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On Distance-Regularity in Graphs

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If A is the adjacency matrix of a graph G, then A_i is the adjacency matrix of the graph on the same vertex set in which a pair of vertices is adjacent if and only if their distance apart is i in G. If G is distance-regular, then A_i is a polynomial of degree i in A. It is shown that the converse is also true. If A_i is a polynomial in A, not necessarily of degree i, G is said to be distance-polynomial. It is shown that this is a larger class of graphs and some of its properties are investigated.

1. INTRODUCTION

A graph G (finite, undirected, without loops or multiple edges) is called *distance-transitive* if whenever u, v, x, y are vertices of G and the distance from u to v is the same as that from x to y, then there is an automorphism of G sending u to x and v to y. Distance-transitivity is a *global* property of a graph which induces a *local* property usually referred to as a type of *regularity*. In order to give a precise definition we need the concept of *intersection number* of a graph. Let u, v be a pair of vertices of a graph G and denote by $\partial(u, v)$ the *distance* from u to v in G. That is, $\partial(u, v)$ is the length of the shortest path from u to v. The *intersection number* $s_{ijl}(u, v)$ is the number of vertices w, satisfying $\partial(u, w) = i$, $\partial(v, w) = j$ with $\partial(u, v) = l$.

DEFINITION. A graph G is called *distance-regular* if for all pairs of vertices u, v of G, with $\partial(u, v) = l$, the intersection numbers $s_{ijl}(u, v)$ depend only on i, j, and l and not on u, v.

Distance-transitive graphs are distance-regular (see [1, Chap. 20]). The converse is not true. A counterexample may be found on p. 139 of [1] or in [2]. In fact it seems clear from available evidence that no regularity (local) property implies *any* transitivity (global) property.

In this article we will examine distance-regularity and some variations of it. The principal device that we will use is a class of matrices which describe

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"distance" in a graph G and a polynomial algebra defined by one of these matrices.

DEFINITION. Let G be a graph with n vertices, $\{u_j\}_{j=1}^{n}$. The ith *distance matrix* A_i is an $n \times n$ matrix defined by

$$\begin{aligned}(A_i)_{rs} &= 1 \qquad \text{if} \quad \partial(u_r, u_s) = i\\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

Clearly, A_0 is the identity matrix and $A_1 = A$ is the so-called adjacency matrix of G. A_i is defined for $0 \leqslant i \leqslant d$, d the diameter of G. The relation between the matrices $\{A_i\}$ and the intersection numbers is given by $(A_iA_j)_{rs} = s_{ijl}(u_r, u_s)$, where $\partial(u_r, u_s) = l$.

The set of all polynomials with complex coefficients in A forms an algebra called the *adjacency algebra*. It has dimension s, as a vector space, where s is the number of distinct eigenvalues of the matrix A. (See [1, p. 12].) The key fact that will be the starting point of our investigation is that when G is distance-regular, then each A_i is a polynomial of degree i in the matrix A. (See [1, pp. 136–137].) In Section 2 we will show, among other things, that the converse also holds. In particular distance-regularity may be characterized by the fact that A acts, by left multipication, as a linear operator on the vector space spanned by $\{A_i\}_{i=0}^{d}$. A similar result for association schemes can be found in [3, p. 660].

In Section 3 we relax the condition that A_i has degree i in A and show that there exist graphs in which each A_i is a polynomial of degree possibly greater than i in A. We call such a graph *distance-polynomial.* It remains an open question whether this property is associated with some sort of transitivity.

In Section 4 we consider two other classes of graphs: those in which the A_i commute as matrices and those in which a specialized version of commutativity holds. This latter class of graphs are called *super-regular* and they are related to vertex-transitive graphs.

We will show by examples that all of these classes are distinct.

2. DISTANCE-REGULAR GRAPHS

In this section we will prove that distance-regularity can be characterized in several ways in terms of the matrices A_i.

2.1. THEOREM. *Let G be a graph of diameter d and A_i, the ith distance matrix of G. Then the following are equivalent:*

(a) *G is distance-regular.*

(b) *A_i is a polynomial of degree i in A for $i = 0, \dots, d$.*

(c) *A acts by left multiplication as a linear operator on the vector space $\langle I, A_1, A_2, \dots, A_d \rangle$.*

Proof. 1. (b) $\Rightarrow$ (c). Since each A_i is a polynomial of degree i in A it follows from the proof of 20.7 in [1] that $\{A_0, A_1, \dots, A_d\}$ is a basis for the adjacency algebra of A. Thus (c) follows.

2. (c) $\Rightarrow$ (a). For each $i = 0, \dots, d$, $AA_i = \sum_{j=0}^{d} \alpha_{ij} A_j$.

Let v_1, v_2 be a pair of vertices satisfying $\partial(v_1, v_2) = r \leqslant i - 2$. Then $(AA_i)_{12} = s_{1ir}(v_1, v_2)$ is the number of vertices of G a distance of 1 from v_1 and a distance of i from v_2. But since $\partial(v_1, v_2) = r \leqslant i - 2$, there are no such vertices and $(AA_i)_{12} = 0$. Hence,

$$0 = (AA_i)_{12} = \sum_{j=0}^{d} \alpha_{ij}(A_j)_{12} = \alpha_{ir}$$

since $(A_r)_{12} = 1$ and $(A_i)_{12} = 0$ for all $i \neq r$. Thus for all $r \leqslant i - 2$, $\alpha_{ir} = 0$. Now assume that v_3, v_4 are a pair of vertices of G with $\partial(v_3, v_4) = s \geqslant i + 2$. By a similar argument, $\alpha_{is} = 0$ for all $s \geqslant i + 2$. Hence we have

$$AA_i = \alpha_{i(i-1)} A_{i-1} + \alpha_{ii} A_i + \alpha_{i(i+1)} A_{i+1}.$$

It is now routine to conclude from this relationship that G is distance-regular.

3. (a) $\Rightarrow$ (b). This is just Theorem 20.7 of [1]. ∎

3. Distance-Polynomial Graphs

Definition. Let G be a graph with A_i a polynomial in A for each $i = 0, 1, \dots, d$, with d the diameter of G. Then G is called a *distance-polynomial* graph.

Every distance-regular graph is, of course, distance-polynomial. We now show that these two classes are distinct.

3.1. Lemma. *If the graph G is regular, connected and of diameter* 2, *then G is distance-polynomial.*

Proof. Consider the sum $I + A_1 + A_2 = J$. Since G is regular and connected, J is a polynomial in A_1 say $J = q(A)$. Then $A_2 = J - I - A_1 = J - I - A = q(A) - I - A$, a polynomial in A. Thus G is distance-polynomial. ∎

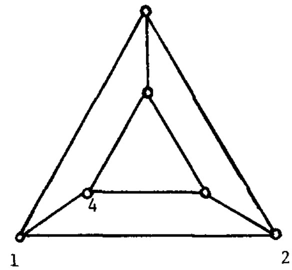

FIGURE 1

The simplest example (pointed out to me by Paul Terwilliger) of a distance-polynomial graph which is not distance-regular is the prism G (Fig. 1). G clearly has diameter 2, is connected and is regular. Thus G is distance-polynomial. It is straightforward to check that G is not distance-regular and that A_2 is a *cubic* in A.

A distance-polynomial graph which is not distance-regular need not have diameter 2. One can show, for example, that the graph consisting of two p-cycles, with neighboring vertices joined as in Fig. 2, is distance-polynomial but not distance-regular whenever p is an odd prime.

4. SUPER-REGULAR GRAPHS

A further weakening of the requirement that A_i is a polynomial in A is the condition that the A_i's commute with one another. Since each A_i is a real symmetric matrix, this condition implies that each A_i is a polynomial in a fixed matrix B which is itself a polynomial in the A_i's, but which can be different from A.

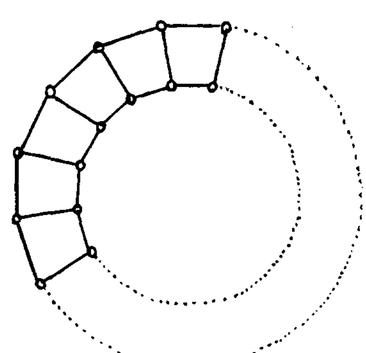

FIGURE 2

We have not been able to characterize these graphs, but we will show that a somewhat larger class of graphs can be completely described in terms of a regularity property.

DEFINITION. Let v be a vertex in the graph G of diameter d. The *generalized degree* of v is the d-tuple $(k_1, \ldots, k_d)$, where k_i is the number of vertices whose distance from v is i. The graph G is called *super-regular* if each vertex has the same generalized degree.

4.1. THEOREM. *Let G be a connected graph of diameter d. G is super-regular if and only if $(A_iA_j)_{rs} = (A_jA_i)_{rs}$ for every pair of adjacent vertices u_r, u_s.*

Proof. Let G be super-regular and consider the sum $J = A_0 + A_1 + \cdots + A_d$. Then $A_iJ = \sum_{j=0}^{d} A_iA_j$ and clearly $k_i = (A_iJ)_{rs} = \sum_{j=0}^{d} (A_iA_j)_{rs}$ for any pair (r, s). Now suppose that $\partial(u_r, u_s) = 1$. Then, since $(A_iA_j)_{rs} = s_{ij1}(u_r, u_s)$, and $s_{ij1} = 0$ if $|i-j| > 1$ (triangle inequality), we have $k_i = s_{i(i-1)1} + s_{ii1} + s_{i(i+1)1}$ at (u_r, u_s). Since G is super-regular $(A_iJ)_{rs} = (JA_i)_{rs}$. Thus we also have $k_i = s_{(i-1)i1} + s_{ii1} + s_{(i+1)i1}$ at (u_r, u_s). Subtracting the last equations we get

$$0 = s_{i(i-1)1} - s_{(i-1)i1} + s_{i(i+1)1} - s_{(i+1)i1} \qquad \text{at } (u_r, u_s).$$

If $i = 1$, $0 = (s_{101} - s_{011}) + (s_{121} - s_{211})$. But $s_{101} = s_{011} = 1$, and so $s_{121} = s_{211}$. We will now show that $s_{i(i+1)1} = s_{(i+1)i1}$ by induction on i. Suppose $s_{(i-1)i1} = s_{i(i-1)1}$. Then since $0 = (s_{i(i-1)1} - s_{(i-1)i1}) + (s_{i(i+1)1} - s_{(i+1)i1})$, we have $s_{i(i+1)1} = s_{(i+1)i1}$. Thus $(A_iA_j)_{rs} = (A_jA_i)_{rs}$ for all r, s such that u_r is adjacent to u_s, and $j = i-1$, i, $i+1$. But for all other values of j, $(A_iA_j)_{rs} = 0$ and the proof is complete in one direction. Now suppose, conversely, that $s_{ij1}(u_r, u_s) = s_{ji1}(u_r, u_s)$ for every pair of adjacent vertices

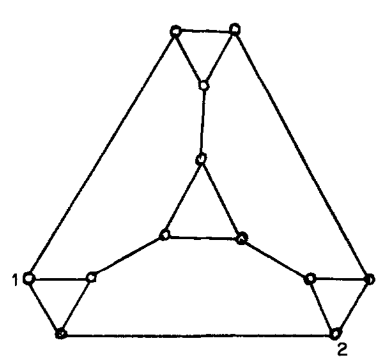

FIGURE 3

u_r, u_s. Let $k_i(r) = (A_iJ)_{rs}$ be the number of vertices of G a distance of i from u_r. Then as before, $k_i(r) = s_{i(i-1)1} + s_{ii1} + s_{i(i+1)1}$ at (u_r, u_s). Also $k_i(s) = (JA_i)_{rs} = s_{(i-1)i1} + s_{ii1} + s_{(i+1)i1}$ at (u_r, u_s).

It follows from the hypothesis that $k_i(s) = k_i(r)$. We can now choose a vertex u_t adjacent to u_s and different from u_r and show that $k_i(r) = k_i(s) = k_i(t)$. Since G is connected, k_i is the same for all vertices. Repeating the argument for all $i = 1,\dots, d$ we get that G is super-regular. ∎

A graph which is super-regular but does not have commuting A_i's is given by Fig. 3. It is easy to check that the generalized degree of any vertex is $(3, 4, 4)$. But $s_{122}(1, 2) = 1$ while $s_{212}(1, 2) = 0$ and hence $A_1A_2 \neq A_2A_1$ even if we were to restrict attention to entries representing vertices a distance of 2 apart.

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