Topology and its Applications 159 (2012) 3453–3460

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Some asymptotic stationary point theorems in topological spaces

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ARTICLE INFO

Article history:
Received 17 January 2012
Received in revised form 10 August 2012
Accepted 10 August 2012

MSC:
primary 54H25, 54C60
secondary 47H08, 47H10

Keywords:
Stationary point
Topological contraction
Measure of noncompactness
Generalized μ-set contraction

ABSTRACT

In this paper, we present some asymptotic stationary point results for topological contraction mappings by relaxing the compactness of the space. Moreover, some classes of topological contractions are characterized.

1. Introduction and preliminaries

Let X be a nonempty set and $T : X \multimap X$ be a set-valued map with nonempty values. An element $x \in X$ is said to be a stationary point (or endpoint or strict fixed point) of T, if $T(x) = \{x\}$. By "asymptotic stationary point theory" we mean results in which the existence of stationary points of a set-valued map T is established with the aid of assumptions on the iterates T^n of T. The existence of stationary points of set-valued maps has significant applications in the optimization theory, fixed point theory and Ekeland's variational principle; for more details see [2,6,7,9,10,16]. The most of the existence of stationary point results and asymptotic stationary points results are in metric spaces and uniform spaces (see [2–6,8–16] and references therein). In the most of the methods for obtaining those results, the authors have used the ideas of Banach contraction principle and its generalizations. Recently Tarafdar and Yuan [9] introduced the notion of topological contraction and proved that every upper semicontinuous set-valued topological contraction with closed values on compact topological spaces has a unique stationary point. Our goal in this work is to derive an asymptotic version of this result and extend our result by relaxing the compactness of the space for generalized μ-set contraction mappings. Also, we show that any generalized sequence of iterations (x_n) with an arbitrary initial point x_1 of set-valued map T on a complete first countable Hausdorff uniform space, converges to stationary point of T. Furthermore, we prove that some recent classes of set-valued maps the stationary points of which have been studied satisfy the assumptions of our results.

Let us introduce some definitions and facts which will be used in the sequel. Suppose that X is a topological space, a set-valued map $T : X \multimap X$ is said to be a topological contraction if for every nonempty compact subset A of X with

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0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2012.08.007

$T(A) = A$, A is singleton, i.e., A is a stationary point of T. Let X and Y be topological spaces, a set-valued map $T : X \multimap Y$ is said to be:

(i) closed, if $T(A)$ is a closed set for any closed subset A of X,
(ii) upper semicontinuous, if for each closed set $B \subseteq Y$, $T^-(B) = \{x \in X\colon\ T(x) \cap B \neq \emptyset\}$ is closed in X.

Let X be a topological space and $(\mathcal{C}, \tau)$ be a topological lattice with minimal element which we denote by 0. Suppose that $\mathcal{B}$ is a collection of nonempty sets of X such that $\overline{A}, A \cup B \in \mathcal{B}$ for any $A, B \in \mathcal{B}$. A measure of noncompactness on X with respect to $\mathcal{B}$ is simply any functional $\mu : \mathcal{B} \to \mathcal{C}$ such that:

(i) $\mu(\overline{A}) = \mu(A)$ for all $A \in \mathcal{B}$;
(ii) $\mu(A) = 0$ if and only if A is relatively compact;
(iii) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for all $A, B \in \mathcal{B}$.

It follows immediately that if $A \subseteq B$, then $\mu(A) \leqslant \mu(B)$. A sequence $(A_n)_{n=1}^{\infty}$ of nonempty subsets in $\mathcal{B}$ is called *μ-descending*, if A_n is closed, $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu(A_n) = 0$. We say that μ has the Kuratowski property, if the intersection $A = \bigcap_{n\in\mathbb{N}} A_n$ is nonempty and compact for any μ-descending sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{B}$.

Let X be a topological space and $\mu : 2^X \to [0, \infty]$ be a measure of noncompactness on X. A set-valued map $T : X \multimap X$ is said to be generalized μ-set contraction, if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for $A \subseteq X$ with $\varepsilon \leqslant \mu(A) < \varepsilon + \delta$, there exists $n \in \mathbb{N}$ such that $\mu(T^n(A)) < \varepsilon$.

The following lemma characterizes a generalized μ-set contraction.

Lemma 1.1. *([1]) Let X be a topological space and μ be a measure of noncompactness on X. Then T is a generalized μ-set contraction on X if and only if for every subset A of X such that $T(A) \subseteq A$ and $\mu(A) < \infty$, we have*

$$\lim_{n\to\infty} \mu\big(T^n(A)\big) = 0.$$

Motivated by the above lemma, we introduce the concept of a generalized μ-set contraction whenever the image of the measure of noncompactness μ contained in a topological lattice $\mathcal{C}$.

Definition 1.2. Let X be a topological space and $(\mathcal{C}, \tau)$ be a topological lattice with minimal element 0. Suppose that $\mathcal{B}$ is a collection of nonempty subsets of X such that $\overline{A}, A \cup B \in \mathcal{B}$ for any $A, B \in \mathcal{B}$. If $\mu : \mathcal{B} \to \mathcal{C}$ is a measure of noncompactness, then a set-valued map $T : X \multimap X$ is said to be a generalized μ-set contraction with respect to (w.r.t.) $\mathcal{B}$, if $\lim_{n\to\infty} \mu(T^n(A)) = 0$ for every $A \in \mathcal{B}$.

2. Stationary points in an arbitrary topological space

In this section, we present some stationary point results for set-valued maps on noncompact topological spaces. Since we decided to study asymptotic stationary point theory for topological contraction maps, then it is a natural question to pose that under which conditions a set-valued map T is a topological contraction whenever its iterates T^n of T is a topological contraction for some n. The following lemma gives an answer to this question.

Lemma 2.1. *Let X be a topological space and $T : X \multimap X$ be a set-valued map, then the following statements hold*:

(i) *If T^n, for some $n \in \mathbb{N}$, is a topological contraction, then T is a topological contraction.*
(ii) *If T is a topological contraction, $A \subseteq X$ compact, $T(A) \subseteq A$ and $T^n(A) = A$, then A is singleton.*
(iii) *If T is a topological contraction and $T(A)$ is compact for any compact subset A of X, then T^n is a topological contraction for each $n \in \mathbb{N}$.*

Proof. (i) Let A be a compact subset of X and $T(A) = A$, then $T^n(A) = A$. Since T^n is a topological contraction, then A is singleton.

(ii) By our assumption $A \supseteq T(A) \supseteq T^2(A) \supseteq \cdots \supseteq T^n(A) = A$, then $T(A) = A$. Since T is a topological contraction, thus A is singleton.

(iii) Let A be a compact subset of X and $T^2(A) = A$, then $T(T(A) \cup A) = T(A) \cup A$. Since $T(A) \cup A$ is compact and T is a topological contraction, then $T(A) \cup A$ is singleton. Therefore, T^2 is a topological contraction. By induction, we deduce that T^n is also a topological contraction. □

In the following example we show that T^n is not in general a topological contraction for $n \geqslant 2$ when T is a topological contraction. Hence, parts (ii) and (iii) of Lemma 2.1 do not hold without extra conditions on T.

Example 2.2. Let $X := [0, \infty)$ and $T : X \multimap X$ be defined as follows:

$$T(x) = \begin{cases} 0 & x = 0, \\ \{10x, 12x\} \cup \{100, 180\} & 0 < x < 15, \\ 0 & 15 \leqslant x < 100, \\ ]10, 15[& 100 \leqslant x \leqslant 180, \\ 0 & x > 180. \end{cases}$$

It is clear that T is a topological contraction, but we have

$$T^2\big([100, 180]\big) = [100, 180].$$

Therefore, T^2 is not a topological contraction.

In the following we give an existence result of stationary points on compact topological spaces.

Theorem 2.3. *Let X be a compact Hausdorff topological space and $T : X \multimap X$ be a topological contraction. Suppose that there exists $p \in \mathbb{N}$ such that T^p is upper semicontinuous and closed values. Then T has a unique stationary point x_0 and $\{x_0\} = \bigcap_{n=0}^{\infty} T^n(X)$.*

Proof. Let $X_n := T^{pn}X$ for each $n \in \mathbb{N}$ and $X_0 := X$. Then the sequence $(X_n)_{n=1}^{\infty}$ is compact and decreasing. Therefore, $K = \bigcap_{n=0}^{\infty} X_n$ is nonempty and compact. We show that $T^p(K) = K$. It is clear that $T^p(K) \subseteq K$. Suppose that $z \in K$, then $z \in T^{pn+p}(X)$ and so $(T^p)^-(z) \cap T^{pn}(X) \neq \emptyset$ for any $n \in \mathbb{N}$. If $Y_n := (T^p)^-(z) \cap T^{pn}(X)$, then (Y_n) is a decreasing sequence of nonempty compact subsets of X (note that T^p is upper semicontinuous with closed values). Therefore, $\bigcap_{n=0}^{\infty} Y_n \neq \emptyset$. If $x \in \bigcap_{n=0}^{\infty} Y_n$, then $z \in T^p(x) \subseteq T^p(\bigcap_{n=0}^{\infty} Y_n) \subseteq T^p(K)$. So $K \subseteq T^p(K)$ and we have $T^p(K) = K$.

On the other hand $X \supseteq TX \supseteq T^2X \supseteq \cdots$, thus $K = \bigcap_{n=0}^{\infty} T^n X$ and $T(K) \subseteq K$. Since T is a topological contraction, hence by part (ii) of Lemma 2.1, K is a singleton. Therefore, T has a unique stationary point. □

The following example shows that Theorem 2.3 is an improvement of Theorem 1 in [9] and Theorem 2.2 of [13].

Example 2.4. Let $X := [0, 1]$ and $T : X \multimap X$ be defined as follows:

$$T(x) = \begin{cases} \{1\} & \text{if } x \text{ is rational}, \\ \mathbb{Q} \cap [0, \frac{1}{2}] & \text{otherwise}. \end{cases}$$

Then, T is not upper semicontinuous and $T(x)$ is not closed for any $x \notin \mathbb{Q} \cap [0, 1]$. But $T^2(x) = \{1\}$ for all $x \in X$. Therefore, T satisfies all of the assumptions of Theorem 2.3, but it does not satisfy the assumptions of Theorem 2.2 in [13].

Theorem 2.5. *Let X be a Hausdorff topological space, $(\mathcal{C}, \tau)$ be a topological lattice with minimal element 0 and $\mu : 2^X \to \mathcal{C}$ be a measure of noncompactness on X with the Kuratowski property. Suppose that $T : X \multimap X$ is a topological contraction and a generalized μ-set contraction w.r.t. $\mathcal{B} = 2^X$. Let there exists $p \in \mathbb{N}$ such that T^p is upper semicontinuous and closed. Then T has a unique stationary point x_0 and $\{x_0\} = \bigcap_{n=0}^{\infty} T^n(X)$.*

Proof. Let $X_n := T^{pn}(X)$ for all $n \in \mathbb{N} \cup \{0\}$, where $X_0 = X$. Clearly, X_n is closed and $X_{n+1} \subseteq X_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is a generalized μ-set contraction w.r.t. $\mathcal{B} = 2^X$, then $\mu(X_n) = \mu(T^{pn}(X)) \to 0$. As μ has the Kuratowski property, then $K := \bigcap_{n=0}^{\infty} X_n$ is nonempty and compact. We show that $T^p(K) = K$. It is clear that $T^p(K) \subseteq K$. For the converse, let $x \in K$ and $Y_n := (T^p)^-(x) \cap X_n$ for each $n \in \mathbb{N}$. Since $x \in K$, then $x \in T^p(X_n)$, so there exists $z \in X_n$ such that $x \in T^p(z)$. Therefore, $z \in X_n \cap (T^p)^-(x) = Y_n$, that is Y_n is nonempty. Since T^p is upper semicontinuous and X_n is closed, then Y_n is closed. Also, $\mu(Y_n) \leqslant \mu(X_n) \to 0$. Therefore, $\bigcap_{n=1}^{\infty} Y_n \neq \emptyset$. If $y \in \bigcap_{n=1}^{\infty} Y_n$, then $y \in K$ and $x \in T^p(y)$. Hence, $K \subseteq T^p(K)$. Furthermore, $K = \bigcap_{n=0}^{\infty} T^n(X)$ and so $T(K) \subseteq K$. Then by part (ii) of Lemma 2.1, K is singleton. Thus, T has a unique stationary point. □

As a consequence of Theorem 2.5, we obtain the following fixed point theorem which improves Corollary 4 of [10].

Corollary 2.6. *Let X be a Hausdorff topological space, $(\mathcal{C}, \tau)$ be a topological lattice with minimal element 0 and $\mu : 2^X \to \mathcal{C}$ be a measure of noncompactness on X with the Kuratowski property. Suppose that $f : X \to X$ is a topological contraction and a generalized μ-set contraction with respect 2^X. If there exists $p \in \mathbb{N}$ such that f^p is continuous and closed, then f has a unique fixed point.*

3. Stationary points in uniform spaces

In this section, we focus our intention on finding stationary points of set-valued maps on uniform spaces. Hence, let E be a Hausdorff uniform space with uniformity defined by a saturated family $\mathcal{D} = \{d_\alpha : \alpha \in \mathcal{A}\}$ of pseudo-metrics d_α, $\alpha \in \mathcal{A}$, uniformly continuous on $E \times E$. We denote by $B(E)$ the set of all nonempty bounded subsets of E, $C(E)$ the set of all nonempty closed subsets of E and by $CB(E)$ the set of all nonempty closed and bounded subsets of E. For $A \in B(E)$ and $\alpha \in \mathcal{A}$, we denote its diameter as

$$\delta_\alpha(A) = \sup\{d_\alpha(x, y): x, y \in A\} = \delta_\alpha(\overline{A}).$$

Then, the Kuratowski measure of noncompactness γ defined on $B(E)$ for each $\alpha \in \mathcal{A}$, as: $[\gamma(A)](\alpha) = \inf\{\delta > 0$: A admits a finite partition into subsets whose diameters with respect to the pseudo-metric d_α are no larger than $\delta\}$.

Let $x_1 \in X$, then a sequence (x_n) such that $x_n \in T^n(x_1)$, $T^n = T \circ T \circ \cdots \circ T$ (n-times), $n > 1$, is called a generalized sequence of iterations with initial point x_1. In Theorems 2.3 and 2.5, we established conditions guaranteeing the existence and uniqueness of stationary points. Now, we are going to show that under these conditions all generalized sequences of iterations of a set-valued map on a complete first countable Hausdorff uniform space converge to these stationary points. In order to prove these results, we need the following lemma:

Lemma 3.1. *Let E be a complete first countable Hausdorff uniform space and γ be the Kuratowski measure of noncompactness on E. Suppose that $T : E \multimap E$ is a generalized γ-set contraction with respect to $\mathcal{B} = 2^E$ and $\bigcap_{n=1}^\infty \overline{T^n(E)} = \{x_0\}$, then $\delta_\alpha(T^n(E)) \to 0$ for all $\alpha \in \mathcal{A}$. Furthermore, if $\bigcap_{n=1}^\infty T^n(E) = \{x_0\}$, then each generalized sequence of iterations (x_n) with an arbitrary initial point x_1, converges to x_0.*

Proof. Let $\alpha \in \mathcal{A}$ be arbitrary and fixed. Since T is a generalized γ-set contraction w.r.t. 2^E, then $\gamma(T^n(E)) \to 0$. Therefore, $\delta_\alpha(T^n(E))$ is finite for sufficiently large n. Without loss of generality, assume that $\delta_\alpha(T^n(E))$ is finite for any $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$, there exist $x_n, u_n \in T^n(E)$ such that

$$\delta_\alpha\big(T^n(E)\big) \leqslant d_\alpha(x_n, u_n) + \frac{1}{n}. \tag{1}$$

Now, we consider two decreasing sequences of sets (C_n) and (D_n) given by $C_n = \{x_i: i \geqslant n\}$ and $D_n = \{u_i: i \geqslant n\}$. Obviously $C_n, D_n \subseteq T^n(E)$ and $\gamma(C_1)(\alpha) = \gamma(C_n)(\alpha) \leqslant \gamma(T^n(E))(\alpha)$ and $\gamma(D_1)(\alpha) = \gamma(D_n)(\alpha) \leqslant \gamma(T^n(E))(\alpha)$ for every $n \in \mathbb{N}$. Therefore, $\gamma(C_1)(\alpha) = 0$ and $\gamma(D_1)(\alpha) = 0$. Since α is arbitrary, then C_1 and D_1 are relatively compact. Let (x_{n_k}) and (u_{n_k}) be subsequences of (x_n) and (u_n) respectively, such that $x_{n_k} \to \bar{x}$, $u_{n_k} \to \bar{u}$. Then $\bar{x}, \bar{u} \in \overline{T^n(E)}$ for all $n \in \mathbb{N}$. Since $\bigcap_{n=1}^\infty \overline{T^n(E)} = \{x_0\}$, then $\bar{x} = \bar{u} = x_0$. Therefore, from (1), we have $\delta_a(T^{n_k}(E))(\alpha) \to 0$. Since $(\delta_\alpha(T^n(E)))$ is a decreasing sequence, then $\lim_{n\to\infty} \delta_\alpha(T^n(E)) = 0$. Furthermore, if $\bigcap_{n=1}^\infty T^n(E) = \{x_0\}$, then $d_\alpha(x_n, x_0) \leqslant \delta_\alpha(T^n(E))$ for all $n > 1$ and $\alpha \in \mathcal{A}$ (note that $x_n \in T^n(x_1)$ for all $n > 1$). Hence, (x_n) converges to x_0. $\square$

By applying Theorem 2.5 and Lemma 3.1, we obtain the following result.

Theorem 3.2. *Let E be a complete first countable Hausdorff uniform space and γ be the Kuratowski measure of noncompactness on E. Suppose that $T : E \multimap E$ is a topological contraction and a generalized γ-set contraction w.r.t. $\mathcal{B} = 2^E$. Let there exists $p \in \mathbb{N}$ such that T^p is upper semicontinuous and closed. Then T has a unique stationary point x_0 in E and each generalized sequence of iterations (x_n) with an arbitrary initial point x_1 converges to x_0.*

Recently Włodarczyk et al. [11–15] obtained the existence of stationary points for some classes of set-valued maps which are defined on metric spaces or uniform spaces. We shall show that the classes of set-valued maps which were introduced in [11–15] are topological contraction and generalized γ-set contraction with respect to a subfamily of 2^E, where E is a uniform space or a metric space and γ is the Kuratowski measure of noncompactness.

In the following result we give a sufficient condition for topological contraction and generalized μ-set contraction set-valued mappings in uniform spaces.

Lemma 3.3. *Let $T : E \multimap E$ be a set-valued map and $\mathcal{B}$ be a collection of nonempty subsets of E which contains nonempty compact subsets of E. Suppose that $\lim_n \delta_\alpha(T^n(A)) = 0$ for any $\alpha \in \mathcal{A}$ and for every $A \in \mathcal{B}$. Then T is topological contraction and generalized γ-set contraction w.r.t. $\mathcal{B}$, where γ is the Kuratowski measure of noncompactness.*

Proof. Assume that on the contrary that T is not a topological contraction. Then there exists a nonempty compact subset A of E such that $T(A) = A$ and A is not singleton. Therefore, there exists $\alpha \in \mathcal{A}$ such that $\delta_\alpha(A) = r > 0$. Since $r = \delta_\alpha(A) = \delta_\alpha(T^n(A))$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \delta_\alpha(T^n(A)) = r > 0$, which is a contradiction.

On the other hand, for any $A \in \mathcal{B}$ we have

$$\gamma\big(T^n(A)\big)(\alpha) \leqslant \delta_\alpha\big(T^n(A)\big) \quad \forall \alpha \in \mathcal{A}.$$

Then $\gamma(T^n(A)) \to 0$ and so T is a generalized γ-set contraction w.r.t. $\mathcal{B}$. $\square$

The following example shows that the converse of Lemma 3.3 is not valid.

Example 3.4. Let c_0 be the null sequences space equipped with its canonical norm

$$\|(x_n)\|_\infty = \sup\{|x_n|: n \in \mathbb{N}\} \quad \forall (x_n) \in c_0.$$

Let B be the closed unit ball of c_0 and $T : B \to B$ be defined as follows:

For each $x = (x_1, x_2, \ldots, x_i, \ldots) \in B$,

$$T(x) = \begin{cases} (0, 0, \ldots) & \text{if } x_i = 0,\ \forall i \geqslant 2, \\ (x_1, \frac{1}{2}x_2, \frac{1}{2}x_3, \ldots) & \text{otherwise.} \end{cases}$$

We show that T is a topological contraction and a generalized γ-set contraction, where γ is the Kuratowski measure of noncompactness.

Let $A = \{x \in B\colon \exists i \geqslant 2,\ x_i \neq 0\}$ and $C \subset B$, then we have

$$\begin{aligned}\gamma(T(C)) &= \gamma\big(T\big((C\cap A)\cup(C\cap A^c)\big)\big) = \gamma\big(T(C\cap A)\cup T(C\cap A^c)\big) = \max\{\gamma(T(C\cap A)), \gamma(T(C\cap A^c))\} \\ &= \gamma(T(C\cap A)) = \gamma\left(\frac{1}{2}(C\cap A^c) + \frac{1}{2}(C\cap A)\right) \leqslant \gamma\left(\frac{1}{2}(C\cap A)\right) = \frac{1}{2}\gamma(C\cap A) \leqslant \frac{1}{2}\gamma(C).\end{aligned}$$

Therefore, T is a $\frac{1}{2}$-set contraction and by Proposition 2.3 of [1], T is a generalized γ-set contraction. Suppose that A is a compact subset of B and $T(A) = A$. Then, we get

$$T^n(A) = \left\{\left(x_1, \frac{1}{2^n}x_2, \frac{1}{2^n}x_3, \ldots\right);\ x = (x_1, x_2, \ldots) \in A\right\} = A.$$

Hence,

$$A = \lim_n T^n(A) = \{(x_1, 0, \ldots),\ x_1 \in [-1, 1],\ x \in A\}.$$

On the other hand $T(A) = \{(0, 0, \ldots)\} = A$, thus A is singleton and so T a is topological contraction.

Let $x = (0, 0, \ldots)$ and $y = (1, 1, 0, \ldots)$, then

$$\|T^n(x) - T^n(y)\| = \left\|\left(1, \frac{1}{2^n}, 0, \ldots\right)\right\| = 1.$$

Therefore, $\delta(T^n(B)) \nrightarrow 0$.

Now, we characterize some classes of set-valued maps which are fulfilled the assumptions of Lemma 3.3.

Definition 3.5. [11] The family

$$\mathcal{V} = \{V_\alpha : 2^E \to [0, \infty],\ \alpha \in \mathcal{A}\}$$

is said to be a $\mathcal{V}$-family of generalized pseudo-distances on E ($\mathcal{V}$-family, for short) if the following conditions hold:

($\mathcal{V}_1$) $\forall \alpha \in \mathcal{A}\ \forall A_1, A_2 \in 2^E\ \{A_1 \subseteq A_2 \Rightarrow V_\alpha(A_1) \leqslant V_\alpha(A_2)\}$,
($\mathcal{V}_2$) $\forall \alpha \in \mathcal{A}\ \forall x, y, z \in X,\ V_\alpha(\{x, z\}) \leqslant V_\alpha(\{x, y\}) + V_\alpha(\{y, z\})$,
($\mathcal{V}_3$) for any sequence (x_m) in E such that

$$\forall \alpha \in \mathcal{A}, \quad \lim_n \sup_m V_\alpha(\{x_n, x_m\}) = 0,$$

if there exists a sequence (y_m) in E satisfying

$$\forall \alpha \in \mathcal{A}, \quad \lim_m V_\alpha(\{x_m, y_m\}) = 0,$$

then

$$\forall \alpha \in \mathcal{A}, \quad \lim_m d_\alpha(\{x_m, y_m\}) = 0,$$

($\mathcal{V}_4$) $\exists \alpha_0 \in \mathcal{A}$ such that $V_{\alpha_0}(E) > 0$.

Definition 3.6. ([11]) Let $T : E \multimap E$ be a set-valued map and $\mathcal{V} = \{V_\alpha : 2^E \to [0,\infty],\ \alpha \in \mathcal{A}\}$ be a $\mathcal{V}$-family on E. Suppose that for each $\alpha \in \mathcal{A}$,

$$D_{\alpha;T,\mathcal{V}} = \big\{V_\alpha(A):\ A \subset E,\ T(A) \subset A,\ V_\alpha(A) > 0\big\}$$

and

$$H_{\alpha;T,\mathcal{V}} = \big\{V_\alpha\big(T^n(E)\big):\ V_\alpha\big(T^n(E)\big) > 0,\ n \in \{0\} \cup \mathbb{N}\big\}$$

where $T^0(E) = E$.

An Ω-family of generalized gauge maps (Ω-family, for short) is by definition a family $\Omega = \{\omega_{m;\alpha}\}_{\alpha\in\mathcal{A}}$ of maps $\omega_{m;\alpha} : H_{\alpha;T,\mathcal{V}} \to (0,\infty]$, $m \in \mathbb{N}$, $\alpha \in \mathcal{A}$, such that

$$\forall\alpha \in \mathcal{A}\ \forall\varepsilon > 0\ \exists\eta > 0\ \exists m \in \mathbb{N}\ \forall t \in [\varepsilon, \varepsilon+\eta),\quad \big\{t \in H_{\alpha;T,\mathcal{V}} \Rightarrow \omega_{m;\alpha}(t) \leqslant \varepsilon\big\}. \tag{2}$$

Definition 3.7. ([11]) Let $T : E \multimap E$ be a set-valued map. If there exist a $\mathcal{V}$-family and an Ω-family such that

$$\forall\alpha \in \mathcal{A}\ \forall m \in \mathbb{N}\ \forall n \in \{0\} \cup \mathbb{N},\quad \big\{V_\alpha\big(T^n(E)\big) > 0 \Rightarrow V_\alpha\big(T^m\big(T^n(E)\big)\big) < \omega_{m;\alpha}\big(V_\alpha\big(T^n(E)\big)\big)\big\}. \tag{3}$$

Then we say that T is a $(\mathcal{V},\Omega)$-asymptotic contraction on E.

Lemma 3.8. *If $T : E \multimap E$ is a $(\mathcal{V},\Omega)$-asymptotic contraction on E, then*

$$\forall\alpha \in \mathcal{A},\quad \lim_{n\to\infty} \delta_\alpha\big(T^n(E)\big) = 0.$$

Proof. Let $w_0 \in E$ and $w_m \in T^m(w_0)$ for $m \in \mathbb{N}$, be arbitrary and fixed, then by part (VI) of the proof of Theorem 2.3 in [11], we have

$$\forall\alpha \in A,\quad \lim_n \sup_{m>n} V_\alpha\big(\{w_n, w_m\}\big) = 0. \tag{4}$$

Let (u_n) be a sequence in E such that $u_n \in T^n(E)$, then $\{w_n, u_n\} \subseteq T^n(E)$ for any $n \in \mathbb{N}$. Therefore, by $(\mathcal{V}_1)$,

$$\forall\alpha \in \mathcal{A}\ \forall m \in \mathbb{N},\quad V_\alpha\big(\{w_n, u_n\}\big) \leqslant V_\alpha\big(T^n(E)\big).$$

But in the step (II) of the proof of Theorem 2.3 in [11], it has been shown that $V_\alpha(T^n(E)) \to 0$. Therefore, we have

$$\forall\alpha \in \mathcal{A},\quad \lim_n V_\alpha\big(\{w_n, u_n\}\big) = 0. \tag{5}$$

Hence, from (4), (5) and $(\mathcal{V}_3)$, we conclude that

$$\forall\alpha \in \mathcal{A},\quad \lim_n d_\alpha(w_n, u_n) = 0. \tag{6}$$

Moreover, by $(\mathcal{V}_2)$,

$$\forall\alpha \in \mathcal{A},\quad V_\alpha\big(\{u_m, u_n\}\big) \leqslant V_\alpha\big(\{u_m, w_m\}\big) + V_\alpha\big(\{w_m, w_n\}\big) + V_\alpha\big(\{w_n, u_n\}\big).$$

From (4) and (6), we get that

$$\forall\alpha \in \mathcal{A},\quad \lim_n \sup_{m>n} V_\alpha\big(\{u_m, u_n\}\big) = 0.$$

If (x_n) is another sequence in E such that $x_n \in T^n(E)$, then

$$\forall\alpha \in \mathcal{A},\quad \lim_n d_\alpha(x_n, u_n) = 0. \tag{7}$$

Let $\alpha \in \mathcal{A}$ be arbitrary and fixed. We claim that there exists $N \in \mathbb{N}$ such that $\delta_\alpha(T^N(E)) < \infty$. Assume on the contrary that $\delta_\alpha(T^n(E)) = \infty$ for every $n \in \mathbb{N}$. Hence, for every $n \in \mathbb{N}$ there exist $x_n, u_n \in T^n(E)$ such that $d_\alpha(x_n, u_n) \geqslant n$. Therefore,

$$\lim_{n\to\infty} d_\alpha(x_n, u_n) = \infty,$$

which contradicts (7). Thus, there exists N such that $\delta_\alpha(T^N(E)) < \infty$. Since $T^{n+1}(E) \subseteq T^n(E)$ for any $n \in \mathbb{N}$, then $\delta_\alpha(T^n(E)) < \infty$ for any $n \geqslant N$. Without loss of generality we can assume that $\delta_\alpha(T^n(E)) < \infty$ for any $n \in \mathbb{N}$. Since the sequence $(\delta_\alpha(T^n(E)))$ is nondecreasing, then there exists a real number r such that

$$\lim_{n\to\infty} \delta_\alpha\big(T^n(E)\big) = r \geqslant 0.$$

If $r \neq 0$, then

$$\exists N_0 \in \mathbb{N} \quad \delta_\alpha\big(T^n(E)\big) > \frac{r}{2} \quad \forall n \geqslant N_0.$$

Therefore, for any $n \geqslant N_0$, there exist $x_n, u_n \in T^n(E)$ such that $d_\alpha(x_n, u_n) > \frac{r}{2}$ which is a contradiction and so the proof is completed. □

From Lemmas 3.3 and 3.8 and Theorem 2.5, we can deduce the following facts.

Remark 3.9. Let $T : E \multimap E$ and γ be the Kuratowski measure of noncompactness.

(i) Suppose E is a metric space and T is a set-valued asymptotic contraction [Definition 2.1 of [12]]. According to the proof of Theorem 2.1 of [12], we have

$$\forall A \in B(E), \quad \lim_{n\to\infty} \delta\big(T^n(A)\big) = 0.$$

Then by Lemma 3.3, T is a topological contraction and a generalized γ-set contraction w.r.t. $B(E)$.

(ii) Suppose E is a uniform space and T is an asymptotic contraction of Meir–Keeler type [Definition 2.1 of [13]]. By the proof of Theorem 2.1 of [13], we have

$$\forall \alpha \in \mathcal{A}\ \forall A \in CB(E), \quad \lim_{n\to\infty} \delta_\alpha\big(T^n(A)\big) = 0.$$

Then by Lemma 3.3, T is a topological contraction and a generalized γ-set contraction w.r.t. $CB(E)$.

(iii) Suppose E is a uniform space and T is a contraction of Meir–Keeler type on E [Definition 2.1 of [15]]. In the proof of Theorem 2.1 of [15], it is shown that

$$\forall \alpha \in \mathcal{A}, \quad \lim_{n\to\infty} \delta_\alpha\big(T^n(E)\big) = 0.$$

Then by Lemma 3.3, T is a topological contraction and a generalized γ-set contraction w.r.t. 2^E.

(iv) Suppose E is a uniform space and T is a $(\mathcal{V}, \Omega)$-asymptotic contraction on E. By Lemma 3.8, we have

$$\forall \alpha \in \mathcal{A}, \quad \lim_{n\to\infty} \delta_\alpha\big(T^n(E)\big) = 0.$$

Lemma 3.3 implies that T is a topological contraction and a generalized γ-set contraction w.r.t. 2^E.

Now by Remark 3.9 and Theorem 3.2, we can deduce the following existence result of stationary points which is different from Theorem 2.2 in [14] and Theorems 2.1 of [11,13] and [15].

Theorem 3.10. *Let E be a complete first countable Hausdorff uniform space and $T : E \multimap E$ be a set-valued map. Assume that there exists $n_0 \in \mathbb{N}$ such that $\gamma(T^{n_0}(E))(\alpha) < \infty$, for every $\alpha \in \mathcal{A}$, where γ is the Kuratowski measure of noncompactness on E. Suppose that there exists $p \in \mathbb{N}$ such that T^p is upper semicontinuous and closed. If T^p is one of the mappings in Remark 3.9, then T has a unique stationary point x_0 and each generalized sequence of iterations (x_n) with an arbitrary initial point x_1 converges to x_0.*

Remark 3.11. (a) In the above corollary if T^p or T is one of the mapping satisfying conditions (iii) and (iv) of Remark 3.9, then there exists $n_0 \in \mathbb{N}$ such that $\gamma(T^{n_0}(E))(\alpha) < \infty$, for every $\alpha \in \mathcal{A}$.

(b) Though T^p for $p \geqslant 2$ defined in Example 2.4 satisfies all of the conditions of mappings in Remark 3.9, but T does not.

Acknowledgements

The authors thank the anonymous reviewer for the valuable comments and suggestions which helped in improving the paper. The first author was partially supported by the Center of Excellence for Mathematics, University of Shahrekord and by a grant from IPM (No. 90540010).

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