v_{i,j} = g_i h_j$, and $G^d_{i,j} = g_i g_j$.

Now, let us consider the approximation of a function $f(x,y)$ in the multiresolution spaces. As mentioned earlier, approximation of functions $f(x,y) \in L^2(R^2)$ is performed by appropriate projections onto the spaces $\mathbf{V}_j$ of the multiresolution analysis.

For $f \in L^2(R^2)$, the approximation of $f(x,y)$ in $\mathbf{V}_j$ is given by its orthogonal projection onto $\mathbf{V}_j$,

$$(P_j f)(x,y) = \sum_{k_1,k_2} s^j_{k_1,k_2} \Phi^j_{k_1,k_2}(x,y), \tag{30}$$

where the "averages" $s^j_{k_1,k_2}$ are given by

$$s^j_{k_1,k_2} = \left\langle f, \Phi^j_{k_1,k_2} \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\Phi^j_{k_1,k_2}(x,y)\,dx\,dy. \tag{31}$$

Since $\mathbf{W}_j$ is composed of three orthogonal subspaces (see (15)),

$$\mathbf{W}_j = \mathbf{W}^h_j \oplus \mathbf{W}^v_j \oplus \mathbf{W}^d_j,$$

with

$$\mathbf{W}^h_j = V_j \oplus W_j, \qquad \mathbf{W}^v_j = W_j \oplus V_j, \qquad \mathbf{W}^d_j = W_j \oplus W_j,$$

the orthogonal projection, Q_j, onto $\mathbf{W}_j$ is the sum of the orthogonal projections onto each one of $\mathbf{W}^h_j$, $\mathbf{W}^v_j$, and $\mathbf{W}^d_j$, namely,

$$Q_j = Q^h_j + Q^v_j + Q^d_j. \tag{32}$$

The orthogonal projection of $f(x,y)$ onto $\mathbf{W}_j$ can then be written as

$$\begin{aligned}(Q_jf)(x,y) &= \left(Q_j^hf\right)(x,y) + \left(Q_j^vf\right)(x,y) + \left(Q_j^df\right)(x,y)\\ &= \sum_{k_1,k_2} d^{h,j}_{k_1,k_2}\Psi^{h,j}_{k_1,k_2}(x,y) + \sum_{k_1,k_2} d^{v,j}_{k_1,k_2}\Psi^{v,j}_{k_1,k_2}(x,y) + \sum_{k_1,k_2} d^{d,j}_{k_1,k_2}\Psi^{d,j}_{k_1,k_2}(x,y),\end{aligned}$$

where the coefficients $d^{h,j}_{k_1,k_2}$, $d^{v,j}_{k_1,k_2}$, and $d^{d,j}_{k_1,k_2}$ are the horizontal, vertical, and diagonal "details", respectively, and, for $\lambda = v,h,d$, are given by

$$d^{\lambda,j}_{k_1,k_2} = \left\langle f, \Psi^{\lambda,j}_{k_1,k_2}\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\Psi^{\lambda,j}_{k_1,k_2}(x,y)\,dx\,dy. \tag{33}$$

The coefficients $d^{\lambda,j}$ ($\lambda = h,v,d$) contain the information (details) to go from an approximation resolution (j) to a higher approximation resolution ($j+1$).

Since $\mathbf{V}_{j-1} \oplus \mathbf{W}_{j-1} = \mathbf{V}_j$, it is easy to see that the averages s^{j-1} and the details $d^{\lambda,j-1}$ ($\lambda = h,v,d$) at resolution $j-1$ are obtained from the averages s^j at resolution j by means of

$$s^{j-1}_{k_1,k_2} = \sum_{m_1=0}^{L-1}\sum_{m_2=0}^{L-1} H_{m_1,m_2} s^j_{m_1+2k_1,m_2+2k_2}, \tag{34}$$

$$d^{j-1,\lambda}_{k_1,k_2} = \sum_{m_1=0}^{L-1}\sum_{m_2=0}^{L-1} G^{\lambda}_{m_1,m_2} s^j_{m_1+2k_1,m_2+2k_2}. \tag{35}$$

The reconstruction of $s^j_{k_1,k_2}$ from $s^{j-1}_{k_1,k_2}$ and the details $d^{h,j-1}_{k_1,k_2}$, $d^{v,j-1}_{k_1,k_2}$, and $d^{d,j-1}_{k_1,k_2}$ are given by

$$\begin{aligned} s^j_{2k_1,2k_2} &= \sum_{m_1,m_2=0}^{M-1} H_{2m_1,2m_2} s^{j-1}_{k_1-m_1,k_2-m_2}\\ &\quad + \sum_{\lambda=h,v,d}\sum_{m_1,m_2=0}^{M-1} G^{\lambda}_{2m_1,2m_2} d^{\lambda,j-1}_{k_1-m_1,k_2-m_2}, \end{aligned}\tag{36}$$

$$\begin{aligned} s^j_{2k_1+1,2k_2+1} &= \sum_{m_1,m_2=0}^{M_1} H_{2m_1+1,2m_2+1} s^{j-1}_{k_1-m_1,k_2-m_2}\\ &\quad + \sum_{\lambda=h,v,d}\sum_{m_1,m_2=0}^{M-1} G^{\lambda}_{2m_1+1,2m_2+1} d^{\lambda,j-1}_{k_1-m_1,k_2-m_2}, \end{aligned}\tag{37}$$

$$\begin{aligned} s^j_{2k_1,2k_2+1} &= \sum_{m_1,m_2=0}^{M-1} H_{2m_1,2m_2+1} s^{j-1}_{k_1-m_1,k_2-m_2}\\ &\quad + \sum_{\lambda=h,v,d}\sum_{m_1,m_2=0}^{M-1} G^{\lambda}_{2m_1,2m_2+1} d^{\lambda,j-1}_{k_1-m_1,k_2-m_2}, \end{aligned}\tag{38}$$

$$\begin{aligned} s^j_{2k_1+1,2k_2} &= \sum_{m_1,m_2=0}^{M-1} H_{2m_1+1,2m_2} s^{j-1}_{k_1-m_1,k_2-m_2}\\ &\quad + \sum_{\lambda=h,v,d}\sum_{m_1,m_2=0}^{M-1} G^{\lambda}_{2m_1+1,2m_2} d^{\lambda,j-1}_{k_1-m_1,k_2-m_2}. \end{aligned}\tag{39}$$

3. THE STANDARD AND NONSTANDARD FORMS

In this section, we discuss the standard and nonstandard forms of the representation of a linear operator as was defined by Beylkin [18] (see also [19]) in a two-dimensional compactly-supported wavelet basis.

The representation of an operator T in wavelet bases is a set of operators acting on the multiresolution spaces. The representation can be in standard or nonstandard form, terms which will be defined in this section. If the multiresolution spaces are finite dimensional, then these operators are represented by finite-dimensional matrices.

We start by defining the standard and nonstandard forms of the representation of a general linear operator T. To this end, consider a multiresolution analysis

$$\{0\} \subset \cdots \subset \mathbf{V}_{-2} \subset \mathbf{V}_{-1} \subset \mathbf{V}_0 \subset \mathbf{V}_1 \subset \mathbf{V}_2 \subset \cdots \subset R^2, \tag{40}$$

in $L^2(R^2)$ generating an orthonormal wavelet basis. Let $\mathbf{W}_j$, $j \in Z$, be the orthogonal complement of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$ ($\mathbf{V}_j \oplus \mathbf{W}_j = \mathbf{V}_{j+1}$), and let P_j and Q_j be the orthogonal projections onto $\mathbf{V}_j$ and $\mathbf{W}_j$, respectively. The standard form of T in (40) is defined as the set of operators

$$T = \left\{ A_j, \left\{ B^j_{j'} \right\}_{j' \le j-1}, \left\{ C^j_{j'} \right\}_{j' \le j-1} \right\}_{j \in Z}, \tag{41}$$

where

$$A_j = Q_j T Q_j : W_j \to W_j, \tag{42}$$

$$B^j_{j'} = Q_j T Q_{j'} : W_{j'} \to W_j, \tag{43}$$

$$C^j_{j'} = Q_{j'} T Q_j : W_j \to W_{j'}. \tag{44}$$

The nonstandard form of T is defined as the set of operators

$$T = \{A_j, B_j, C_j\}_{j \in Z}, \tag{45}$$

where

$$A_j = Q_j T Q_j : \mathbf{W}_j \to \mathbf{W}_j, \tag{46}$$

$$B_j = Q_j T P_j : \mathbf{V}_j \to \mathbf{W}_j, \tag{47}$$

$$C_j = P_j T Q_j : \mathbf{W}_j \to \mathbf{V}_j. \tag{48}$$

As we can see, the difference between the standard and nonstandard forms is that the standard form consists of operators mapping detail spaces onto detail spaces, whereas the nonstandard form consists of three types of operators:

- A_j maps detail spaces onto detail spaces,
- B_j maps approximation spaces onto detail spaces,
- C_j maps detail spaces onto approximation spaces.

In numerical schemes, one considers a finest space $\mathbf{V}_n$ and a coarsest space $\mathbf{V}_{n-J}$, where J is the depth of the multiresolution analysis, that is, we consider a "truncated" version of (40),

$$\mathbf{V}_{n-J} \subset \mathbf{V}_{n-J+1} \subset \cdots \subset \mathbf{V}_{n-1} \subset \mathbf{V}_n. \tag{49}$$

The standard form of T in (49) is then given by the set of operators

$$T_n = \left\{ A_j, \left\{ B^j_{j'} \right\}_{j'=n-J}^{j-1}, \left\{ C^j_{j'} \right\}_{j'=n-J}^{j-1}, E^j_{n-J}, F^j_{n-J}, T_{n-J} \right\}_{n-J \le j \le n-1}, \tag{50}$$

where A_j, $B^j_{j'}$, and $C^j_{j'}$ are as in (42)–(44), and

$$E^j_{n-J} = Q_j T P_{n-J} : \mathbf{V}_{n-J} \to \mathbf{W}_j,$$

$$F^j_{n-J} = P_{n-J} T Q_j : \mathbf{W}_j \to \mathbf{V}_{n-J},$$

$$T_{n-J} = P_{n-J} T P_{n-J} : \mathbf{V}_{n-J} \to \mathbf{V}_{n-J},$$

and the nonstandard form of T in (49) is given by the set of operators

$$T_n = \{\{A_j, B_j, C_j\}_{n-J\le j\le n-1}, T_{n-J}\},$$

where A_j, B_j, and C_j are as in (46)–(48), and

$$T_{n-J} = P_{n-J} T P_{n-J} : \mathbf{V}_{n-J} \to \mathbf{V}_{n-J}. \tag{51}$$

As a final note in this section, we see that the representation of a linear operator T in a finite multiresolution analysis is an expansion in the multiresolution spaces of the approximation of the operator T by $T_n = P_n T P_n$ (an operator mapping $\mathbf{V}_n$ onto itself). Depending upon how T_n is expanded down the multiresolution, we obtain either the standard or nonstandard form of the representation. In the remainder of this paper, we shall be concerned with the nonstandard form of the representation, and the construction of the matrix representation.

4. THE MATRIX REPRESENTATION

In this section, we derive the matrix representation of the general linear operator T in a finite multiresolution analysis. The results of this section will be used in the next two sections to construct the matrices representing the differential operators ∂_x, ∂_y, and, in general, $T = g(\partial_x, \partial_y)$, where g is analytic.

Recall that the nonstandard form of T in a finite multiresolution analysis is the set of operators

$$\{\{A_j, B_j, C_j\}_{n-J\le j\le n-1}, T_{n-J}\}, \tag{52}$$

where $A_j = Q_j T Q_j$, $B_j = P_j T Q_j$, $C_j = Q_j T P_j$, and $T_{n-J} = P_{n-J} T P_{n-J}$. Since $\mathbf{W}_j = \mathbf{W}_j^h \oplus \mathbf{W}_j^v \oplus \mathbf{W}_j^d$, the orthogonal projection Q_j onto $\mathbf{W}_j$ is the sum of the three orthogonal projections Q_j^h, Q_j^v, and Q_j^d onto $\mathbf{W}^h$, $\mathbf{W}^v$, and $\mathbf{W}^d$, respectively, i.e.,

$$Q_j = Q_j^h + Q_j^v + Q_j^d. \tag{53}$$

As a result of (53), the operators A_j, B_j, and C_j in (52) are given by the sums

$$A_j = \sum_{\lambda,\lambda'=h,v,d} A_j^{\lambda,\lambda'} = \sum_{\lambda,\lambda'=h,v,d} Q_j^{\lambda} T Q_j^{\lambda'}, \tag{54}$$

$$B_j = \sum_{\lambda=h,v,d} B_j^{\lambda} = \sum_{\lambda=h,v,d} Q_j^{\lambda} T P_j, \tag{55}$$

$$C_j = \sum_{\lambda=h,v,d} C_j^{\lambda} = \sum_{\lambda=h,v,d} P_j T Q_j^{\lambda}. \tag{56}$$

The nonstandard form of T is then rewritten as the set of operators

$$T_n = \left\{ \left\{ A_j^{\lambda,\lambda'}, B_j^{\lambda}, C_j^{\lambda} \right\}_{n-J\le j\le n-1;\lambda,\lambda'=h,v,d}, T_{n-J} \right\}, \tag{57}$$

where

$$\begin{aligned} A_j^{\lambda,\lambda'} &: \mathbf{W}_j^{\lambda'} :\longrightarrow \mathbf{W}_j^{\lambda}, & \lambda,\lambda' &= h,v,d,\\ B_j^{\lambda} &: \mathbf{V}_j :\longrightarrow \mathbf{W}_j^{\lambda}, & \lambda &= h,v,d,\\ C_j^{\lambda} &: \mathbf{W}_j^{\lambda} :\longrightarrow \mathbf{V}_j, & \lambda &= h,v,d. \end{aligned}$$

In the construction of the matrix representation, we use $[0,1]^2$-periodic wavelets [5] (see also [23] for more details). The subspaces $\mathbf{V}_j$ and $\mathbf{W}_j$ are finite dimensional. The dimension of $\mathbf{V}_j$ is 2^{2j} with orthonormal doubly-indexed basis

$$\left\{ \Phi^j_{k_1,k_2},\ k_1,k_2 = 0,1,2,\dots,2^j-1 \right\},$$

whereas the dimension of $\mathbf{W}_j$ is 3×2^{2j} (three times as large!) with orthonormal doubly-indexed basis

$$\left\{\Psi^{\lambda,j}_{k_1,k_2},\ \lambda = h, v, d;\ k_1, k_2 = 0, 1, 2, \ldots, 2^j - 1\right\}.$$

Each one of the operators in (57) acts on a finite dimensional subspace of $L^2(R^2)$ with doubly-indexed basis elements ($\Phi^j_{k_1,k_2}$ for $\mathbf{V}_j$ and $\Psi^j_{k_1,k_2}$ for $\mathbf{W}_j$). Of course, they can be represented by ordinary matrices if the basis with doubly-indexed elements is converted into a basis with singly-indexed elements, by reordering. However, it is best to work with the original basis, and to represent the operators by four-dimensional structures.

In order to see how the four-dimensional structures come about, consider a function f. The action of, for example, $T_j = P_jTP_j$, on f gives

$$\begin{aligned}(T_jf)(x,y) = P_jTP_j(f) &= P_jT \sum_{k_1,k_2=0}^{2^j-1} s^j_{k_1,k_2}\Phi^j_{k_1,k_2}(x,y)\\ &= \sum_{k_3,k_4=0}^{2^j-1}\left[\sum_{k_1,k_2=0}^{2^j-1}\left\langle \Phi^j_{k_3,k_4}, T\left(\Phi^j_{k_1,k_2}\right)\right\rangle s^j_{k_1,k_2}\right]\Phi^j_{k_3,k_4}(x,y)\\ &= \sum_{k_3,k_4=0}^{2^j-1} \tilde{s}^j_{k_3,k_4}\Phi^j_{k_3,k_4}(x,y),\end{aligned} \tag{58}$$

where $s^j_{k_1,k_2} = \langle f, \Phi^j_{k_1,k_2}\rangle$ are the coordinates of P_jf, the projection of f onto $\mathbf{V}_j$, and

$$\tilde{s}^j_{k_3,k_4} = \sum_{k_1,k_2=0}^{2^j-1}\left\langle \Phi^j_{k_3,k_4}, T\left(\Phi^j_{k_1,k_2}\right)\right\rangle s^j_{k_1,k_2} \tag{59}$$

are the coordinates of $T_j(f)$ in $\mathbf{V}_j$. Define the *structure* T^j to be the $2^j \times 2^j$ block matrix with matrix entries T^{j,k_3,k_4} $(k_3, k_4 = 0, 1, \ldots, 2^j - 1)$,

$$T^j = \begin{bmatrix} T^{j,0,0} & T^{j,0,1} & \ldots & T^{j,0,2^j-1}\\ T^{j,1,0} & T^{j,1,1} & \ldots & T^{j,1,2^j-1}\\ \vdots & \vdots & \ldots & \vdots\\ T^{j,2^j-1,0} & T^{j,2^j-1,1} & \ldots & T^{j,2^j-1,2^j-1}\end{bmatrix}, \tag{60}$$

with the entries of each matrix T^{j,k_3,k_4} given by

$$T^{j,k_3,k_4}_{k_1,k_2} = \left\langle \Phi^j_{k_3,k_4}, T\left(\Phi^j_{k_1,k_2}\right)\right\rangle, \qquad k_1, k_2 = 0, 1, \ldots, 2^j - 1.$$

Then, the action of T_j on f is represented by an operation of the structure T^j on the matrix $s^j = (s^j_{k_1,k_2})$, given in terms of the following.

Given a $k \times k$ block matrix Γ with $m \times m$ blocks $\Gamma^{i,j}$ and an $m \times m$ matrix A, define the operation

$$\Gamma \odot A = B,$$

where B is $k \times k$ with entries b_{ij} given by

$$b_{ij} = \sum_{k,l}\left(\Gamma^{i,j} \circ A\right)_{kl},$$

and $\Gamma^{i,j} \circ A$ is the Hadamard (element by element) product of $\Gamma^{i,j}$ and A, i.e., $(\Gamma^{i,j} \circ A)_{kl} = \Gamma^{i,j}_{kl}A_{kl}$.

In terms of the above, the matrix $\tilde{s}^j$ of $T_j(f)$ in $\mathbf{V}_j$ is given by

$$\tilde{s}^j = T^j \odot s^j.$$

All the operators $A_j^{\lambda,\lambda'}$, B_j^{λ}, C_j^{λ}, and T_j are represented by structures $A^{j,\lambda,\lambda'}$, $B^{j,\lambda}$, $C^{j,\lambda}$, and T^j, the entries of whose blocks are given by

$$A_{k_1,k_2}^{j,\lambda,\lambda',k_3,k_4} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_{k_3,k_4}^{\lambda,j}(x,y) T\left(\Psi_{k_1,k_2}^{\lambda',j}\right)(x,y)\,dx\,dy, \tag{61}$$

$$B_{k_1,k_2}^{j,\lambda,k_3,k_4} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_{k_3,k_4}^{\lambda,j}(x,y) T\left(\Phi_{k_1,k_2}^{j}\right)(x,y)\,dx\,dy, \tag{62}$$

$$C_{k_1,k_2}^{j,\lambda,k_3,k_4} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{k_3,k_4}^{j}(x,y) T\left(\Psi_{k_1,k_2}^{\lambda,j}\right)(x,y)\,dx\,dy, \tag{63}$$

$$T_{k_1,k_2}^{j,k_3,k_4} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{k_3,k_4}^{j}(x,y) T\left(\Phi_{k_1,k_2}^{j}\right)(x,y)\,dx\,dy. \tag{64}$$

We remark that, because the wavelets are periodic, the structures discussed above are considered to be periodic. More precisely, if S is an $m\times m$ structure with $m\times m$ blocks $P^{i,j}$ $(0\le i,j\le m-1)$, then S is said to be m-periodic if $P^{i+m,j}=P^{i,j}$, $P^{i,j+m}=P^{i,j}$, $P^{i,j}_{k_1+m,k_2}=P^{i,j}_{k_1,k_2}$, and $P^{i,j}_{k_1,k_2+m}=P^{i,j}_{k_1,k_2}$ for all i, j, k_1, k_2 $(0\le i,j,\,k_1,k_2\le m-1)$.

Using equations (28) and (29), we find that all the structures $A^{j,\lambda,\lambda'}$, $B^{j,\lambda}$, $C^{j,\lambda}$, and T^j $(n-J\le j\le n-1)$ are obtained from the structure T^n recursively by means of the formulae

$$A_{k_1,k_2}^{j,\lambda,\lambda',k_3,k_4} = \sum_{m_1,m_2,m_3,m_4=0}^{L-1} G_{m_1,m_2}^{\lambda'} G_{m_3,m_4}^{\lambda} T_{m_1+2k_1,m_2+2k_2}^{j+1,m_3+2k_3,m_4+2k_4}, \tag{65}$$

$$B_{k_1,k_2}^{j,\lambda,k_3,k_4} = \sum_{m_1,m_2,m_3,m_4=0}^{L-1} H_{m_1,m_2} G_{m_3,m_4}^{\lambda} T_{m_1+2k_1,m_2+2k_2}^{j+1,m_3+2k_3,m_4+2k_4}, \tag{66}$$

$$C_{k_1,k_2}^{j,\lambda,k_3,k_4} = \sum_{m_1,m_2,m_3,m_4=0}^{L-1} G_{m_1,m_2}^{\lambda} H_{m_3,m_4} T_{m_1+2k_1,m_2+2k_2}^{j+1,m_3+2k_3,m_4+2k_4}, \tag{67}$$

$$T_{k_1,k_2}^{j,k_3,k_4} = \sum_{m_1,m_2,m_3,m_4=0}^{L-1} H_{m_1,m_2} H_{m_3,m_4} T_{m_1+2k_1,m_2+2k_2}^{j+1,m_3+2k_3,m_4+2k_4}. \tag{68}$$

Therefore, if we have the structure of T_n in $\mathbf{V}_n$, all the lower scale structures are determined.

The application of an operator T to a function f is approximated by the sum of the applications of the operators $A_j^{\lambda,\lambda'}$, B_j^{λ}, C_j^{λ}, $n-J\le j\le n-1$, $\lambda,\lambda'=h,v,d$, and T_{n-J} (or, equivalently, by the application of the operator T_n to f). First, the function is approximated by its projection P_nf onto $\mathbf{V}_n$ to obtain the coordinates of P_nf in $\mathbf{V}_n$ as a $2^n\times 2^n$ matrix s^n. The matrix s^n is decomposed down the multiresolution spaces, using equations (34) and (35), recursively, to arrive at the matrices $d^{h,j}$, $d^{v,j}$, $d^{d,j}$, and s^j for $j=n-J,\dots,n-1$. Finally, the structures $A^{j,\lambda,\lambda'}$, $B^{j,\lambda}$, $C^{j,\lambda}$, $n-J\le j\le n-1$, and T^{n-J} are constructed. The approximation $T(f)\simeq T_n(f)$ is then given by

$$T_n(f)(x,y) = \sum_{j=n-J}^{n-1}\left[\sum_{\lambda=h,v,d}\left(\sum_{k_3,k_4=0}^{2^j-1} \tilde{d}_{k_3,k_4}^{\lambda,j}\Psi_{k_3,k_4}^{\lambda,j}(x,y)\right) + \sum_{k_3,k_4=0}^{2^j-1}\tilde{s}^j\Phi_{k_3,k_4}^{j}(x,y)\right], \tag{69}$$

where $\tilde{d}^{\lambda,j}$, $\lambda=h,v,d$, and $\tilde{s}^j$ are given by

$$\begin{aligned}
\tilde{d}^{h,j} &= A^{j,h,h}\odot d^{h,j} + A^{j,v,h}\odot d^{v,j} + A^{j,d,h}\odot d^{d,j} + B^{j,h}\odot s^j,\\
\tilde{d}^{v,j} &= A^{j,h,v}\odot d^{h,j} + A^{j,v,v}\odot d^{v,j} + A^{j,d,v}\odot d^{d,j} + B^{j,v}\odot s^j,\\
\tilde{d}^{d,j} &= A^{j,h,d}\odot d^{h,j} + A^{j,v,d}\odot d^{v,j} + A^{j,d,d}\odot d^{d,j} + B^{j,d}\odot s^j,\\
\tilde{s}^{j} &= C^{j,h}\odot d^{h,j} + C^{j,v}\odot d^{v,j} + C^{j,d}\odot d^{d,j}, \qquad \text{for } n-J+1\le j\le n-1,\\
\tilde{s}^{n-J} &= C^{n-J,h}\odot d^{h,n-J} + C^{n-J,v}\odot d^{v,n-J} + C^{n-J,d}\odot d^{d,n-J} + T^{n-J}\odot s^{n-J}.
\end{aligned}$$

The function $\tilde{f} = T_n(f)$ in (69) can then be reconstructed back to the finest space $\mathbf{V}_n$, i.e., expressed as

$$\tilde{f}(x,y) = \sum_{k_1=0}^{2^n-1} \sum_{k_2=0}^{2^n-1} s^n_{k_1,k_2} \Phi^n_{k_1,k_2}(x,y). \tag{70}$$

This is accomplished by using the inverse of the decomposition procedure to obtain the coordinate matrix $s^n = (s^n_{k_1,k_2})$. The procedure of constructing s^n is as follows. Given the matrices $\tilde{d}^{j,\lambda}$ and $\tilde{s}^j$, for $n-J \le j \le n-1$, $\lambda = h, v, d$, we start at the coarsest scale $(n-J)$ and reconstruct $\tilde{d}^{\lambda,n-J}$ $(\lambda = h,v,d)$ and $\tilde{s}^{n-J}$ into a matrix $\hat{s}^{n-J+1}$, using (36)–(39), and form the sum $s^{n-J+1} = \hat{s}^{n-J+1} + \tilde{s}^{n-J+1}$. Then, at each scale $j = n-J+1, n-J+2, \dots, n-1$, we reconstruct $\tilde{d}^{\lambda,j}$ and $s^j (= \tilde{s}^j + \hat{s}^j)$ into $\hat{s}^{j+1}$ and form the sum $s^{j+1} = \hat{s}^{j+1} + \tilde{s}^{j+1}$. The final reconstruction (at scale $n-1$) of s^{n-1} and $\tilde{d}^{\lambda,n-1}$ gives the coordinate matrix s^n in (70).

From equations (65)–(68), we see that the structures of $A^{j,\lambda,\lambda'}$, $B^{j,\lambda}$, $C^{j,\lambda}$, and T^j for $n-J \le j \le n-1$, and hence the matrix representation of the nonstandard form of a linear operator T, are completely determined by the construction of the structure T^n, the matrix representation of T_n in $\mathbf{V}_n$. We therefore turn to the construction of T^n for the cases $T = \partial_x$, $T = \partial_y$, and $T = g(\partial_x, \partial_y)$. In Section 5, we consider ∂_x and ∂_y and then, in Section 6, we consider the general differential operator $L = g(\partial_x, \partial_y)$.

5. THE MATRIX REPRESENTATION OF THE NS-FORMS OF ∂_x AND ∂_y

For $T = \partial_x$, equation (64) with $j = n$ gives

$$\begin{aligned} T^{n,k_3,k_4}_{k_1,k_2} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi^n_{k_3,k_4}(x,y) \frac{\partial}{\partial x} \left[\Phi^n_{k_1,k_2}(x,y)\right] dx\, dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{n,k_3}(x)\phi_{n,k_4}(y) \frac{\partial}{\partial x} [\phi_{n,k_1}(x)\phi_{n,k_2}(y)]\, dx\, dy \\ &= \left[\int_{-\infty}^{\infty} \phi_{n,k_3}(x) \frac{\partial}{\partial x}(\phi_{n,k_1}(x))\, dx\right] \left[\int_{-\infty}^{\infty} \phi_{n,k_2}(y)\phi_{n,k_4}(y)\, dy\right]. \end{aligned}$$

Since $\phi_{n,k}$ is an orthonormal basis,

$$\int_{-\infty}^{\infty} \phi_{n,k_2}(y)\phi_{n,k_4}(y)\, dy = \delta_{k_2,k_4}.$$

As for the x-integral, we have

$$\int_{-\infty}^{\infty} \phi_{n,k_3}(x) \frac{\partial}{\partial x}(\phi_{n,k_1}(x))\, dx = 2^n r_{k_3-k_1},$$

where

$$r_l = \int_{-\infty}^{\infty} \phi(x-l)\phi'(x)\, dx. \tag{71}$$

Thus, $T^{n,k_3,k_4}_{k_1,k_2} = 2^n r_{k_3-k_1} \delta_{k_2,k_4}$, and therefore, the block matrix T^n is completely determined by the coefficients r_l for which we have the following proposition (see [18]).

PROPOSITION 1. *If integral (71) exists, then the coefficients r_l satisfy the following system of linear equations:*

$$r_l = 2r_{2l} + \sum_{k=1}^{L/2} a_{2k-1}(r_{2l-2k+1} + r_{2l+2k-1}), \tag{72}$$

with

$$\sum l r_l = -1, \tag{73}$$

where the coefficients a_n (the autocorrelation of the filter coefficients h_k) are given by

$$a_n = 2 \sum_{i=0}^{L-1-n} h_i h_{i+1}, \qquad n = 1, 2, \ldots, L-1. \tag{74}$$

The proof of this proposition can be found in [18]. It is also proved in [18] that if the number of vanishing moments M of the mother wavelet $\psi(x)$ satisfies $M \geq 2$, (72) and (73) have a unique solution with $r_l \neq 0$ for $-L+2 \leq l \leq L-2$, and $r_{-l} = -r_l$. Note that the range of the nonzero r_l is easy to see, since the scaling function $\phi(x)$ is supported in $[0, L-1]$. For an efficient method of solution for the coefficients r_l, see [23].

With a minor change in notation in (60), the block matrix T_x^n ($= T^n$ for $T = \partial_x$) becomes

$$T_x^n = \begin{bmatrix} X^{1,1} & X^{1,2} & \ldots & X^{1,2^n-1} & X^{1,2^n} \\ X^{2,1} & X^{2,2} & \ldots & X^{2,2^n-1} & X^{2,2^n} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ X^{2^n-1,1} & X^{2^n-1,2} & \ldots & X^{2^n-1,2^n-1} & X^{2^n-1,2^n} \\ X^{2^n,1} & X^{2^n,2} & \ldots & X^{2^n,2^n-1} & X^{2^n,2^n} \end{bmatrix}, \tag{75}$$

where $X_{k_1,k_2}^{k_3,k_4} = T_{k_1,k_2}^{n,k_3,k_4} = 2^n r_{k_3-k_1} \delta_{k_2,k_4}$. It then follows that each X^{k_3,k_4} contains only one nonzero column, the k_4^{th} column. Explicitly,

$$X_{k_1,k_2}^{k_3,k_4} = \begin{cases} 2^n r_{k_3,k_1}, & \text{if } k_2 = k_4, \\ 0, & \text{otherwise.} \end{cases}$$

The nonzero column of $X^{1,1}$ is given by

$$c = 2^n \left(0, r_{-1}, r_{-2}, \ldots, r_{-(L-2)}, 0, \ldots, 0, r_{L-2}, \ldots, r_1\right)^t,$$

from which all the matrices X^{k_3,k_4} are obtained by means of the formulae

$$\begin{aligned} X^{k_3,k_4+1} &= \text{FSRWR}\left(X^{k_3,k_4}\right), \qquad k_4 = 1, 2, \ldots, 2^n - 1, \\ X^{k_3+1,k_4} &= \text{FSCWR}\left(X^{k_3,k_4}\right), \qquad k_3 = 1, 2, \ldots, 2^n - 1, \end{aligned}$$

where FSRWR and FSCWR stand for forward-shift-row-wraparound and forward-shift-column-wraparound, defined by the following: for

$$A = \begin{pmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{pmatrix}, \qquad \text{FSRWR}(A) = \begin{pmatrix} \vec{c}_n & \vec{c}_1 & \cdots & \vec{c}_{n-1} \end{pmatrix},$$

and for

$$A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_n \end{bmatrix}, \qquad \text{FSCWR}(A) = \begin{bmatrix} \vec{r}_n \\ \vec{r}_1 \\ \vdots \\ \vec{r}_{n-1} \end{bmatrix},$$

where $\vec{c}_i$ is the i^{th} column of A, and $\vec{r}_j$ is the j^{th} row.

For $T = \partial_y$, we obtain, in an analogous fashion,

$$T_{k_1,k_2}^{n,k_3,k_4} = 2^n r_{k_4-k_2} \delta_{k_1,k_3}.$$

As in (75), we obtain

$$T_y^n = \begin{bmatrix} Y^{1,1} & Y^{1,2} & \ldots & Y^{1,2^n-1} & Y^{1,2^n} \\ Y^{2,1} & Y^{2,2} & \ldots & Y^{2,2^n-1} & Y^{2,2^n} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ Y^{2^n-1,1} & Y^{2^n-1,2} & \ldots & Y^{2^n-1,2^n-1} & Y^{2^n-1,2^n} \\ Y^{2^n,1} & Y^{2^n,2} & \ldots & Y^{2^n,2^n-1} & Y^{2^n,2^n} \end{bmatrix}, \tag{76}$$

where each Y^{k_3,k_4} is a $2^n \times 2^n$ matrix equal to the transpose of X^{k_3,k_4} in (75). Thus, each Y^{k_3,k_4} contains only one nonzero row (the k_3^{th} row)

$$Y^{k_3,k_4}_{k_1,k_2} = \begin{cases} 2^n r_{k_4-k_2}, & \text{if } k_1 = k_3, \\ 0, & \text{otherwise.} \end{cases}$$

The nonzero row of $Y^{1,1}$ is

$$r = 2^n \left(0, r_{-1}, r_{-2}, \dots, r_{-(L-2)}, 0, \dots, 0, r_{L-2}, \dots, r_1\right),$$

from which all the matrices Y^{k_3,k_4} are obtained by means of the formulae

$$\begin{aligned} Y^{k_3,k_4+1} &= \mathrm{FSRWR}\left(Y^{k_3,k_4}\right), & k_4 &= 1, 2, \dots, 2^n - 1, \\ Y^{k_3+1,k_4} &= \mathrm{FSCWR}\left(Y^{k_3,k_4}\right), & k_3 &= 1, 2, \dots, 2^n - 1. \end{aligned}$$

6. THE MATRIX REPRESENTATION OF THE NS-FORM OF $g(\partial_x, \partial_y)$

We now consider the general case where $T = g(\partial_x, \partial_y)$. In order to obtain the matrix representation of $T = g(\partial_x, \partial_y)$ in $\mathbf{V}_n$, we represent (approximate) the operator $g(\partial_x, \partial_y)$ in $\mathbf{V}_n$ in such a way that its matrix representation can be obtained from the matrix representations of $P_n \partial_x P_n$ and $P_n \partial_y P_n$.

We should mention that there are two approaches to representing $T = g(\partial_x, \partial_y)$ in $\mathbf{V}_n$:

(i) by computing the projection of (∂_x, ∂_y) onto $\mathbf{V}_n$,

$$T_n = P_n g(\partial_x, \partial_y) P_n, \tag{77}$$

or, in contrast,

(ii) by computing the function of the projections of ∂_x and ∂_y onto $\mathbf{V}_n$,

$$T_n = g(T_{n,x}, T_{n,y}), \tag{78}$$

where $T_{n,x} = P_n \partial_x P_n$ and $T_{n,y} = P_n \partial_y P_n$.

Thus, T_n in (78) is no longer a projection, but a different representation of $T = g(\partial_x, \partial_y)$ in $\mathbf{V}_n$. The difference between these two approaches depends upon $|\hat{\phi}(\xi)|^2$, the magnitude of the Fourier transform of the scaling function ϕ (see [20]). We adopt the second approach and represent $T = g(\partial_x, \partial_y)$ in $\mathbf{V}_n$ by (78).

If we denote by X and Y the matrix representations of $P_n \partial_x P_n$ and $P_n \partial_y P_n$, respectively, then the matrix representation, T^n, of $T_n = g(\partial_x, \partial_y)$ is obtained by applying the function g to X and Y, i.e.,

$$T^n = g(X, Y).$$

Now, if we simultaneously diagonalize X and Y and write

$$X = P^{-1} D_x P, \qquad Y = P^{-1} D_y P,$$

where P is a diagonalizing matrix for both X and Y and D_x and D_y are diagonal matrices containing the eigenvalues of X and Y, respectively, then, since g is analytic, we obtain

$$T_n = g(X, Y) = P^{-1} g(D_x, D_y) P.$$

Therefore, the problem reduces to simultaneously diagonalizing X and Y.

To this end, it is necessary to restructure the structures T_x^n and T_y^n into ordinary matrices X and Y, respectively, and to express the $\odot$ operation, described in Section 4, in terms of ordinary matrix multiplication. Express $s^n = (s_{ij})$ as the column vector

$$\vec{s} = \begin{pmatrix} \vec{r}_1^{\,t} \\ \vdots \\ \vec{r}_{2^n}^{\,t} \end{pmatrix},$$

where $\vec{r}_i$ is the i^{th} row of s^n. From the matrix $T_x^n = (X^{i,j})$, $1 \le i, j \le 2^n$, construct the new $2^{2n} \times 2^{2n}$ matrix X as follows. Let $\vec{r}_k^{\,i,j}$ denote the k^{th} row of the matrix $X^{i,j}$. Then, the first row of X consists of

$$(\vec{r}_1^{\,1,1} \quad \vec{r}_2^{\,1,1} \quad \cdots \quad \vec{r}_{2^n}^{\,1,1}),$$

the second row consists of

$$(\vec{r}_1^{\,1,2} \quad \vec{r}_2^{\,1,2} \quad \cdots \quad \vec{r}_{2^n}^{\,1,2}),$$

the $(2^n+1)^{\text{th}}$ row is

$$(\vec{r}_1^{\,2,1} \quad \vec{r}_2^{\,2,1} \quad \cdots \quad \vec{r}_{2^n}^{\,2,1}),$$

and so on, with the last row of X given by

$$(\vec{r}_1^{\,2^n,2^n} \quad \vec{r}_2^{\,2^n,2^n} \quad \cdots \quad \vec{r}_{2^n}^{\,2^n,2^n}).$$

In other words, the matrix X is formed by laying out the rows of $X^{1,1}$ next to one another to compose the first row of X, laying out the rows of $X^{1,2}$ next to one another to compose the second row, and so on. The result of such a restructuring is that the action of T_x^n on s^n is now given simply by the matrix-vector multiplication

$$T_x^n \odot s^n \sim X\vec{s}.$$

Similarly,

$$T_y^n \odot s^n \sim Y\vec{s},$$

where Y is obtained from T_y^n via the same procedure. The symbol $\sim$ indicates that the operation on the left-hand side is equivalent to the one on the right, the vector on the right being a restructuring of the matrix on the left.

At this point, we need to recall a few definitions.

DEFINITION 1. *A square matrix $n \times n$ is circulant if it is of the form*

$$C = \operatorname{circ}(c_1, c_2, \dots, c_n) = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & & \vdots \\ c_2 & c_3 & \cdots & c_1 \end{bmatrix},$$

that is,

$$C_{i+1,j+1} = C_{i,j}, \quad C_{i,1} = C_{i-1,n}, \qquad \text{for } i = 2, 3, \dots, n.$$

DEFINITION 2. *The Fourier matrix F of order N (or of size $(N \times N)$) is the matrix with entries*

$$F_{k,l} = \frac{1}{\sqrt{N}} W_N^{-(k-1)(l-1)}, \qquad 1 \le k, l \le N,$$

where

$$W_N = e^{2\pi i/N}.$$

The inverse Fourier matrix F^* of order N is the adjoint (the conjugate transpose) of $F(FF^* = F^*F = I)$, with entries

$$F^*_{k,l} = \frac{1}{\sqrt{N}} W_N^{(k-1)(l-1)}, \qquad 1 \le k, l \le N.$$

We have the following proposition concerning the eigenvalues of a circulant matrix.

PROPOSITION 2. *Let $C = \operatorname{circ}(c_1, c_2, \ldots, c_N)$ be a circulant matrix. Then, the eigenvalues of C are given explicitly by*

$$\lambda_k = \sum_{l=1}^{N} c_l e^{2\pi i(k-1)(l-1)/N}, \qquad 1 \le k \le N. \tag{79}$$

REMARK 1. The row vector $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ containing the eigenvalues of a circulant matrix $C = \operatorname{circ}(c_1, c_2, \ldots, c_N)$ is given by the vector-matrix product

$$(\lambda_1, \lambda_2, \ldots, \lambda_N) = \sqrt{N}(c_1, c_2, \ldots, c_N)F^*,$$

where F^* is the inverse Fourier matrix of order N.

It is now necessary to diagonalize X and Y simultaneously. Since, the operators ∂_x and ∂_y commute, their representations X and Y ought to be commuting matrices, and this can be deduced from the structures of X and Y, to which we now turn. The matrix X turns out to have the following structure:

$$X = \begin{bmatrix} \bar{X}^{1,1} & \bar{X}^{1,2} & \cdots & \bar{X}^{1,2^n} \\ \vdots & \vdots & & \vdots \\ \bar{X}^{2^n,1} & \bar{X}^{2^n,2} & \cdots & \bar{X}^{2^n,2^n} \end{bmatrix},$$

where each $\bar{X}^{i,j}$ is circulant and X is block circulant, i.e., $X = \operatorname{bccb}(\bar{X}^{1,1} \cdots \bar{X}^{1,2^n})$, where "bccb" stands for block circulant with circulant blocks (see, e.g., [24]). The matrix Y has the same structure, $Y = (\bar{Y}^{i,j})$, with each $\bar{Y}^{i,j}$ circulant and Y block circulant, with $Y = \operatorname{bccb}(\bar{Y}^{1,1} \cdots \bar{Y}^{1,2^n})$. Since every $\bar{X}^{i,j}$ commutes with every $\bar{Y}^{k,l}$ (being circulant matrices of the same size), it follows that X and Y commute [24, p. 128], as expected, and are therefore simultaneously diagonalizable.

To this end, we employ the following.

DEFINITION 3. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ and $p \times q$, respectively. Then, the Kronecker product (or tensor, or direct) product of A and B is the $mn \times pq$ matrix C defined by*

$$A \otimes B = C = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

A bccb matrix M is said to be of *type* (m, n) if M is $m \times m$ and the blocks are $n \times n$. In terms of these notions, we have (see, e.g., [24, p. 128]) the following result.

THEOREM 1. *All* bccb *matrices of type (m, n) are simultaneously diagonalizable by the unitary matrix $F_m \otimes F_n$, where F_n is the Fourier matrix of order n. If the eigenvalues of the circulant blocks are given by the diagonal matrices Λ_k, $k = 1, 2, \ldots, m$, the $mn \times mn$ diagonal matrix Λ of the eigenvalues of the* bccb *matrix is given by*

$$\Lambda = \sum_{k=1}^{m} \Omega_m^{k-1} \otimes \Lambda_k, \tag{80}$$

where Ω_m is the $m \times m$ diagonal matrix given by

$$\Omega_m = \operatorname{diag}\left(1, W_m, W_m^2, \ldots, W_m^{m-1}\right), \qquad W_m = e^{2\pi i/m}.$$

Conversely, any matrix of the form

$$A = (F_m^* \otimes F_n^*)\,\Lambda(F_m \otimes F_n),$$

where Λ is diagonal, is bccb.

We employ the above theorem to diagonalize $X = (\bar{X}^{i,j})$ and $Y = (\bar{Y}^{i,j})$ simultaneously. Let $\Lambda^{x,k}$, $\Lambda^{y,k}$ be the diagonal matrices containing the eigenvalues of $\bar{X}^{1,k}$ and $\bar{Y}^{1,k}$, respectively, for $k = 1, 2, \ldots, 2^n$. Then, we have

$$X = (F_{2^n}^* \otimes F_{2^n}^*)\,\Lambda^x\,(F_{2^n} \otimes F_{2^n}), \qquad \Lambda^x = \sum_{k=1}^{2^n} \Omega_{2^n}^{k-1} \otimes \Lambda^{x,k}, \tag{81}$$

$$Y = (F_{2^n}^* \otimes F_{2^n}^*)\,\Lambda^y\,(F_{2^n} \otimes F_{2^n}), \qquad \Lambda^y = \sum_{k=1}^{2^n} \Omega_{2^n}^{k-1} \otimes \Lambda^{y,k}. \tag{82}$$

It follows that the matrix representation T^n (in bccb form) for $T = g(\partial_x, \partial_y)$ is given by

$$T^n = (F_{2^n}^* \otimes F_{2^n}^*)\, g(\Lambda^x, \Lambda^y)\,(F_{2^n} \otimes F_{2^n}), \tag{83}$$

where $g(\Lambda^x, \Lambda^y) \equiv \Lambda$ is the $2^{2n} \times 2^{2n}$ diagonal matrix with entries given by

$$\Lambda_{k,k} = g\left(\Lambda^x_{k,k}, \Lambda^y_{k,k}\right).$$

By Theorem 1, T^n is block circulant with circulant blocks, so it is completely determined by its first row. Let

$$T_1^n = (\vec{r}_1 \quad \vec{r}_2 \quad \cdots \quad \vec{r}_{2^n})$$

be the first row of T^n, where each $\vec{r}_k$ is a row vector of length 2^n. Let $\tilde{\Lambda}$ be the $2^n \times 2^n$ matrix containing the eigenvalues $\Lambda_{k,k}$, $k = 1, \ldots, 2^{2n}$,

$$\tilde{\Lambda} = \begin{bmatrix} \tilde{\Lambda}^1 \\ \tilde{\Lambda}^2 \\ \vdots \\ \tilde{\Lambda}^{2^n} \end{bmatrix}, \tag{84}$$

with

$$\tilde{\Lambda}^k = (\Lambda_{j+1,j+1}, \Lambda_{j+2,j+2}, \ldots, \Lambda_{j+2^n,j+2^n}), \qquad j = (k-1)2^n.$$

By equation (83), we have

$$\vec{r}_k = \frac{1}{2^n\sqrt{2^n}} \sum_{m=1}^{2^n} \tilde{\Lambda}^m \left(W_{2^n}^{-(m-1)(k-1)} F_{2^n}\right). \tag{85}$$

Next, we wish to place the matrix T^n (presently in bccb form) into a structure form so that it acts by the operation $\odot$ defined earlier. This is accomplished by the exact inverse of the procedure, described earlier, for obtaining the bccb form. Then, the structure T^n, that is, the representation of $T_n = g(\partial_x, \partial_y)$ in $\mathbf{V}_n$, has the same form as those for ∂_x and ∂_y, with $T^{n,k_3,k_4+1} = \mathrm{FSRWR}\,(T^{n,k_3,k_4})$ for $1 \le k_4 \le 2^n - 1$ and $T^{n,k_3+1,k_4} = \mathrm{FSCWR}\,(T^{n,k_3,k_4})$ for $1 \le k_3 \le 2^n - 1$. Therefore, all the matrices T^{n,k_3,k_4} are obtained from $T^{n,1,1}$ by applications of the functions FSRWR and FSCWR. Finally, the matrix $T^{n,1,1}$ is given by

$$T^{n,1,1} = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_{2^n} \end{bmatrix}, \tag{86}$$

where the $\vec{r}_k$ are as in (85). From equation (85), we have

$$\vec{r}_k = \frac{1}{2^n} \frac{1}{\sqrt{2^n}} \left(1, W_{2^n}^{-(k-1)}, W_{2^n}^{-2(k-1)}, \ldots, W_{2^n}^{-(2^n-1)(k-1)}\right) \begin{bmatrix} \tilde{\Lambda}^1 \\ \tilde{\Lambda}^2 \\ \vdots \\ \tilde{\Lambda}^{2^n} \end{bmatrix} F_{2^n}.$$

Since

$$\frac{1}{\sqrt{2^n}}\left(1, W_{2^n}^{-(k-1)}, W_{2^n}^{-2(k-1)}, \ldots, W_{2^n}^{-(2^n-1)(k-1)}\right)$$

is the k^{th} row of the Fourier matrix F_{2^n} (of order 2^n), it follows that

$$T^{n,1,1} = \frac{1}{2^n} F_{2^n} \tilde{\Lambda} F_{2^n}.$$

The first block, $T^{n,1,1}$, of the structure T^n for $T = g(\partial_x, \partial_y)$ can be expressed in terms of the first blocks, $T_x^{n,1,1}$ and $T_y^{n,1,1}$, of the structures T_x^n and T_y^n for $T = \partial_x$, and $T = \partial_y$, respectively. To this end, we proceed as follows. Reshape the $2^{2n} \times 2^{2n}$ diagonal matrices Λ^x and Λ^y (in equations (81) and (82)) into two $2^n \times 2^n$ matrices $\tilde{\Lambda}^x$ and $\tilde{\Lambda}^y$, by the same procedure for obtaining the matrix $\tilde{\Lambda}$ in (84) from the diagonal matrix $\Lambda = g(\Lambda_x, \Lambda_y)$. A careful examination of the expressions for Λ^x and Λ^y shows that

$$\tilde{\Lambda}^x = \sqrt{2^n} F_{2^n}^* \begin{bmatrix} \Lambda^{x,1} \\ \Lambda^{x,2} \\ \vdots \\ \Lambda^{x,2^n} \end{bmatrix}, \qquad \tilde{\Lambda}^y = \sqrt{2^n} F_{2^n}^* \begin{bmatrix} \Lambda^{y,1} \\ \Lambda^{y,2} \\ \vdots \\ \Lambda^{y,2^n} \end{bmatrix},$$

where $\Lambda^{x,k}$ and $\Lambda^{y,k}$ are now row vectors containing the eigenvalues of the circulant blocks $\bar{X}^{1,k}$ and $\bar{Y}^{1,k}$, respectively. Each of the rows $\Lambda^{x,k}$ and $\Lambda^{y,k}$ is given by the product of the first rows of the blocks $\bar{X}^{1,k}$ and $\bar{Y}^{1,k}$ and the matrix $\sqrt{2^n} F_{2^n}^*$, respectively (see Remark 1). Since the first row of $\bar{X}^{1,k}$ is the k^{th} row of $T_x^{n,1,1}$ and the first row of $\bar{Y}^{1,k}$ is the k^{th} row of $T_y^{n,1,1}$, it follows that

$$\begin{aligned} \tilde{\Lambda}^x &= \sqrt{2^n} F_{2^n}^* T_x^{n,1,1} \sqrt{2^n} F_{2^n}^* = 2^n F_{2^n}^* T_x^{n,1,1} F_{2^n}^*, \\ \tilde{\Lambda}^y &= \sqrt{2^n} F_{2^n}^* T_y^{n,1,1} \sqrt{2^n} F_{2^n}^* = 2^n F_{2^n}^* T_y^{n,1,1} F_{2^n}^*. \end{aligned}$$

The matrix $\tilde{\Lambda}$ in (84) is therefore given by

$$\tilde{\Lambda} = g\left(\tilde{\Lambda}^x, \tilde{\Lambda}^y\right),$$

where g is applied element-wise. Finally, the matrix $T^{n,1,1}$ for $g(\partial_x, \partial_y)$ can be expressed in terms of $T_x^{n,1,1}$ and $T_y^{n,1,1}$ as follows:

$$T^{n,1,1} = \frac{1}{2^n} F_{2^n} g\left(2^n F_{2^n}^* T_x^{n,1,1} F_{2^n}^*, 2^n F_{2^n}^* T_y^{n,1,1} F_{2^n}^*\right) F_{2^n}. \tag{87}$$

Note that if $g(\partial_x, \partial_y) = \partial_x$, i.e., $T = \partial_x$, then the above formula gives

$$T^{n,1,1} = \frac{1}{2^n} F_{2^n}\left(2^n F_{2^n}^* T_x^{n,1,1} F_{2^n}^*\right) F_{2^n} = I T_x^{n,1,1} I = T_x^{n,1,1},$$

as expected. Similarly, if $g(\partial_x, \partial_y) = \partial_y$, i.e., $T = \partial_y$, equation (87) gives $T^{n,1,1} = T_y^{n,1,1}$, as expected.

REMARK 2. We remark that, by construction, the matrix $T^{n,1,1}$ for $g(\partial_x, \partial_y)$ is real. Therefore, one should take the real part of the result to suppress any unnecessary imaginary parts due to numerical fluctuations.

There are special cases of the function $g(\partial_x, \partial_y)$ worth considering. First, we note that if g is a function of ∂_x only, i.e., $g(\partial_x, \partial_y) = g(\partial_x)$, then the first block, $T^{n,1,1}$, given by (87), has only the first column as nonzero. Similarly, if g is a function of ∂_y only, i.e., $g(\partial_x, \partial_y) = g(\partial_y)$, then the first block, $T^{n,1,1}$, has only the first row as nonzero. Also, the first blocks of all the structures

$A^{j,\lambda,\lambda'}$, $B^{j,\lambda}$, $C^{j,\lambda}$, and T^j, for $n-J \leq j \leq n-1$ and $\lambda = h, v, d$, will have only the first column or the first row as nonzero.

As a consequence of the above, it follows that in cases $T = g(\partial_x)$ and $T = g(\partial_y)$, the action of the structure T^n of the operators $T = g(\partial_x)$ or $T = g(\partial_y)$, defined by

$$\tilde{s} = T^n \odot s^n,$$

is equivalent to normal matrix multiplication, that is,

$$\tilde{s} = Xs, \qquad \text{for } T = g(\partial_x), \tag{88}$$

$$\tilde{s} = sY, \qquad \text{for } T = g(\partial_y), \tag{89}$$

where X is a circulant matrix whose first row is the transpose of the first column of $T^{n,1,1}$ for $T = g(\partial_x)$, and Y is also a circulant matrix whose first column is the transpose of the first row of $T^{n,1,1}$ for $T = g(\partial_y)$. Note that X multiplies s on the left and Y multiplies s on the right.

Another special case is when the function $g(\partial_x, \partial_y)$ is separable, that is, $g(\partial_x, \partial_y) = g_1(\partial_x) \times g_2(\partial_y)$. In such cases the first block, $T^{n,1,1}$, of the structure representation, T^n, of the operator $T = g(\partial_x, \partial_y)$ is the product of the first block of T_x^n and the first block of T_y^n, where T_x^n and T_y^n are the structure representations of the operators $g_1(\partial_x)$ and $g_2(\partial_y)$, respectively. More precisely, if $X^{1,1}$ and $Y^{1,1}$ are the first blocks of the structures T_x^n and T_y^n, respectively, then

$$T^{n,1,1} = X^{1,1} Y^{1,1}.$$

But, since $X^{1,1}$ has only the first column as nonzero and $Y^{1,1}$ has only the first row as nonzero, the action of the structure T^n of the operator $T = g_1(\partial_x) g_2(\partial_y)$ on a coordinate matrix s^n is equivalent to an ordinary matrix multiplication,

$$\tilde{s} = T^n \odot s^n = XsY,$$

where X is a circulant matrix whose first row is the transpose of the first column of $X^{1,1}$ and Y is also a circulant matrix whose first column is the transpose of the first row of $Y^{1,1}$.

The above observations apply to the various structures $A^{j,\lambda,\lambda'}$, $B^{j,\lambda}$, $C^{j,\lambda}$, and T^j, for $n-J \leq j \leq n-1$ and $\lambda = h, v, d$.

The importance of the representation of differential operators in a wavelet multiresolution analysis is that, for some differential operators, although the matrix representation on the finest space $\mathbf{V}_n$ may be dense, the lower scale matrices are sparse in the sense that many entries will be less than a certain threshold ε. In addition, the wavelet representation of a function will have detail coefficients $d^{\lambda,j}$ smaller than ε where the function is smooth and significant coefficients in regions where the function has a large gradient. This way, only the significant detail coefficients of the function and significant entries in the matrices are used in the computation.

As an example, consider the differential operator

$$T = g(\partial_x, \partial_y) = e^{\Delta t(\partial_x^2 + \partial_y^2)},$$

where Δt is the time step in the numerical solution of evolution partial-differential equations. The structure representation, T^n, of T in the finest space $\mathbf{V}_n$ is completely determined by the first block (top left), $T^{n,1,1}$. As mentioned earlier, T^n may be dense, but the representation (decomposition) of T^n down the multiresolution spaces gives rise to sparse structures. Figure 1 shows the entries of $T^{n,1,1}$ which are bigger than ε, Figures 2 and 3 show the entries of the first block of the lower scale structures, $A^{j,\lambda,\lambda'}$, $B^{j,\lambda}$, $C^{j,\lambda}$, whose entries are bigger than ε, and Figure 4 shows the first blocks of the structures $A^{n-J,\lambda,\lambda'}$, $B^{n-J,\lambda}$, $C^{n-J,\lambda}$, and T^{n-J}.

The wavelet representations of functions are also sparse. As an example, consider the function $f(x,y) = \cos(2\pi(x+y))$. The scaling coefficients $s^n_{k_1,k_2}$, $k_1,k_2 = 0,1,2,\ldots,2^n-1$, of f may be approximated by the function samples at the discrete mesh points $(x_i,y_j) = 2^{-n}(i,j)$, $i,j = 0,1,2,\ldots,2^n-1$. The resulting matrix, s^n, is dense, as seen in Figure 5, whose entries bigger than $\varepsilon = 10^{-6}$ are shown in grey. However, the wavelet decomposition of this function is sparse

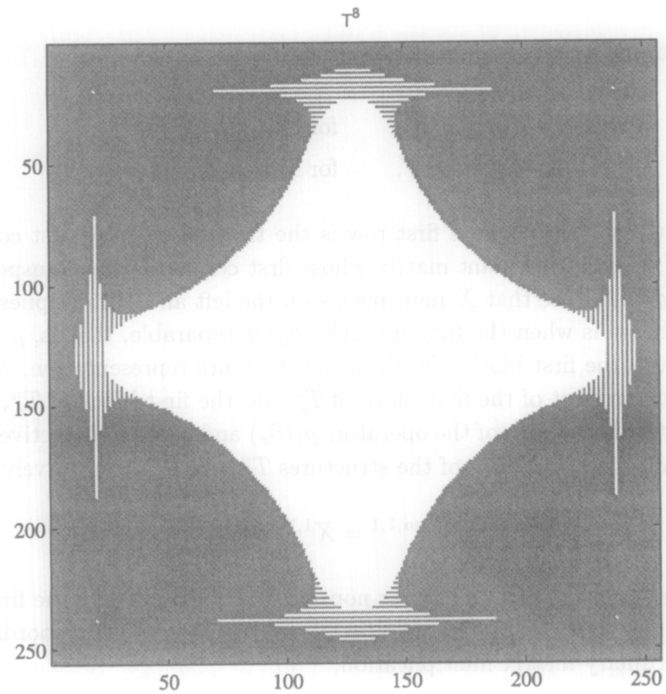

Figure 1. Entries of the first block $T^{8,1,1}$ of the operator $T = e^{\Delta t(\partial_x^2+\partial_y^2)}$ above $\varepsilon = 10^{-6}$. Daubechies' wavelets with $M = 6$ have been used.

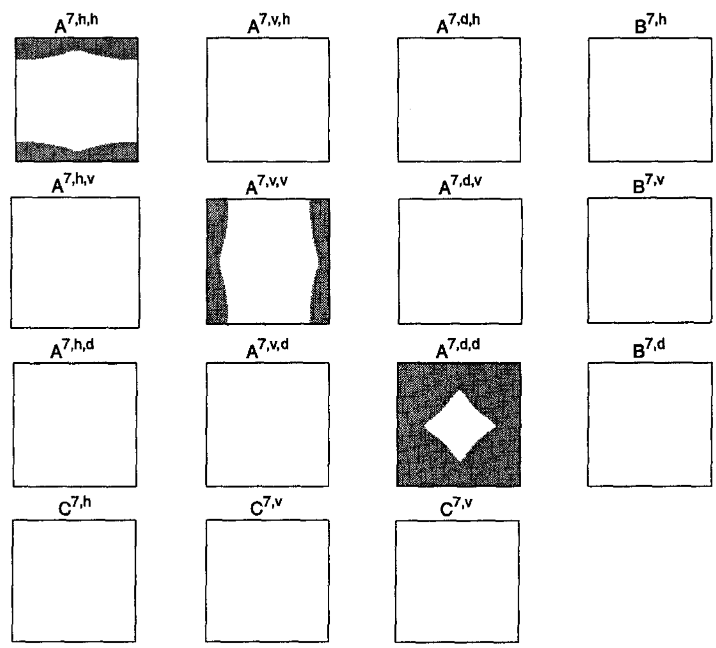

Figure 2. Entries of the first block of $A^{7,\lambda,\lambda'}$, $B^{7,\lambda}$, and $C^{7,\lambda}$ above $\varepsilon = 10^{-6}$. Daubechies' wavelets with $M = 6$ have been used.

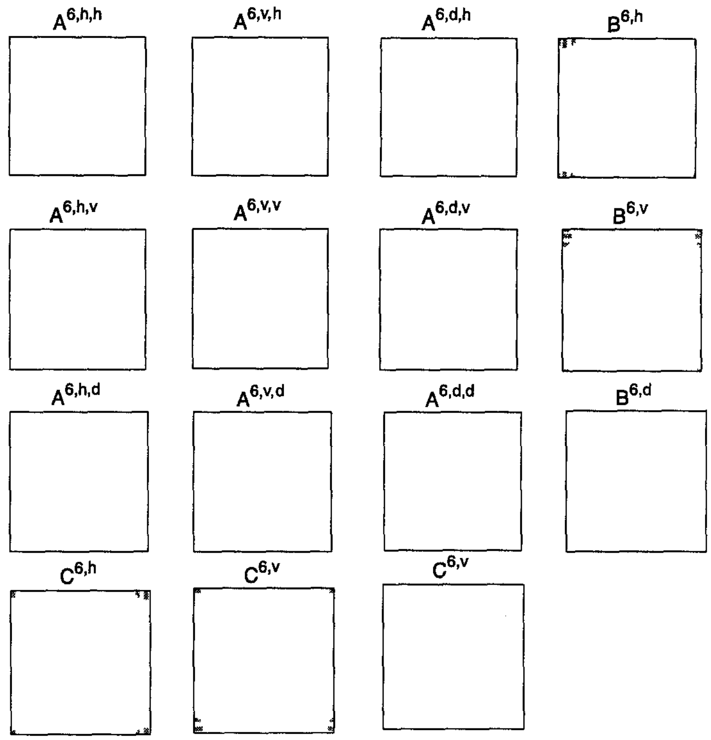

Figure 3. Entries of the first block of $A^{6,\lambda,\lambda'}$, $B^{6,\lambda}$, and $C^{6,\lambda}$ above $\varepsilon = 10^{-6}$. Daubechies' wavelets with $M = 6$ have been used.

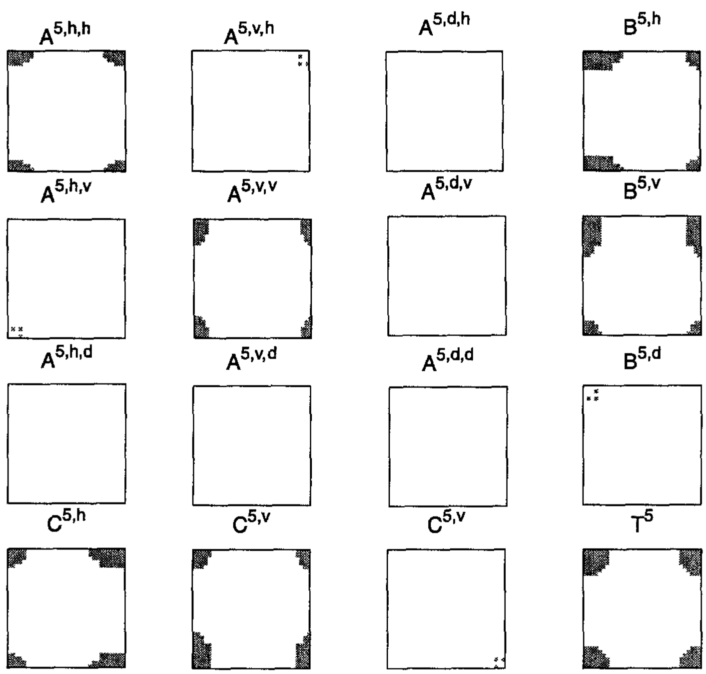

Figure 4. Entries of the first block of $A^{5,\lambda,\lambda'}$, $B^{5,\lambda}$, and $C^{5,\lambda}$ above $\varepsilon = 10^{-6}$. Daubechies' wavelets with $M = 6$ have been used.

as shown in Figure 6, where the wavelet coefficients, $d^{\lambda,j} > \varepsilon$, for $\lambda = h, v, d$, are shown in gray. As we can see, all of the wavelet coefficients at the lower scales $j = 7$ and $j = 6$ are smaller than ε. Only at the coarsest level ($j = 5$) are there significant wavelet coefficients.

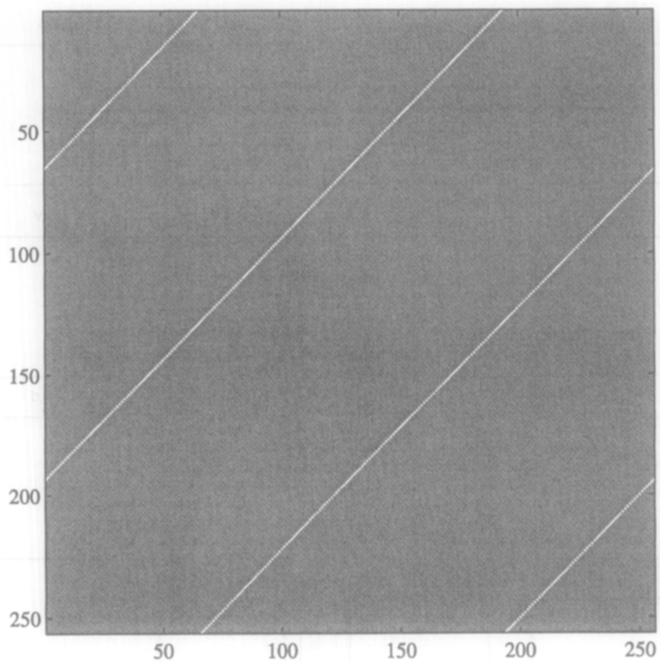

Figure 5. Scaling coefficients of the function $f(x, y) = \cos(2\pi(x + y))$ on the finest space $\mathbf{V}_8$. Daubechies' wavelets with $M = 6$ have been used.

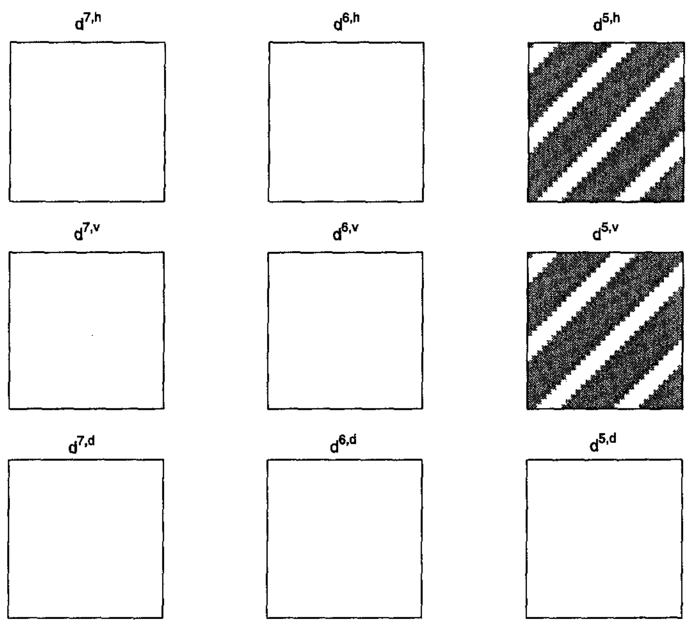

Figure 6. The horizontal, vertical, and diagonal wavelet coefficients, $d^{j,\lambda}$ for $j = 7, 6, 5$ of the function $f(x, y) = \cos(2\pi(x + y))$. Coefficients above $\varepsilon = 10^{-6}$ are shown in gray. Daubechies' wavelets with $M = 6$ have been used.

7. CONCLUDING REMARKS

In this paper, we have generalized the work done by Beylkin [18] on the representation of differential operators in one-dimensional wavelet bases to two-dimensional wavelet bases. We have used $[0, 1]^2$-periodic separable Daubechies wavelets to construct the matrix representation. The periodicity property of these wavelets made it possible to arrive at a closed form formula for the matrix representation of a general differential operator $L = g(\partial_x, \partial_y)$ in terms of the matrix representations of ∂_x and ∂_y. These representations in periodic wavelets are serving as a good tool in developing wavelet-based algorithms for solving partial-differential evolution equations subject to periodic boundary conditions on the unit square [25]. A future research direction could be to consider nonseparable periodic two-dimensional wavelets. Also, one might investigate the use of wavelets on the interval or the square. These are currently being investigated by the authors.

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