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Nonlinear impulsive problems for fractional differential equations and Ulam stability[☆]

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A R T I C L E I N F O

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A B S T R A C T

In this paper, the first purpose is treating Cauchy problems and boundary value problems for nonlinear impulsive differential equations with Caputo fractional derivative. We introduce the concept of piecewise continuous solutions for impulsive Cauchy problems and impulsive boundary value problems respectively. By using a new fixed point theorem, we obtain many new existence, uniqueness and data dependence results of solutions via some generalized singular Gronwall inequalities. The second purpose is discussing Ulam stability for impulsive fractional differential equations. Some new concepts in stability of impulsive fractional differential equations are offered from different perspectives. Some applications of our results are also provided.

1. Introduction

The interest in the study of differential equations of fractional order lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. With this advantage, the fractional order models become more realistic and practical than the classical integer order models, in which such effects are not taken into account. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics and fitting of experimental data. For more recent development on this hot topic, one can see the monographs of Baleanu et al. [1], Diethelm [2], Kilbas et al. [3], Lakshmikantham et al. [4], Miller and Ross [5], Michalski [6], Podlubny [7] and Tarasov [8]. Fractional differential equations involving the Riemann–Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attention (see for example [9–19]).

The theory of impulsive differential equations of integer order has found its extensive applications in realistic mathematical modeling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. For the general theory and applications of impulsive differential equations, we refer the reader to [20–22]. However, impulsive differential equations of fractional order have not been much studied and many aspects of these

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equations are yet to be explored. For some recent work on impulsive fractional differential equations, see [23–27] and the references therein. However, Fečkan et al. [28] study a Cauchy problem for a fractional differential equation with linear impulsive conditions and make a counterexample to illustrate the concepts of piecewise continuous solutions used in current papers are not appropriate. Motivated by these remarks, it is necessary to reconsider these interesting problems.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The problem posed by Ulam was the following: "Under what conditions does there exist an additive mapping near an approximately additive mapping?" (for more details see [29]). The first answer to the question of Ulam was given by Hyers in 1941 in the case of Banach spaces in [30]: Let E_1, E_2 be two real Banach spaces and $\epsilon > 0$. Then for every mapping $f : E_1 \to E_2$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E_1$ there exists a unique additive mapping $g : E_1 \to E_2$ with the property

$$\|f(x) - g(x)\| \le \epsilon, \quad \text{for all } x \in E_1.$$

Thereafter, this type of stability is called the *Ulam–Hyers* stability. In 1978, Rassias [31] provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. The stability properties of all kinds of equations have attracted the attention of many mathematicians. In particular, the Ulam–Hyers stability and Ulam–Hyers–Rassias stability have been taken up by a number of mathematicians and the study of this area has the grown to be one of the central subjects in the mathematical analysis area. For more details on the recently advanced on the Ulam–Hyers stability and Ulam–Hyers–Rassias stability of differential equations, one can see the monographs of Cădariu [32], Hyers [33] and Jung [34] and the research papers of Jung [35], Miura et al. [36,37], Rus [38], Obłoza [39,40], Takahasi et al. [41], Cimpean and Popa [42] and Lungu and Popa [43]. Meanwhile, it is worth remark that Wang et al. [44–46] discuss four type Ulam stability of fractional differential equations and obtain some new and interesting stability results. Unfortunately, Ulam stability of impulsive fractional differential equations is still not studied until now.

The main purpose are three folds. We firstly consider Cauchy problems for nonlinear impulsive fractional differential equations

$$\begin{cases} {}^cD_{0,t}^q u(t) := {}^cD_t^q u(t) = f(t, u(t)), \quad t \in J' := J \setminus \{t_1, \dots, t_m\},\ J := [0, T], \\ \Delta u(t_k) := u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), \quad k = 1, 2, \dots, m, \\ u(0) = u_0, \end{cases} \tag{1}$$

where ${}^cD_t^q$ is the Caputo fractional derivative of order $q \in (0, 1)$ with the lower limit zero, $u_0 \in R, f : J \times R \to R$ is jointly continuous, $I_k : R \to R$ and t_k satisfy $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T, u(t_k^+) = \lim_{\epsilon \to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim_{\epsilon \to 0^-} u(t_k + \epsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$.

Secondly, we discuss Ulam stability for impulsive fractional differential equations

$$\begin{cases} {}^cD_t^q u(t) = f(t, u(t)), \quad t \in J', \\ \Delta u(t_k) = I_k(u(t_k^-)), \quad k = 1, 2, \dots, m. \end{cases} \tag{2}$$

Lastly, we consider boundary problems for nonlinear impulsive fractional differential equations

$$\begin{cases} {}^cD_t^q u(t) = f(t, u(t)), \quad t \in J', \\ \Delta u(t_k) = I_k(u(t_k^-)), \quad k = 1, 2, \dots, m, \\ au(0) + bu(T) = c, \end{cases} \tag{3}$$

where a, b, c are real constants with $a + b \neq 0$.

The rest of this paper is organized as follows. In Section 2, we give some notations, recall some concepts and preparation results. In Section 3, we introduce a concept of a piecewise continuous solution for Cauchy problems and give existence, uniqueness and data dependence results of solutions for the problem (1) by using a new fixed point theorem which is linking degree theory for condensing maps via generalized Gronwall inequality. In Section 4, we introduce four types of Ulam stability definitions for fractional differential equations: Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability and generalized Ulam–Hyers–Rassias stability. We present the four types of Ulam stability results for impulsive fractional differential equation (2). In Section 5, we also introduce a concept of a piecewise continuous solution for boundary value problems and give existence and data dependence results of solutions by using the method applied in Section 3 via a generalized singular Gronwall inequality with mixed integral term. Examples are given in Section 6 to demonstrate the application of our main results.

To end this section, we recall some basic definitions and properties of the fractional calculus theory which are used further in this paper. For more details, see [3].

Definition 1.1. The fractional order integral of the function $h \in L^1([a,b],R)$ of order $q \in R^+$ is defined by

$$I_a^q h(t) = \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds$$

where Γ is the Gamma function.

Definition 1.2. For a function h given on the interval $[a,b]$, the qth Riemann–Liouville fractional order derivative of h, is defined by

$$(D_{a,t}^q h)(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-q-1} h(s)ds,$$

here $n = [q] + 1$ and $[q]$ denotes the integer part of q.

Definition 1.3. For a function h given on the interval $[a,b]$, the Caputo fractional order derivative of h, is defined by

$$({}^cD_{a,t}^q h)(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} h^{(n)}(s)ds,$$

where $n = [q] + 1$ and $[q]$ denotes the integer part of q.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts. Throughout this paper, let $C(J,R)$ be the Banach space of all continuous functions from J into R with the norm $\|u\|_C = \sup\{|u(t)| : t \in J\}$ for $u \in C(J,R)$. We also introduce the Banach space $PC(J,R) = \{u : J \to R : u \in C((t_k, t_{k+1}], R),\ k = 0,\dots,m$ and there exist $u(t_k^-)$ and $u(t_k^+),\ k = 1,\dots,m,$ with $u(t_k^-) = u(t_k)\}$ with the norm $\|u\|_{PC} = \sup\{|u(t)| : t \in J\}$.

For a minute description of the following notions we refer the reader to Deimling [47]. In the following, X will be a Banach spaces and $\mathcal{B} \in \mathcal{P}(X)$ will be the family of all its bounded sets. The function $\alpha : \mathcal{B} \to R^+$ defined by $\alpha(B) = \inf\{d > 0 : B$ admits a finite cover by sets of diameter $\le d\}$, $B \in \mathcal{B}$, is called the Kuratowski measure of noncompactness.

Definition 2.1. Consider $\Omega \subset X$ and $F : \Omega \to X$ a continuous bounded map. We say that F is α-Lipschitz if there exists $\kappa \ge 0$ such that $\alpha(F(B)) \le \kappa\alpha(B)$ for all $B \subset \Omega$ bounded. If, in addition, $\kappa < 1$, then we say that F is a strict α-contraction.

We say that F is α-condensing if $\alpha(F(B)) < \alpha(B)$ for all $B \subset \Omega$ bounded with $\alpha(B) > 0$. In other words, $\alpha(F(B)) \ge \alpha(B)$ implies $\alpha(B) = 0$. The class of all strict α-contractions $F : \Omega \to X$ is denoted by $\mathcal{S}C_\alpha(\Omega)$ and the class of all α-condensing maps $F : \Omega \to X$ is denoted by $C_\alpha(\Omega)$.

We remark that $\mathcal{S}C_\alpha(\Omega) \subset C_\alpha(\Omega)$ and every $F \in C_\alpha(\Omega)$ is α-Lipschitz with constant $\kappa = 1$. We also recall that $F : \Omega \to X$ is Lipschitz if there exists $\kappa > 0$ such that $\|Fx - Fy\| \le \kappa\|x - y\|$ for all $x, y \in \Omega$ and that F is a strict contraction if $\kappa < 1$. Next, we collect some properties of the applications defined above.

Proposition 2.2. *If $F, G : \Omega \to X$ are α-Lipschitz maps with constants κ, respectively κ', then $F + G : \Omega \to X$ are α-Lipschitz with constants $\kappa + \kappa'$.*

Proposition 2.3. *If $F : \Omega \to X$ is compact, then F is α-Lipschitz with constant $\kappa = 0$.*

Proposition 2.4. *If $F : \Omega \to X$ is Lipschitz with constant κ, then F is α-Lipschitz with the same constant κ.*

Now we state a fixed point theorem which will be used in the proofs of the main existence results. For more details, see [48].

Theorem 2.5 (*Theorem 2, [48]*). *Let $F : X \to X$ be α-condensing and*

$$\mathcal{S} = \{x \in X : \text{exists } \lambda \in [0,1] \text{ such that } x = \lambda Fx\}.$$

If $\mathcal{S}$ is a bounded set in X, so there exists $r > 0$ such that $\mathcal{S} \subset B_r(0)$, then F has at least one fixed point and the set of the fixed points of F lies in $B_r(0)$.

The following generalized Gronwall inequalities with Caputo singular kernel will be widely used to deal with our problems in the sequence.

Lemma 2.6 (*Theorem 1, [49]*). *Suppose $\beta > 0$, $\widetilde{a}(t)$ is a nonnegative function locally integrable on $[a, b)$ and $\widetilde{g}(t)$ is a nonnegative, nondecreasing continuous function defined on $\widetilde{g}(t) \le M$, $t \in [a, b)$, and suppose $y(t)$ is nonnegative and locally integrable on $[a, b)$ with*

$$y(t) \le \widetilde{a}(t) + \widetilde{g}(t) \int_0^t (t-s)^{\beta-1} y(s) ds, \quad t \in [a, b).$$

Then

$$y(t) \le \widetilde{a}(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\widetilde{g}(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} \widetilde{a}(s) \right] ds, \quad t \in [a, b).$$

Remark 2.7. Under the hypothesis of Lemma 2.6, let $\widetilde{a}(t)$ be a nondecreasing function on $[a, b)$. Then we have

$$y(t) \le \widetilde{a}(t) E_\beta(\widetilde{g}(t)\Gamma(\beta)t^\beta),$$

where E_β is the Mittag-Leffler function [3] defined by

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}, \quad z \in \mathbb{C}, \ \mathfrak{Re}(\beta) > 0.$$

Lemma 2.8. *Let $u \in PC(J, R)$ satisfy the following inequality*

$$|u(t)| \le c_1(t) + c_2 \int_0^t (t-s)^{q-1} |u(s)| ds + \sum_{0<t_k<t} \theta_k |u(t_k^-)|,$$

where $c_1(t)$ is nonnegative continuous and nondecreasing on J, and $c_2, \theta_k \ge 0$ are constants. Then

$$|u(t)| \le c_1(t) \left(1 + \theta E_\beta(c_2\Gamma(\beta)t^\beta)\right)^k E_\beta(c_2\Gamma(\beta)t^\beta) \quad \text{for } t \in (t_k, t_{k+1}], \tag{4}$$

where $\theta = \max\{\theta_k : k = 1, 2, \ldots, m\}$.

Proof. Indeed, from Remark 2.7 we derive

$$|u(t)| \le c_1(t) E_\beta(c_2\Gamma(\beta)t^\beta) \quad \text{for } t \in [0, t_1], \tag{5}$$

$$|u(t)| \le \left(c_1(t) + \sum_{j=1}^{k} \theta_j |u(t_j^-)| \right) E_\beta(c_2\Gamma(\beta)t^\beta) \quad \text{for } t \in (t_k, t_{k+1}]. \tag{6}$$

By (5), inequality (4) holds for $k = 0$. By induction suppose (4) holds for $k = j < m$. Then by (6) and since $c_1(t)$ and $E_\beta(z)$ are nondecreasing, for $t \in (t_{j+1}, t_{j+2}]$, we derive

$$\begin{aligned}
|u(t)| &\le \left(c_1(t) + \sum_{i=1}^{j+1} \theta_i |u(t_i^-)| \right) E_\beta(c_2\Gamma(\beta)t^\beta) \\
&\le \left(c_1(t) + \sum_{i=1}^{j+1} \theta_i c_1(t_i^-) \left(1 + \theta E_\beta(c_2\Gamma(\beta)(t_i^-)^\beta)\right)^{i-1} E_\beta(c_2\Gamma(\beta)(t_i^-)^\beta) \right) E_\beta(c_2\Gamma(\beta)t^\beta) \\
&\le \left(c_1(t) + \theta \sum_{i=1}^{j+1} c_1(t) \left(1 + \theta E_\beta(c_2\Gamma(\beta)t^\beta)\right)^{i-1} E_\beta(c_2\Gamma(\beta)t^\beta) \right) E_\beta(c_2\Gamma(\beta)t^\beta) \\
&= c_1(t) \left(1 + \theta E_\beta(c_2\Gamma(\beta)t^\beta)\right)^{j+1} E_\beta(c_2\Gamma(\beta)t^\beta).
\end{aligned}$$

This finishes the proof. □

Lemma 2.9. *Let $u \in PC(J, R)$ satisfy the following inequality*

$$|u(t)| \le a + b \int_0^t (t-s)^{q-1} |u(s)|^\lambda ds + c \int_0^T (T-s)^{q-1} |u(s)|^\lambda ds \quad \text{for } t \in J', \tag{7}$$

where $q \in (0, 1)$, $\lambda \in [0, 1)$ and $a, b, c \ge 0$ are constants. Then we have

$$|u(t)| \le M,$$

where M is the only positive solution of the equation $M = a + (b+c)\frac{T^q}{q} M^\lambda$.

Proof. Let $m = \max_{t\in J} |u(t)|$. Then (7) implies

$$m \le a + b\int_0^t (t-s)^{q-1} m^{\lambda} ds + c\int_0^T (T-s)^{q-1} m^{\lambda} ds \le a + (b+c)\frac{T^q}{q} m^{\lambda}.$$

So $m \le M$. The proof is finished. □

Remark 2.10. It is obvious that we have the similar results if we replace a by $\widetilde{a}(t)$ on J in Lemma 2.9. Similar results are given in [50, Lemma 3.1] and [51, Lemma 3.1].

3. Existence results for impulsive Cauchy problems

This section deals with the existence, data dependence and uniqueness of solutions for the problem (1).

Definition 3.1. A function $u \in PC(J, R)$ is said to be a solution of the problem (1) if $u(t) = u_k(t)$ for $t \in (t_k, t_{k+1})$ and $u_k \in C([0, t_{k+1}], R)$, $k = 0, 1, 2, \ldots, m$ satisfies ${}^cD_t^q u_k(t) = f(t, u_k(t))$ a.e. on $(0, t_{k+1})$ with the restriction of $u_k(t)$ on $[0, t_k)$ is just $u_{k-1}(t)$, and the conditions $u(t_k^+) = u(t_k^-) + I_k(u(t_k^-))$, $k = 1, 2, \ldots, m$, and $u(0) = u_0$.

Let $h : J \to R$ be continuous. Note that

$$u(t) = u_0 - \frac{1}{\Gamma(q)}\int_0^a (a-s)^{q-1} h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} h(s)ds,$$

solves the Cauchy problems

$$\begin{cases} {}^cD_t^q u(t) = h(t), \quad t \in J, \\ u(0) = u_0 - \dfrac{1}{\Gamma(q)}\displaystyle\int_0^a (a-s)^{q-1} h(s)ds. \end{cases} \tag{8}$$

Then, one can obtain the following result immediately.

Lemma 3.2. *Let $q \in (0, 1)$ and $h : J \to R$ be continuous. A function $u \in C(J, R)$ given by*

$$u(t) = u_0 - \frac{1}{\Gamma(q)}\int_0^a (a-s)^{q-1} h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} h(s)ds,$$

is the only solution of the following fractional Cauchy problems

$$\begin{cases} {}^cD_t^q u(t) = h(t), \quad t \in J, \\ u(a) = u_0, \quad a > 0. \end{cases} \tag{9}$$

As a consequence of Lemma 3.2 we have the following result which is useful in what follows.

Lemma 3.3. *Let $q \in (0, 1)$ and $h : J \to R$ be continuous. A function u given by*

$$u(t) = \begin{cases} u_0 + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1} h(s)ds, \quad \text{for } t \in [0, t_1], \\ u_0 + I_1(u(t_1^-)) + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1} h(s)ds, \quad \text{for } t \in (t_1, t_2], \\ u_0 + I_1(u(t_1^-)) + I_2(u(t_2^-)) + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1} h(s)ds, \quad \text{for } t \in (t_2, t_3], \\ \vdots \\ u_0 + \displaystyle\sum_{i=1}^m I_i(u(t_i^-)) + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1} h(s)ds, \quad \text{for } t \in (t_m, T]. \end{cases} \tag{10}$$

is the only solution of the following impulsive problem

$$\begin{cases} {}^cD_t^q u(t) = h(t), \quad t \in (0, T], \\ \Delta u(t_k) = I_k(u(t_k^-)), \quad k = 1, 2, \ldots, m, \\ u(0) = u_0. \end{cases} \tag{11}$$

Proof. Assume u satisfies (11). If $t \in [0, t_1]$ then

$$ {}^cD_t^q u(t) = h(t), \quad t \in (0, t_1] \text{ with } u(0) = u_0. \tag{12}$$

Integrating the expression (12) from 0 to t by the fractional integral definition, one can obtain

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds.$$

If $t \in (t_1, t_2]$ then

$$ {}^cD_t^q u(t) = h(t), \quad t \in (t_1, t_2] \text{ with } u(t_1^+) = u(t_1^-) + I_1(u(t_1^-)).$$

By Lemma 3.2, one obtain

$$\begin{aligned} u(t) &= u(t_1^+) - \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} h(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds \\ &= u(t_1^-) + I_1(u(t_1^-)) - \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} h(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds \\ &= u_0 + I_1(u(t_1^-)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds. \end{aligned}$$

If $t \in (t_2, t_3]$ then again Lemma 3.2 we get

$$\begin{aligned} u(t) &= u(t_2^+) - \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} h(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds \\ &= u(t_2^-) + I_2(u(t_2^-)) - \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} h(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds \\ &= u_0 + I_1(u(t_1^-)) + I_2(u(t_2^-)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds. \end{aligned}$$

If $t \in (t_k, t_{k+1}]$ then again from Lemma 3.2 we get

$$u(t) = u_0 + \sum_{i=1}^{k} I_i(u(t_i^-)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds, \quad k = 1, 2, \ldots, m.$$

Conversely, assume that u satisfies (10). If $t \in (0, t_1]$ then $u(0) = u_0$ and using the fact that ${}^cD_t^q$ is the left inverse of I_t^q we get (12). If $t \in (t_k, t_{k+1}],\ k = 1, 2, \ldots, m$ and using the fact of the Caputo derivative of a constant is equal to zero, we obtain ${}^cD_t^q u(t) = h(t),\ t \in (t_k, t_{k+1}]$ and $u(t_k^+) - u(t_k^-) = I_k(u(t_k^-))$. This completes the proof. □

Before stating and proving the main results, we introduce the following hypotheses.

[H1]: $f : J \times R \to R$ is jointly continuous.

[H2]: For arbitrary $(t, u) \in J \times R$, there exist $C_f, M_f > 0$ and $q_1 \in [0, 1)$ such that

$$|f(t,u)| \le C_f |u|^{q_1} + M_f.$$

[H3]: There exist constants $L_f > 0$ and $\lambda \in [0, 1)$ such that

$$|f(t,u) - f(t,v)| \le L_f |u-v|^{\lambda}, \quad \text{for each } t \in J, \text{ and all } u, v \in R.$$

[H4]: $I_k: R \to R$ and there exists a constant $K_I^k > 0$ such that

$$|I_k(u) - I_k(v)| \le K_I^k |u-v|, \quad \text{for all } u, v \in R, \text{ and } k = 1, 2, \ldots, m.$$

Moreover, it holds $K_I = \sum_{i=1}^m K_I^i \in [0, 1)$.

[H5]: For arbitrary $u \in R$, there exist $C_I, M_I > 0$ and $q_2 \in [0, 1)$ such that

$$|I_k(u)| \le C_I |u|^{q_2} + M_I, \quad k = 1, 2, \ldots, m.$$

Now, we define operators as follows:
$F : PC(J, R) \to PC(J, R)$ given by

$$(Fu)(t) = \sum_{i=0}^{k} (F^i u)(t), \quad t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \ldots, m,$$

where

$$(F^0u)(t) = u_0, \qquad (F^iu)(t) = I_i(u(t_i^-)), \quad i = 1, 2, \ldots, m.$$

$G : PC(J, R) \to PC(J, R)$ given by

$$(Gu)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))ds, \quad t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \ldots, m.$$

Let $\mathbb{T} : PC(J, R) \to PC(J, R)$ given by

$$\mathbb{T}u = Fu + Gu.$$

Thus, the existence of a solution for the problem (1) is equivalent to the existence of a fixed point for operator $\mathbb{T}$.

Lemma 3.4. *The operator $F : PC(J, R) \to PC(J, R)$ is Lipschitz with constant K_I. Consequently F is α-Lipschitz with the same constant K_I. Moreover, F satisfies the following growth condition:*

$$\|Fu\|_{PC} \le |u_0| + m(C_I\|u\|_C^{q_2} + M_I), \tag{13}$$

for every $u \in PC(J, R)$.

Proof. For $t \in [0, t_1]$, it is obvious that

$$|(Fu)(t) - (Fv)(t)| = |(F^0u)(t) - (F^0v)(t)| = 0$$

for every $u, v \in PC(J, R)$. For $t \in (t_1, t_2]$, using [H4],

$$|(Fu)(t) - (Fv)(t)| = |(F^1u)(t) - (F^1v)(t)| \le K_I^1\|u - v\|_{PC},$$

for every $u, v \in PC(J, R)$. In general, for $t \in (t_k, t_{k+1}],\ k = 2, 3, \ldots, m$, using [H4] step by step, F^i is α-Lipschitz with constant K_I^i. Using Proposition 2.2, F is α-Lipschitz with constant $\sum_{i=1}^m K_I^i$. Relation (13) is a simple consequence of [H5]. □

Lemma 3.5. *The operator $G : PC(J, R) \to PC(J, R)$ is continuous. Moreover, G satisfies the following growth condition:*

$$\|Gu\|_{PC} \le \frac{T^q(C_f\|u\|_{PC}^{q_1} + M_f)}{\Gamma(q+1)}, \tag{14}$$

for every $u \in PC(J, R)$.

Proof. For our purpose, we need to check the continuity of the operators G on $C((t_k, t_{k+1}], R),\ k = 1, 2, \ldots, m$ step by step.

For that, let $\{u_n\}$ be a sequence of a bounded set $\mathfrak{B}_{K_1} := \{\|u\|_{C([0,t_1],R)} \le K_1 : u \in C([0, t_1], R)\} \subseteq C([0, t_1], R)$ such that $u_n \to u$ in $\mathfrak{B}_{K_1}$. We have to show that $\|Gu_n - Gu\|_{C([0,t_1],R)} \to 0$.

It is easy to see that $f_n = \max_{s\in J} |f(s, u_n(s)) - f(s, u(s))| \to 0$ as $n \to \infty$ due to the continuity of f. Then, for all $t \in [0, t_1]$,

$$\begin{aligned} |(Gu_n)(t) - (Gu)(t)| &\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, u_n(s)) - f(s, u(s))|ds \\ &\le \frac{f_n}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds \le \frac{f_nT^q}{\Gamma(q)}. \end{aligned}$$

Therefore, $Gu_n \to Gu$ as $n \to \infty$ which implies that G is continuous on $[0, t_1]$.

Repeat the above process step by step, one can obtain the continuity of the operators G is continuous on $(t_k, t_{k+1}],\ k = 1, 2, \ldots, m$. Relation (14) is a simple consequence of [H2]. □

Lemma 3.6. *The operator $G : PC(J, R) \to PC(J, R)$ is compact. Consequently G is α-Lipschitz with zero constant.*

Proof. In order to prove the compactness of G, we consider a bounded set $\mathcal{D} \subset \mathfrak{B}_{K_1} \subseteq C((t_k, t_{k+1}], R),\ k = 1, 2, \ldots, m$ and we will show that $G(\mathcal{D})$ is relatively compact in $C((t_k, t_{k+1}], R)$ with the help of Arzela–Ascoli Theorem.

For $t \in [0, t_1]$, let $\{u_n\}$ be a sequence on $\mathcal{D} \subset \mathfrak{B}_{K_1} \subseteq C([0, t_1], R)$, for every $u_n \in \mathcal{D}$. By (14), we have

$$\|Gu\|_{C([0,t_1],R)} \le \frac{T^q(C_f\|u\|_{C([0,t_1],R)}^{q_1} + M_f)}{\Gamma(q+1)},$$

for every $u_n \in \mathcal{D}$, so $G(\mathcal{D})$ is bounded in $C([0, t_1], R)$.

Now we prove that $\{Gu_n\}$ is equicontinuous. For $0 \le \tau_1 < \tau_2 \le t_1$, we get

$$\begin{aligned}|(Gu_n)(\tau_1) - (Gu_n)(\tau_2)| &\le \frac{1}{\Gamma(q)}\int_0^{\tau_1}((\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1})|f(s, u_n(s))|ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1}|f(s, u_n(s))|ds \\ &\le \frac{1}{\Gamma(q)}\int_0^{\tau_1}((\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1})(C_f|u_n(s)|^{q_1} + M_f)ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1}(C_f|u_n(s)|^{q_1} + M_f)ds \\ &\le \frac{(C_fK_1^{q_1} + M_f)}{\Gamma(q)}\left[\frac{\tau_1^q}{q} - \frac{\tau_2^q}{q} + \frac{(\tau_2 - \tau_1)^q}{q} + \frac{(\tau_2 - \tau_1)^q}{q}\right] \\ &\le \frac{2(C_fK_1^{q_1} + M_f)(\tau_2 - \tau_1)^q}{\Gamma(q)}.\end{aligned}$$

As $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero. Therefore $\{Gu_n\}$ is equicontinuous. Therefore, $G(\mathcal{D}) \subset C([0, t_1], R)$ satisfies the hypothesis of Arzela–Ascoli Theorem, so $G(\mathcal{D})$ is relatively compact in $C([0, t_1], R)$.

For each $(t_k, t_{k+1}],\ k = 1, 2, \ldots, m$. Repeating the above process again, one can obtain the compactness of the operators G on $C((t_k, t_{k+1}], R),\ k = 1, 2, \ldots, m$.

By Proposition 2.3, G is α-Lipschitz with zero constant. □

Now, we have the possibility to prove the main results of this section.

Theorem 3.7. *Assume that* [H1], [H2], [H4] *and* [H5] *hold, then the problem* (1) *has at least one solution* $u \in PC(J, R)$ *and the set of the solutions of the problem* (1) *is bounded in* $PC(J, R)$.

Proof. Let $F, G, \mathbb{T} : PC(J, R) \to PC(J, R)$ be the operators defined in the beginning of this section. They are continuous and bounded. Moreover, F is α-Lipschitz with constant $K_I \in [0, 1)$ and G is α-Lipschitz with zero constant (see Lemmas 3.4–3.6). Also, Proposition 2.2 shows us that $\mathbb{T}$ is a strict α-contraction with constant K_I.

Set

$$S = \{u \in PC(J, R) : \exists\lambda \in [0, 1] \text{ such that } u = \lambda\mathbb{T}u\}.$$

Next, we prove that S is bounded in $PC(J, R)$. Consider $u \in S$ and $\lambda \in [0, 1]$ such that $u = \lambda\mathbb{T}u$. It follows from (13) and (14) that

$$\begin{aligned}\|u\|_{PC} &\le \lambda(\|Fu\|_{PC} + \|Gu\|_{PC}) \\ &\le \lambda\left[|u_0| + \frac{T^q(C_f\|u\|_{PC}^{q_1} + M_f)}{\Gamma(q+1)} + m(C_I\|u\|_{PC}^{q_2} + M_I)\right].\end{aligned} \tag{15}$$

This inequality, together with $q_1 < 1$ and $q_2 < 1$, shows us that S is bounded in $PC(J, R)$. If not, we suppose by contradiction, $\rho := \|u\|_{PC} \to \infty$. Dividing both sides of (15) by ρ, and taking $\rho \to \infty$, we have

$$1 \le \lim_{\rho\to\infty} \frac{|u_0| + \frac{T^q(C_f\rho^{q_1} + M_f)}{\Gamma(q+1)} + m(C_I\rho^{q_2} + M_I)}{\rho} = 0. \tag{16}$$

This is a contradiction. Consequently, by Theorem 2.5 we deduce that $\mathbb{T}$ has at least one fixed point and the set of the fixed points of $\mathbb{T}$ is bounded in $PC(J, R)$. □

Remark 3.8. (i) If the growth condition [H2] is formulated for $q_1 = 1$, then the conclusions of Theorem 3.7 remain valid provided that $\frac{T^qC_f}{\Gamma(q+1)} < 1$;
(ii) If the growth condition [H5] is formulated for $q_2 = 1$, then the conclusions of Theorem 3.7 remain valid provided that $mC_I < 1$;
(iii) If the growth conditions [H2] and [H5] are formulated for $q_2 = 1$ and $q_3 = 1$, then the conclusions of Theorem 3.7 remain valid provided that $\frac{T^qC_f}{\Gamma(q+1)} + mC_I < 1$.
(iv) For $q_1 = q_2 = 1$, one can also obtain the boundedness of the set S by virtue of Lemma 2.6 and Remark 2.7.
(v) For $q_1, q_2 \in [0, 1)$, one can also obtain the boundedness of the set S by virtue of Lemma 2.9.

Now, we give the following data dependence result.

Theorem 3.9. *Assume that* [H1]–[H5] *hold. Let* $u(\cdot)$ *and* $v(\cdot)$ *be the solutions of problem* (1) *with initial values* u_0 *and* v_0*, respectively. Then it holds*

$$\|u - v\|_{PC} \le M^*,$$

where M^* *is the only positive solution of the equation*

$$M^* = \frac{|u_0 - v_0|}{1 - K_I} + \frac{L_f T^q M^{*\lambda}}{(1 - K_I)\Gamma(q + 1)}.$$

Proof. Note that [H3] and [H4], we obtain

$$|u(t) - v(t)| \le |u_0 - v_0| + K_I\|u - v\|_{PC} + \frac{L_f}{\Gamma(q)}\int_0^t (t - s)^{q-1}|u(s) - v(s)|^{\lambda} ds,$$

which implies that

$$(1 - K_I)|u(t) - v(t)| \le (1 - K_I)\|u - v\|_{PC} \le |u_0 - v_0| + \frac{L_f}{\Gamma(q)}\int_0^t (t - s)^{q-1}|u(s) - v(s)|^{\lambda} ds.$$

Thus,

$$|u(t) - v(t)| \le \frac{|u_0 - v_0|}{1 - K_I} + \frac{L_f}{(1 - K_I)\Gamma(q)}\int_0^t (t - s)^{q-1}|u(s) - v(s)|^{\lambda} ds.$$

Lemma 2.9 completes the proof. □

In order to obtain the uniqueness of solutions, we revise [H3] to following assumption:

[H3′] There exists a constant $L_f > 0$ such that

$$|f(t, u) - f(t, v)| \le L_f|u - v|, \quad \text{for each } t \in J, \text{ and all } u, v \in R.$$

Finally, we give the following uniqueness result.

Theorem 3.10. *Assume that* [H1], [H2], [H3′], [H4] *and* [H5] *hold, then problem* (1) *has a unique solution* $u \in PC(J, R)$.

Proof. By Theorem 3.7, the problem (1) has a solution $u(\cdot)$ in $PC(J, R)$. Let $v(\cdot)$ be another solution of problem (1) with initial value v_0. Note that [H3′] and [H4], repeating the same process of Theorem 3.9, we obtain

$$|u(t) - v(t)| \le \frac{1}{1 - K_I}|u_0 - v_0| + \frac{L_f}{(1 - K_I)\Gamma(q)}\int_0^t (t - s)^{q-1}|u(s) - v(s)| ds.$$

This yields that the uniqueness of $u(\cdot)$ due to Lemma 2.6 and Remark 2.7. □

4. Ulam stability results for impulsive fractional differential equations

In this section, we consider the Ulam stability of impulsive fractional differential equation (2). Let ϵ be a positive real number and $\varphi : J \to R^+$ be a continuous function. We consider the following inequalities

$$\begin{cases} |{}^cD_t^q y(t) - f(t, y(t))| \le \epsilon, & t \in J', \\ |\Delta u(t_k) - I_k(y(t_k^-))| \le \epsilon, & k = 1, 2, \ldots, m, \end{cases} \tag{17}$$

$$\begin{cases} |{}^cD_t^q y(t) - f(t, y(t))| \le \varphi(t), & t \in J', \\ |\Delta u(t_k) - I_k(y(t_k^-))| \le \varphi(t), & k = 1, 2, \ldots, m, \end{cases} \tag{18}$$

$$\begin{cases} |{}^cD_t^q y(t) - f(t, y(t))| \le \epsilon\varphi(t), & t \in J', \\ |\Delta u(t_k) - I_k(y(t_k^-))| \le \epsilon\varphi(t), & k = 1, 2, \ldots, m. \end{cases} \tag{19}$$

Definition 4.1. Eq. (2) is Ulam–Hyers stable if there exists a real number $c_{f,m} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in PC(J, R)$ of the inequality (17) there exists a solution $x \in PC(J, R)$ of Eq. (2) with

$$|y(t) - x(t)| \le c_{f,m}\epsilon, \quad t \in J.$$

Definition 4.2. Eq. (2) is generalized Ulam–Hyers stable if there exists $\theta_{f,m} \in PC(J, R^+)$, $\theta_{f,m}(0) = 0$ such that for each solution $y \in PC(J, R)$ of the inequality (17) there exists a solution $x \in PC(J, R)$ of Eq. (2) with

$$|y(t) - x(t)| \le \theta_{f,m}(\epsilon), \quad t \in J.$$

Definition 4.3. Eq. (2) is Ulam–Hyers–Rassias stable with respect to φ if there exists $c_{f,m,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in PC(J,R)$ of the inequality (19) there exists a solution $x \in PC(J,R)$ of Eq. (2) with

$$|y(t) - x(t)| \le c_{f,m,\varphi}\epsilon\varphi(t), \quad t \in J.$$

Definition 4.4. Eq. (2) is generalized Ulam–Hyers–Rassias stable with respect to φ if there exists $c_{f,m,\varphi} > 0$ such that for each solution $y \in PC(J,R)$ of the inequality (18) there exists a solution $x \in PC(J,R)$ of Eq. (2) with

$$|y(t) - x(t)| \le c_{f,m,\varphi}\varphi(t), \quad t \in J.$$

Remark 4.5. It is clear that: (i) Definition 4.1 $\Longrightarrow$ Definition 4.2; (ii) Definition 4.3 $\Longrightarrow$ Definition 4.4; (iii) Definition 4.3 for $\varphi(t) = 1 \Longrightarrow$ Definition 4.1.

Remark 4.6. A function $y \in PC(J,R)$ is a solution of the inequality (17) if and only if there exist a function $g \in PC(J,R)$ and a sequence g_k, $k = 1, 2, \ldots, m$ (which depend on y) such that

(i) $|g(t)| \le \epsilon,\ t \in J$ and $|g_k| \le \epsilon,\ k = 1, 2, \ldots, m$;
(ii) ${}^cD_t^q y(t) = f(t, y(t)) + g(t),\ t \in J'$;
(iii) $\Delta y(t_k) = I_k(y(t_k^-)) + g_k,\ k = 1, 2, \ldots, m$.

One can have similar remarks for the inequalities (18) and (19).

So, the Ulam stabilities of the impulsive fractional differential equations are some special types of data dependence of the solutions of impulsive fractional differential equations.

Remark 4.7. Let $0 < q < 1$, if $y \in PC(J,R)$ is a solution of the inequality (17) then y is a solution of the following integral inequality

$$\left|y(t) - y(0) - \sum_{i=1}^{k} I_i(y(t_i^-)) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, y(s))ds\right| \le \left(m + \frac{t^q}{\Gamma(q+1)}\right)\epsilon, \quad t \in J.$$

Indeed, by Remark 4.6 we have that

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t)) + g(t), & t \in J', \\ \Delta y(t_k) = I_k(y(t_k^-)) + g_k, & k = 1, 2, \ldots, m. \end{cases}$$

Then

$$\begin{aligned} y(t) &= y(0) + \sum_{i=1}^{k} I_i(y(t_i^-)) + \sum_{i=1}^{k} g_i \\ &\quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, y(s))ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g(s)ds, \quad t \in (t_k, t_{k+1}]. \end{aligned}$$

From this it follows that

$$\begin{aligned} \left|y(t) - y(0) - \sum_{i=1}^{k} I_i(y(t_i^-)) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, y(s))ds\right| &= \left|\sum_{i=1}^{k} g_i + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g(s)ds\right| \\ &\le \sum_{i=1}^{m} |g_i| + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|g(s)|ds \\ &\le m\epsilon + \frac{\epsilon}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\ &\le \left(m + \frac{t^q}{\Gamma(q+1)}\right)\epsilon. \end{aligned}$$

We have similar remarks for the solutions of the inequalities (18) and (19).

Now, we give the main results, generalized Ulam–Hyers–Rassias stable results, in this section.

Theorem 4.8. *Assumptions* [H1], [H2], [H3′], [H4] *and* [H5] *hold. Suppose there exists* $\lambda_\varphi > 0$ *such that*

$$\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\varphi(s)ds \le \lambda_\varphi\varphi(t), \quad \text{for each } t \in J,$$

where $\varphi \in C(J, R^+)$ *is nondecreasing. Then Eq.* (2) *is generalized Ulam–Hyers–Rassias stable.*

Proof. Let $y \in PC(J, R)$ be a solution of the inequality (18). By Theorem 3.10 there x is a unique solution of the impulsive Cauchy problem

$$\begin{cases} {}^cD_t^q x(t) = f(t, x(t)), & t \in J', \\ \Delta x(t_k) = I_k(x(t_k^-)), & k = 1, 2, \ldots, m, \\ x(0) = y(0). \end{cases} \tag{20}$$

Then we have

$$x(t) = \begin{cases} y(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds, & \text{for } t \in [0, t_1], \\ y(0) + I_1(x(t_1^-)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds, & \text{for } t \in (t_1, t_2], \\ y(0) + I_1(x(t_1^-)) + I_2(x(t_2^-)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds, & \text{for } t \in (t_2, t_3], \\ \vdots \\ y(0) + \sum_{k=1}^{m} I_k(x(t_k^-)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds, & \text{for } t \in (t_m, T]. \end{cases}$$

By differential inequality (18), for each $t \in (t_k, t_{k+1}]$, we have

$$\left| y(t) - y(0) - \sum_{i=1}^{k} I_k(y(t_i^-)) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s)) ds \right| \leq \sum_{k=1}^{m} |g(t_k^-)| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \varphi(s) ds$$
$$\leq (m + \lambda_\varphi)\varphi(t), \quad t \in J.$$

Hence for each $t \in (t_k, t_{k+1}]$, it follows

$$\begin{aligned} |y(t) - x(t)| &\leq \left| y(t) - y(0) - \sum_{i=1}^{k} I_i(x(t_i^-)) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds \right| \\ &\leq \left| y(t) - y(0) - \sum_{i=1}^{k} I_i(y(t_i^-)) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s)) ds \right| + \sum_{i=1}^{k} |I_i(x(t_i^-)) - I_i(y(t_i^-))| \\ &\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, y(s)) - f(s, x(s))| ds \\ &\leq (m + \lambda_\varphi)\varphi(t) + \frac{L_f}{\Gamma(q)} \int_0^t (t-s)^{q-1} \, |y(s) - x(s)| \, ds + \sum_{k=1}^{m} K_I^k |y(t_k^-) - x(t_k^-)|. \end{aligned}$$

By Lemma 2.8, there exists a constant $M_{f,m}^* > 0$ independent of $\lambda_\varphi \varphi(t)$ such that

$$|y(t) - x(t)| \leq M_{f,m}^* (m + \lambda_\varphi)\varphi(t) := c_{f,m,\varphi}\varphi(t), \quad t \in J.$$

Thus, Eq. (2) is generalized Ulam–Hyers–Rassias stable. The proof is completed. □

Remark 4.9. (i) Under the assumptions of Theorem 4.8, we consider Eq. (2) and the inequality (19). One can repeat the same process to verify that Eq. (2) is Ulam–Hyers–Rassias stable.

(ii) Under the assumptions of Theorem 4.8, we consider Eq. (2) and the inequality (17). One can repeat the same process to verify that Eq. (2) is Ulam–Hyers stable.

(iii) One can extend the above results to case of Eq. (2) with $T = +\infty$.

5. Existence results for impulsive boundary value problems

This section deals with the existence and data dependence of solutions for the problem (3).

Definition 5.1. A function $u \in PC(J, R)$ is said to be a solution of the problem (3) if $u(t) = u_k(t)$ for $t \in (t_k, t_{k+1})$ and $u_k \in C([0, t_{k+1}], R)$, $k = 0, 1, 2, \ldots, m$ satisfies ${}^cD_t^q u_k(t) = f(t, u_k(t))$ a.e. on $(0, t_{k+1})$ with the restriction of $u_k(t)$ on $[0, t_k)$ is just $u_{k-1}(t)$, and the conditions $u(t_k^+) = u(t_k^-) + I_k(u(t_k^-))$, $k = 1, 2, \ldots, m$, and $au(0) + bu(T) = c$.

Lemma 5.2. *Let $q \in (0, 1)$ and $h : J \to R$ be continuous. A function u is a solution of the fractional integral equation*

$$u(t) = \begin{cases} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds \\ \quad - \frac{1}{a+b}\left[\sum_{i=1}^m bI_i(u(t_i^-)) + \frac{b}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds - c\right], \quad \text{for } t \in [0, t_1], \\ I_1(u(t_1^-)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds \\ \quad - \frac{1}{a+b}\left[\sum_{i=1}^m bI_i(u(t_i^-)) + \frac{b}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds - c\right], \quad \text{for } t \in (t_1, t_2], \\ I_1(u(t_1^-)) + I_2(u(t_2^-)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds \\ \quad - \frac{1}{a+b}\left[\sum_{i=1}^m bI_i(u(t_i^-)) + \frac{b}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds - c\right], \quad \text{for } t \in (t_2, t_3], \\ \vdots \\ \sum_{i=1}^m I_i(u(t_i^-)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds \\ \quad - \frac{1}{a+b}\left[\sum_{i=1}^m bI_i(u(t_i^-)) + \frac{b}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds - c\right], \quad \text{for } t \in (t_m, T], \end{cases} \tag{21}$$

if and only if u is a solution of the following impulsive boundary value problems

$$\begin{cases} {}^cD_t^q u(t) = h(t), \quad t \in (0, T], \\ \Delta u(t_k) = I_k(u(t_k^-)), \quad k = 1, 2, \ldots, m, \\ au(0) + bu(T) = c. \end{cases} \tag{22}$$

Proof. Assume u satisfies (22). If $t \in [0, t_1]$ then

$${}^cD_t^q u(t) = h(t), \quad t \in (0, t_1]. \tag{23}$$

Integrating the expression (12) from 0 to t by the fractional integral definition, one can obtain

$$u(t) = u(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds.$$

If $t \in (t_1, t_2]$ then

$${}^cD_t^q u(t) = h(t), \quad t \in (t_1, t_2] \text{ with } u(t_1^+) = u(t_1^-) + I_1(u(t_1^-)).$$

By Lemma 3.2, one obtain

$$\begin{aligned} u(t) &= u(t_1^+) - \frac{1}{\Gamma(q)}\int_0^{t_1} (t_1-s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds \\ &= u(t_1^-) + I_1(u(t_1^-)) - \frac{1}{\Gamma(q)}\int_0^{t_1} (t_1-s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds \\ &= u(0) + I_1(u(t_1^-)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds. \end{aligned}$$

If $t \in (t_2, t_3]$ then again from Lemma 3.2 we get

$$\begin{aligned} u(t) &= u(t_2^+) - \frac{1}{\Gamma(q)}\int_0^{t_2} (t_2-s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds \\ &= u(t_2^-) + I_2(u(t_2^-)) - \frac{1}{\Gamma(q)}\int_0^{t_2} (t_2-s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds \\ &= u(0) + I_1(u(t_1^-)) + I_1(u(t_2^-)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds. \end{aligned}$$

If $t \in (t_m, T]$ then again from Lemma 3.2 we derive

$$u(t) = u(0) + \sum_{i=1}^{m} I_i(u(t_i^-)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds.$$

From the boundary value condition $au(0) + bu(T) = c$ we get

$$u(0) = -\frac{1}{a+b} \left[\sum_{i=1}^{m} bI_i(u(t_i^-)) + \frac{b}{\Gamma(q)} \int_0^T (T-s)^{q-1} h(s) ds - c \right].$$

This gives formula (21).

Conversely, assume that u satisfies (21). If $t \in (0, t_1]$ and using the fact that ${}^cD_t^q$ is the left inverse of I_t^q we get (23). If $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$ and using the fact of the Caputo derivative of a constant is equal to zero, we obtain ${}^cD_t^q u(t) = h(t)$, $t \in (t_{k-1}, t_k]$ and $u(t_k^+) - u(t_k^-) = I_k(u(t_k^-))$, $k = 1, 2, \ldots, m$. For $t \in (t_m, T]$, we have $au(0) + bu(T) = c$. This completes the proof. □

Now, define operators as follows:

$F_1 : PC(J, R) \to PC(J, R)$ given by

$$(F_1 u)(t) = \frac{a}{a+b} \sum_{i=1}^{k} I_i(u(t_i^-)) - \frac{b}{a+b} \sum_{i=k+1}^{m} I_i(u(t_i^-)) + \frac{c}{a+b}, \quad t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \ldots, m.$$

$G_{11} : PC(J, R) \to PC(J, R)$ given by

$$(G_{11} u)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, \quad t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \ldots, m.$$

$G_{12} : PC(J, R) \to PC(J, R)$ given by

$$(G_{12} u)(t) = -\frac{b}{(a+b)\Gamma(q)} \int_0^T (T-s)^{q-1} f(s, u(s)) ds, \quad t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \ldots, m.$$

Let $\mathbb{T}_1 : PC(J, R) \to PC(J, R)$ given by

$$\mathbb{T}_1 u = F_1 u + G_{11} u + G_{12} u.$$

Thus, the existence of a solution for the problem (3) is equivalent to the existence of a fixed point for operator $\mathbb{T}_1$.

Proofs of the following lemmas follow like above, so we omit them.

Lemma 5.3. *Under assumptions* [H1], [H4] *and* [H5], *the operator* $F_1 : PC(J, R) \to PC(J, R)$ *is Lipschitz with constant* $\bar{K}_I = \frac{\max\{|a|, |b|\}}{|a+b|} \sum_{i=1}^{m} K_I^i$. *Consequently* F_1 *is* α*-Lipschitz with the same constant* $\bar{K}_I$. *Moreover,* F_1 *satisfies the following growth condition:*

$$\|F_1 u\|_{PC} \le \frac{\max\{|a|, |b|\}}{|a+b|} m(C_I \|u\|_C^{q_2} + M_I) + \frac{|c|}{|a+b|}, \tag{24}$$

for every $u \in PC(J, R)$.

Lemma 5.4. *Under assumptions* [H1] *and* [H2], *the operators* $G_{11}, G_{12} : PC(J, R) \to PC(J, R)$ *are continuous and compact satisfying the following growth conditions:*

$$\|G_{11} u\|_{PC} \le \frac{T^q (C_f \|u\|_{PC}^{q_1} + M_f)}{\Gamma(q+1)}, \qquad \|G_{12} u\|_{PC} \le \frac{|b| T^q (C_f \|u\|_{PC}^{q_1} + M_f)}{|a+b|\Gamma(q+1)}, \tag{25}$$

for every $u \in PC(J, R)$.

Now, we are ready to prove the main result of this section.

Theorem 5.5. *Assume that* [H1], [H2], [H4] *and* [H5] *hold. Suppose* $\bar{K}_I < 1$. *Then the problem* (3) *has at least one solution* $u \in PC(J, R)$ *and the set of the solutions of the problem* (3) *is bounded in* $PC(J, R)$.

Proof. Let $F_1, G_{11}, G_{12}, \mathbb{T}_1 : PC(J, R) \to PC(J, R)$ be the operators defined in the beginning of this section. They are continuous and bounded. Moreover, F_1 is α-Lipschitz with the constant $\bar{K}_I$ and $G_{11} + G_{12}$ is α-Lipschitz with zero constant. Also, Proposition 2.2 shows us that $\mathbb{T}_1$ is a strict α-contraction with the constant $\bar{K}_I$.

Set

$$S_1 = \{u \in PC(J, R) : (\exists)\lambda \in [0, 1] \text{ such that } u = \lambda \mathbb{T}_1 u\}.$$

Next, we prove that S_1 is bounded in $PC(J, R)$. Consider $u \in S_1$ and $\lambda \in [0, 1]$ such that $u = \lambda \mathbb{T}_1 u$. It follows from (24) and (25) that

$$\begin{aligned} \|u\|_{PC} &= \lambda \|\mathbb{T}_1 u\|_{PC} \\ &\le \lambda(\|F_1 u\|_{PC} + \|G_{11} u\|_{PC} + \|G_{12} u\|_{PC}) \\ &\le \frac{m \max\{|a|, |b|\}(C_I \|u\|_C^{q_2} + M_I)}{|a+b|} + \frac{|c|}{|a+b|} + \left(1 + \frac{|b|}{|a+b|}\right) \frac{T^q (C_f \|u\|_C^{q_1} + M_f)}{\Gamma(q+1)}. \end{aligned} \tag{26}$$

This inequality (26), together with $q_1 < 1, q_2 < 1$, shows us that S_1 is bounded in $PC(J, R)$. Consequently, by Theorem 2.5 we deduce that $\mathbb{T}_1$ has at least one fixed point and the set of the fixed points of $\mathbb{T}_1$ is bounded in $PC(J, R)$. □

Remark 5.6. (i) If the growth condition [H2] is formulated for $q_1 = 1$, then the conclusions of Theorem 5.5 remain valid provided that $\left(1 + \frac{|b|}{|a+b|}\right) \frac{T^q C_f}{\Gamma(q+1)} < 1$;

(ii) If the growth condition [H5] is formulated for $q_2 = 1$, then the conclusions of Theorem 5.5 remain valid provided that $\frac{m \max\{|a|,|b|\} C_I}{|a+b|} < 1$;

(iii) If the growth conditions [H2] and [H5] are formulated for $q_1 = 1$ and $q_2 = 1$, then the conclusions of Theorem 5.5 remain valid provided that $\left(1 + \frac{|b|}{|a+b|}\right) \frac{T^q C_f}{\Gamma(q+1)} + \frac{m \max\{|a|,|b|\} C_I}{|a+b|} < 1$.

(iv) For $q_1, q_2 \in [0, 1)$, one can also obtain the boundedness of the set S_1 by virtue of Lemma 2.9.

(v) We do not obtain the boundedness of the set S_1 without putting any restrict condition.

To end this section, we give the following data dependence results.

Theorem 5.7. *Assume that* [H1]–[H5] *hold and* $\bar{K}_I < 1$. *Let* $u(\cdot)$ *and* $v(\cdot)$ *be solutions of the problem* (3) *with boundary conditions* $au(0) + bu(T) = c$ *and* $av(0) + bv(T) = c$, *respectively. Then it holds*

$$\|u - v\|_{PC} \le \left[\frac{L_f T^q}{(1 - \bar{K}_I)\Gamma(q+1)} \left(1 + \frac{\max\{|a|, |b|\}}{|a| + |b|}\right) \right]^{\frac{1}{1-\lambda}}.$$

Proof. Without loss of generality, for $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$, using [H3] and [H4], we obtain

$$\begin{aligned} |u(t) - v(t)| \le \; & \frac{\max\{|a|, |b|\} \sum_{i=1}^{k} K_I^i}{|a+b|} \|u - v\|_{C((t_k, t_{k+1}], R)} + \frac{L_f}{\Gamma(q)} \int_0^t (t-s)^{q-1} |u(s) - v(s)|^{\lambda} ds \\ & + \frac{|b| L_f}{|a+b| \Gamma(q)} \int_0^T (T-s)^{q-1} |u(s) - v(s)|^{\lambda} ds, \end{aligned}$$

which implies that

$$\left(1 - \bar{K}_I\right) |u(t) - v(t)| \le \frac{L_f}{\Gamma(q)} \int_0^t (t-s)^{q-1} |u(s) - v(s)|^{\lambda} ds + \frac{|b| L_f}{|a+b| \Gamma(q)} \int_0^T (T-s)^{q-1} |u(s) - v(s)|^{\lambda} ds.$$

By Lemma 2.9, we obtain

$$\|u - v\|_{PC} \le M^*,$$

where M^* is the only positive solution of the equation

$$M^* = \frac{L_f T^q M^{*\lambda}}{(1 - \bar{K}_I)\Gamma(q+1)} \left(1 + \frac{\max\{|a|, |b|\}}{|a| + |b|}\right).$$

Note that $M^* = \left[\frac{L_f T^q}{(1-\bar{K}_I)\Gamma(q+1)} \left(1 + \frac{\max\{|a|,|b|\}}{|a|+|b|}\right) \right]^{\frac{1}{1-\lambda}}$. This completes the proof. □

Remark 5.8. Under the assumptions of Theorem 5.7, we do not obtain the uniqueness of the solutions.

6. Applications

In this section, we present two examples to indicate how our theorems can be applied to concrete problems.

Example 6.1. Let us consider Cauchy problems for nonlinear impulsive fractional differential equations

$$\begin{cases} {}^cD^{\frac{2}{3}}u(t) = \dfrac{|u(t)|^{\frac{1}{2}}}{(1+99e^t)\left(1+|u(t)|^{\frac{1}{2}}\right)}, \quad t \in (0,1] \setminus \left\{\dfrac{1}{2}\right\}, \\ \Delta u\left(\dfrac{1}{2}\right) = \dfrac{\left|u\left(\frac{1}{2}^-\right)\right|}{100\left(1+\left|u\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}\right)}, \\ u(0) = 0, \end{cases} \tag{27}$$

and boundary value problems for nonlinear impulsive fractional differential equations

$$\begin{cases} {}^cD^{\frac{2}{3}}u(t) = \dfrac{|u(t)|^{\frac{1}{2}}}{(1+99e^t)\left(1+|u(t)|^{\frac{1}{2}}\right)}, \quad t \in (0,1] \setminus \left\{\dfrac{1}{2}\right\}, \\ \Delta u\left(\dfrac{1}{2}\right) = \dfrac{\left|u\left(\frac{1}{2}^-\right)\right|}{100\left(1+\left|u\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}\right)}, \\ 99u(0) = -u(1). \end{cases} \tag{28}$$

Set

$$f(t,u) = \frac{u^{\frac{1}{2}}}{(1+99e^t)\left(1+u^{\frac{1}{2}}\right)}, \quad (t,u) \in [0,1] \times [0,+\infty).$$

Let $u_1, u_2 \in [0,\infty)$ and $t \in [0,1]$. Then we have

$$\begin{aligned} |f(t,u_1) - f(t,u_2)| &= \frac{1}{(1+99e^t)} \left| \frac{u_1^{\frac{1}{2}}}{1+u_1^{\frac{1}{2}}} - \frac{u_2^{\frac{1}{2}}}{1+u_2^{\frac{1}{2}}} \right| \\ &= \frac{\left|u_1^{\frac{1}{2}} - u_2^{\frac{1}{2}}\right|}{(1+99e^t)\left(1+u_1^{\frac{1}{2}}\right)\left(1+u_2^{\frac{1}{2}}\right)} \\ &\le \frac{1}{100}|u_1 - u_2|^{\frac{1}{2}}. \end{aligned}$$

Obviously, for all $u \in [0,\infty)$ and each $t \in [0,1]$,

$$|f(t,u)| = \frac{1}{1+99e^t}\left|\frac{u^{\frac{1}{2}}}{1+u^{\frac{1}{2}}}\right| \le \frac{1}{100}\,|u|^{\frac{1}{2}}\,.$$

Set

$$I_1\left(u\left(\frac{1}{2}^-\right)\right) = \frac{\left|u\left(\frac{1}{2}^-\right)\right|}{100\left(1+\left|u\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}\right)}.$$

Also,

$$\left|I_1\left(u_1\left(\frac{1}{2}^-\right)\right) - I_1\left(u_2\left(\frac{1}{2}^-\right)\right)\right| \le \frac{\left|u_1\left(\frac{1}{2}^-\right) - u_2\left(\frac{1}{2}^-\right)\right|}{100},$$

$$\left|I_1\left(u\left(\frac{1}{2}^-\right)\right)\right| \le \frac{\left|u\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{100}.$$

Denote $q = \frac{2}{3}, q_1 = \lambda = \frac{1}{2}, q_2 = \frac{1}{2}, m = 1, L_f = C_f = \frac{1}{100}, K_I = C_I = \frac{1}{100}, M_I = M_f = 0, a = 99, b = 1, c = 0.$

- It is not difficult to see that all the assumptions in Theorem 3.7 are satisfied. Thus, the problem (27) has a solution in $PC([0,1],R)$.
- It is obvious that all the assumptions in Theorem 5.5 are satisfied. Thus, the problem (28) has a solution in $PC([0,1],R)$.

Example 6.2. We consider the impulsive fractional differential equation

$$\begin{cases} {}^cD_t^q x(t) = 0, \quad t \in (0,1] \setminus \left\{\frac{1}{2}\right\}, \\ \Delta x\left(\frac{1}{2}\right) = \dfrac{\left|x\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{1+\left|x\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}, \end{cases} \tag{29}$$

and the inequalities

$$\begin{cases} |{}^cD_t^q y(t)| \le \epsilon, \quad t \in (0,1] \setminus \left\{\frac{1}{2}\right\}, \\ \left|\Delta y\left(\frac{1}{2}\right) - \dfrac{\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{1+\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}\right| \le \epsilon, \quad \epsilon > 0. \end{cases} \tag{30}$$

Let $y \in PC([0,1],R)$ be a solution of the inequality (30). Then there exist $g \in PC([0,1],R)$ and $g_1 \in R$ such that:

(i) $|g(t)| \le \epsilon, \quad t \in [0,1], \qquad |g_1| \le \epsilon$

(ii) $${}^cD_t^q y(t) = g(t), \quad t \in [0,1] \setminus \left\{\frac{1}{2}\right\}, \tag{31}$$

(iii) $$\Delta y\left(\frac{1}{2}\right) = \frac{\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{1+\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}} + g_1. \tag{32}$$

Integrating (31) from 0 to t by virtue of Definition 1.1 via (32), we have

$$y(t) = y(0) + \chi_{(1/2,1]}(t)\left(\frac{\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{1+\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}} + g_1\right) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g(s)ds$$

for the characteristic function $\chi_{(1/2,1]}(t)$ of $(1/2, 1]$.

Let us take the unique solution $x(t)$ of (29) given by

$$x(t) = y(0) + \chi_{(1/2,1]}(t)\frac{|x(\frac{1}{2}^-)|^{\frac{1}{2}}}{1+|x(\frac{1}{2}^-)|^{\frac{1}{2}}}.$$

Then we have

$$\begin{aligned} |y(t)-x(t)| &= \left|\chi_{(1/2,1]}(t)\left(\frac{\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}}{1+\left|y\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}}} - \frac{|x(\frac{1}{2}^-)|^{\frac{1}{2}}}{1+|x(\frac{1}{2}^-)|^{\frac{1}{2}}} + g_1\right) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g(s)ds\right| \\ &\le \chi_{(1/2,1]}(t)\left|y\left(\frac{1^-}{2}\right) - x\left(\frac{1^-}{2}\right)\right|^{\frac{1}{2}} + |g_1| + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|g(s)|ds \\ &\le \chi_{(1/2,1]}(t)\left|y\left(\frac{1^-}{2}\right) - x\left(\frac{1^-}{2}\right)\right|^{\frac{1}{2}} + \epsilon + \frac{\epsilon}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\ &\le \chi_{(1/2,1]}(t)\left|y\left(\frac{1^-}{2}\right) - x\left(\frac{1^-}{2}\right)\right|^{\frac{1}{2}} + \epsilon + \frac{\epsilon}{\Gamma(q+1)}, \quad t \in [0,1], \end{aligned}$$

which gives

$$|y(t) - x(t)| \leq \left(1 + \frac{1}{\Gamma(q+1)}\right)\epsilon + \sqrt{1 + \frac{1}{\Gamma(q+1)}}\sqrt{\epsilon}, \quad t \in [0, 1].$$

So, Eq. (29) is generalized Ulam–Hyers stable.

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