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Hankel convolution operators on entire functions and distributions ☆

M. Belhadj and J.J. Betancor *

Departamento de Análisis Matemático, Universidad de La Laguna, Islas Canarias, La Laguna, Tenerife 38271, Spain

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Abstract

In this paper we study the Hankel convolution operators on the space of even and entire functions and on Schwartz distribution spaces. We characterize the Hankel convolution operators as those ones that commute with Hankel translations and with a Bessel operator. Also we prove that the Hankel convolution operators are hypercyclic and chaotic on the spaces under consideration.

Keywords: Schwartz distributions; Hankel transformation; Convolution; Cyclic and chaotic operators

1. Introduction

The Hankel integral transformation appears taking different forms in the literature (see, for instance, [23,27,35]). Here we define the Hankel transformation h_μ through [23]

$$h_\mu(\phi)(x) = \int_0^\infty (xy)^{-\mu} J_\mu(xy)\phi(y)y^{2\mu+1}\,dy, \quad x \in (0, \infty),$$

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* Corresponding author.

E-mail address: jbetanco@ull.es (J.J. Betancor).

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where by J_μ we represent the Bessel function of the first kind and order μ. Throughout this paper we will always assume that the order μ is greater than $-1/2$. If $n \in \mathbf{N}$ the Hankel transform $h_{(n-2)/2}$ of order $(n-2)/2$ appears when it calculates the Euclidean Fourier transform of functions defined on $\mathbf{R}^n$ having radial symmetry.

The convolution operation by the Hankel h_μ-transformation was investigated by Hirschman [24], Haimo [22] and Cholewinski [13].

To simplify we denote by $L_{1,\mu}$ the space $L^1((0,\infty), x^{2\mu+1}\,dx)$, where dx represents the Lebesgue measure on $(0,\infty)$, that is, a measurable function f is in $L_{1,\mu}$ if, and only if, $\int_0^\infty |f(x)| x^{2\mu+1}\,dx < \infty$.

If $f, g \in L_{1,\mu}$ the Hankel convolution $f \#_\mu g$ of f and g of order μ is defined by

$$(f \#_\mu g)(x) = \int_0^\infty f(y)({}_\mu\tau_x g)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\,dy, \quad \text{a.e. } x \in (0,\infty),$$

where the Hankel translation operator ${}_\mu\tau_x$, $x \in (0,\infty)$, is given through

$$({}_\mu\tau_x g)(y) = \int_0^\infty g(z) D_\mu(x,y,z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\,dz, \quad \text{a.e. } y \in (0,\infty).$$

Also ${}_\mu\tau_0 g = g$. Here a.e. is understood to be with respect to the Lebesgue measure and the kernel D_μ is defined by

$$D_\mu(x,y,z) = \left(2^\mu \Gamma(\mu+1)\right)^2 \int_0^\infty (xt)^{-\mu} J_\mu(xt)(yt)^{-\mu} J_\mu(yt)(zt)^{-\mu} \times J_\mu(zt) t^{2\mu+1}\,dt, \quad x,y,z \in (0,\infty).$$

The Hankel transformation satisfies the following interchange formula with respect to $\#_\mu$-convolution [24, Theorem 2.d]

$$h_\mu(f \#_\mu g) = h_\mu(f) h_\mu(g), \quad f, g \in L_{1,\mu}.$$

In the sequel, since any confusion is unliked, we write #, τ_x, $x \in (0,\infty)$, and D instead of $\#_\mu$, ${}_\mu\tau_x$, $x \in (0,\infty)$, and D_μ, respectively.

Zemanian [33–35] studied the Hankel transformation on distribution spaces. He considered, for the Hankel transformation, the following form

$$H_\mu(\phi)(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy)\phi(y)\,dy, \quad x \in (0,\infty).$$

It is clear that h_μ and H_μ are closely connected. Then, all the results obtained by Zemanian for H_μ can be transmitted to h_μ.

Inspired by Zemanian's investigations, Altenburg [1] developed a distributional theory for Hankel's h_μ transformation.

In [1] it was defined the space $\mathcal{H}$ consists of all those complex and smooth functions ϕ on $(0,\infty)$ such that, for every $m,n \in \mathbf{N}$, the quantity

$$\gamma_{m,n}(\phi) = \sup_{x\in(0,\infty)} \left(1+x^2\right)^m \left| \left(\frac{1}{x}D\right)^n \phi(x) \right|$$

is finite. $\mathcal{H}$ is equipped with the topology generated by the family $\{\gamma_{m,n}\}_{m,n\in\mathbf{N}}$ of seminorms. Thus $\mathcal{H}$ is a Fréchet space and h_μ is an automorphism of $\mathcal{H}$ [1, Satz 5]. The Hankel transformation is defined on $\mathcal{H}'$, the dual space of $\mathcal{H}$, by transposition.

Let $a > 0$. According to [1], the space $\mathcal{B}^a$ consists of all the functions $\phi \in \mathcal{H}$ such that $\phi(x) = 0$, $x \geqslant a$. $\mathcal{B}^a$ is a complete subspace of $\mathcal{H}$. Moreover $\mathcal{B}^a$ is continuously contained in $\mathcal{B}^b$, provided that $0 < a < b < \infty$. The union space $\mathcal{B} = \bigcup_{a>0} \mathcal{B}^a$ is equipped with the inductive topology. The Hankel transform $h_\mu(\mathcal{B}^a)$ of $\mathcal{B}^a$, $a > 0$, can be characterized by using [34, Theorem 1].

According to [18, Corollary 4.8] the space $\mathcal{H}$ coincides with the space $\mathcal{S}_{\text{even}}$ of all the even functions in the Schwartz space $\mathcal{S}$. Moreover, for every $a > 0$, the space $\mathcal{B}^a$ agrees with the space $\mathcal{D}_a$ considered by Trimèche [31] and that is constituted by all the functions $\phi \in \mathcal{S}_{\text{even}}$ such that $\phi(x) = 0$, $|x| \leqslant a$. Then, the space $\mathcal{D}_* = \bigcup_{a>0} \mathcal{D}_a$ [31] coincides with the space $\mathcal{B}$.

As in [31], $\mathcal{E}_*$ denotes the space of all those complex valued, smooth and even functions defined on $\mathbf{R}$. $\mathcal{E}_*$ is endowed with the usual topology and it coincides with the space $x^{-\mu-1/2}\mathcal{E}_\mu$ where $\mathcal{E}_\mu$ is the space introduced in [5] as it was defined as follows. A complex and smooth function f defined on $(0,\infty)$ is in $\mathcal{E}_\mu$ if and only if, for every $k \in \mathbf{N}$, there exists the following limit

$$\lim_{x\to 0^+} \left(\frac{1}{x}\frac{d}{dx}\right)^k f(x).$$

The convolution for the Hankel H_μ transformation can be defined by making a straightforward modification in the convolution # defined by Hirschman [24]. The study of the distributional Hankel convolution was started by de Sousa-Pinto [28] who considered only the order $\mu = 0$. In a series of papers Betancor and Marrero [5–7,25] have investigated the Hankel convolution on the Zemanian's distribution spaces. More recently, Betancor and Rodríguez-Mesa [9] have defined the Hankel convolution of distributions with exponential growth.

In this paper we study Hankel convolution operators on the Schwartz distribution spaces and on the space $\mathcal{H}_e(\mathbf{C})$ of even and entire functions. It is organized as follows. In Section 2 we define the Hankel transformation on the dual space $\mathcal{H}_e(\mathbf{C})'$ of $\mathcal{H}_e(\mathbf{C})$. The Hankel convolution operators on $\mathcal{H}_e(\mathbf{C})$, $\mathcal{E}_*$ and their duals are studied in Section 3. We characterize the linear and continuous mappings from $\mathcal{H}_e(\mathbf{C})$ into itself that commute with the Hankel translation τ_z, for each $z \in \mathbf{C}$, as the Hankel convolution operators defined by the functionals in $\mathcal{H}_e(\mathbf{C})'$. The corresponding result on the space $\mathcal{D}_*$ was obtained in Section 4.

Suppose now that X is a topological linear space and T is a continuous linear operator from X into itself. An element $x \in X$ is called hypercyclic for T when the set $\{T^n x\colon n \in \mathbf{N}\}$ is dense in X. The importance of hypercyclic vectors derives from the study of closed invariant subsets. The paper of Grosse-Erdman [21] is a excellent survey of the state of art concerning hypercyclic operators, that is, operators having hypercyclic vectors. According to Bonet [11] (see also Devaney [16] and Banks et al. [2]), we say that T is a chaotic operator if T satisfies the following two conditions:

(i) T is topologically transitive, that is, for every pair of open sets U and V of X there exists $n \in \mathbf{N}$ for which $T^n(U) \cap V \neq \emptyset$.
(ii) The set of periodic vectors of T is dense in X. As usual, we say that a vector $x \in X$ is periodic for T when there exists $n \in \mathbf{N}$ such that $T^n x = x$.

Note that each hypercyclic operator is topologically transitive.

Godefroy and Shapiro [20] extended the celebrated classical results of Birkhoff [10] and MacLane [26] proving that every partial differential operator that is not a scalar multiple of the identity operator is hypercyclic and chaotic on $C^\infty(\mathbf{R}^n)$. Recently, Bonet [11] established that the usual convolution operators that are not scalar multiples of the Dirac δ-functional are hypercyclic and chaotic on the Beurling ultradifferentiable functions.

In Sections 3 and 4 we establish, inspired by the ideas of Bonet [11], that the Hankel convolution operators defined by functionals in $\mathcal{E}'_*$ are hypercyclic and chaotic on $\mathcal{E}_*$ and $\mathcal{D}'_*$, when on $\mathcal{D}'_*$ the strong topology is considered. Recently, Betancor and Bonilla [4] investigated the hypercyclicity of Hankel and Fourier convolution operators on certain Banach spaces.

Throughout this paper we always denote by C a positive constant that can change from a line to the other one. We need to use some properties of the Bessel functions that can be encountered in the extensive monograph of Watson [32].

2. The Hankel transformation on the space $\mathcal{H}_e(\mathbf{C})'$ the dual of $\mathcal{H}_e(\mathbf{C})$

By $\mathcal{H}_e(\mathbf{C})$ we denote the space of the even and entire functions. We equip $\mathcal{H}_e(\mathbf{C})$, as usual, with the topology of the uniform convergence on the compact subsets of $\mathbf{C}$. If we define, for every $n \in \mathbf{N}$, the norm p_n by

$$p_n(f) = \sup_{|z| \leqslant n+1} \left|f(z)\right|, \quad f \in \mathcal{H}_e(\mathbf{C}),$$

the system $\{p_n\}_{n\in\mathbf{N}}$ generates the topology of $\mathcal{H}_e(\mathbf{C})$. Thus $\mathcal{H}_e(\mathbf{C})$ is a Fréchet space [29, p. 231].

It is clear that $\mathcal{H}_e(\mathbf{C})$ is continuously contained in the space $\mathcal{E}_*$, that is, $\mathcal{H}_e(\mathbf{C})$ is a subspace of $\mathcal{E}_*$ and the topology of $\mathcal{H}_e(\mathbf{C})$ is finer than the one induced in $\mathcal{H}_e(\mathbf{C})$ by $\mathcal{E}_*$.

The dual space of $\mathcal{H}_e(\mathbf{C})$ is represented by $\mathcal{H}_e(\mathbf{C})'$. It is clear that, for every $z \in \mathbf{C}$, the function $f_z(t) = 2^\mu \Gamma(\mu+1)(zt)^{-\mu} J_\mu(zt)$, $t \in \mathbf{C}$, is in $\mathcal{H}_e(\mathbf{C})$. We define the Hankel transform $h_\mu(T)$ of $T \in \mathcal{H}_e(\mathbf{C})'$ by

$$h_\mu(T)(z) = 2^\mu \Gamma(\mu+1)\big\langle T(t), (zt)^{-\mu} J_\mu(zt)\big\rangle, \quad z \in \mathbf{C}.$$

Note that, since for every $z \in \mathbf{C}$, the series

$$(zt)^{-\mu} J_\mu(zt) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+\mu} k! \Gamma(\mu+k+1)} (tz)^{2k}, \quad t \in \mathbf{C},$$

converges in $\mathcal{H}_e(\mathbf{C})$, we can write that, for every $T \in \mathcal{H}_e(\mathbf{C})'$,

$$h_\mu(T)(z) = \Gamma(\mu+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k! \Gamma(\mu+k+1)} z^{2k} \big\langle T(t), t^{2k}\big\rangle, \quad z \in \mathbf{C}.$$

Hence, $h_\mu(T) \in \mathcal{H}_e(\mathbf{C})$ provided that $T \in \mathcal{H}_e(\mathbf{C})'$.

In [1] we defined the Hankel transformation on the space $\mathcal{E}'_*$ that is contained in $\mathcal{H}_e(\mathbf{C})'$. According to [5, Proposition 4.6] the definition given for the Hankel transformation on $\mathcal{H}_e(\mathbf{C})'$ extends the definition of the Hankel transformation on $\mathcal{E}'_*$.

We now characterize the functions in $\mathcal{H}_e(\mathbf{C})$ that belong to the image $h_\mu(\mathcal{H}_e(\mathbf{C})')$ of $\mathcal{H}_e(\mathbf{C})'$ by h_μ. Our next result, that is a Hankel version of the one presented in [30, pp. 474–475] for the Fourier transformation, shows that $h_\mu(\mathcal{H}_e(\mathbf{C})')$ is actually independent of μ.

Proposition 2.1. *Let f be a function in $\mathcal{H}_e(\mathbf{C})$. Then the following assertions are equivalent.*

(i) *There exists $T \in \mathcal{H}_e(\mathbf{C})'$ such that $f = h_\mu(T)$.*
(ii) *The function f is of exponential type, that is, there exist $A, B > 0$ for which $|f(z)| \leqslant Be^{A|z|}$, $z \in \mathbf{C}$.*

Proof. Suppose firstly that $f = h_\mu(T)$, for some $T \in \mathcal{H}_e(\mathbf{C})'$. Since $T \in \mathcal{H}_e(\mathbf{C})'$, there exist $C > 0$ and $r \in \mathbf{N}$ such that

$$\big|\langle T, g\rangle\big| \leqslant C \sup_{|z|\leqslant r} \big|g(z)\big|, \quad g \in \mathcal{H}_e(\mathbf{C}).$$

Hence, by using the Hahn–Banach theorem, duality arguments and by arguing as in [29, p. 231] (see also [6]), we can find a complex measure λ having bounded support such that

$$\langle T, g\rangle = \int_{\mathbf{C}} g(z)\, d\lambda(z), \quad g \in \mathcal{H}_e(\mathbf{C}).$$

In particular, it has

$$h_\mu(T)(z) = 2^\mu \Gamma(\mu+1) \int_{\mathbf{C}} (zt)^{-\mu} J_\mu(zt)\, d\lambda(t), \quad z \in \mathbf{C}.$$

According to [19, (5.3.b)], we can write

$$\left|h_\mu(T)(z)\right| \leqslant C e^{a|z|}, \quad z \in \mathbf{C},$$

where $a > 0$ is such that the support of λ is contained in the disc $D(0,a)$ centered in the origin and of radius a.

Hence $h_\mu(T)$ is an even and entire function of exponential type.

Assume now f is a function in $\mathcal{H}_e(\mathbf{C})$ of exponential type, that is, for certain $A, B > 0$, $|f(z)| \leqslant Be^{A|z|}$, $z \in \mathbf{C}$.

We put

$$f(z) = \sum_{k=0}^{\infty} a_k \frac{z^{2k}}{2^{2k} k! \Gamma(\mu+k+1)}, \quad z \in \mathbf{C}.$$

Note that thus $a_k = (\Delta_\mu^k f)(0)$, for every $k \in \mathbf{N}$, where Δ_μ denotes the Bessel operator $z^{-2\mu-1} D z^{2\mu+1} D$.

According to the Cauchy integral formula, it follows that

$$\frac{|a_k|}{2^{2k} k! \Gamma(\mu+k+1)} \leqslant C e^{AR} R^{-2k}, \quad k \in \mathbf{N} \text{ and } R > 0.$$

Hence, Stirling's formula implies that, for every $k \in \mathbf{N}$ and $R > 0$,

$$|a_k| \leqslant C 2^{2k} (\mu+k)^{\mu+k} e^{-\mu-k} \sqrt{2\pi(\mu+k)}\, k^k e^{-k} \sqrt{2\pi k}\, e^{AR} R^{-2k}.$$

Then, by taking, for every $k \in \mathbf{N} \setminus \{0\}$, $R = \frac{2k}{A}$, it follows

$$|a_k| \leqslant C \left(\frac{\mu+k}{k}\right)^k (\mu+k)^{\mu+1/2} \sqrt{k}\, A^{2k} \leqslant C M^{2k}, \tag{2.1}$$

for some $M > 0$.

Suppose now γ is a closed simple path having the origin in its interior. For every $m \in \mathbf{N}$, we have that

$$\begin{aligned} &\frac{1}{2\pi i} \int_\gamma (zt)^{-\mu} J_\mu(zt) t^{-2m-1}\, dt \\ &\quad = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k+\mu} k! \Gamma(\mu+k+1)} \int_\gamma t^{2k-2m-1}\, dt \\ &\quad = \frac{(-1)^m z^{2m}}{2^{2m+\mu} m! \Gamma(\mu+m+1)}. \end{aligned}$$

Hence, since by (2.1) the series $\sum_{m=0}^{\infty} a_m(-1)^m z^{-2m-1}$ converges for every $z \in \mathbf{C}$ with $|z| > M$, if γ represents the circle with center in 0 and radius $2M$ then

$$f(z) = \frac{1}{2\pi i}\int_{\gamma}(zt)^{-\mu}J_\mu(zt)2^\mu\sum_{m=0}^{\infty}(-1)^m a_m t^{-2m-1}\,dt, \quad z \in \mathbf{C}. \tag{2.2}$$

We now define the functional T on $\mathcal{H}_e(\mathbf{C})$ by

$$\langle T, g\rangle = \frac{1}{2\pi i}\int_{\gamma} g(t)\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(\mu+1)}a_m t^{-2m-1}\,dt, \quad g \in \mathcal{H}_e(\mathbf{C}).$$

Thus $T \in \mathcal{H}_e(\mathbf{C})'$. Indeed, for every $g \in \mathcal{H}_e(\mathbf{C})$, from (2.1) it follows that

$$\bigl|\langle T, g\rangle\bigr| \leqslant C\sup_{|z|\leqslant 2M}\bigl|g(z)\bigr|, \quad g \in \mathcal{H}_e(\mathbf{C}).$$

Moreover (2.2) says that $h_\mu(T) = f$. $\square$

Remark 1. According to Proposition 2.1 the Hankel transformation of an element of $\mathcal{H}_e(\mathbf{C})'$ is always actually the Hankel transform of a complex measure on $\mathbf{C}$ having compact support. The Hankel transforms of measures on $[0, \infty)$ has been studied, for instance, in [15].

Remark 2. Proposition 2.1 can be seen as an extension of [5, Theorem 4.9] where Paley–Wiener type theorem for Hankel transforms of the elements of $\mathcal{E}'_*$ were established.

We now establish a uniqueness theorem for Hankel transforms on $\mathcal{H}_e(\mathbf{C})'$. Our next result will be also useful in the sequel.

Proposition 2.2. *If V is a subset of $\mathbf{C}$ having adherence points, then the linear space*

$$\mathcal{M}_V = \operatorname{span}\bigl\{(.z)^{-\mu}J_\mu(.z)\colon z \in V\bigr\}$$

generated by the functions $(z.)^{-\mu}J_\mu(z.)$, $z \in V$, is dense in $\mathcal{H}_e(\mathbf{C})$.

In particular, if $T \in \mathcal{H}_e(\mathbf{C})'$ and $h_\mu(T) = 0$ then $T = 0$.

Proof. Suppose that $T \in \mathcal{H}_e(\mathbf{C})'$ and $T = 0$ on $\mathcal{M}_V$. There exists a complex measure λ having compact support [29, p. 231] such that

$$\langle T, f\rangle = \int_{\mathbf{C}} f(t)\,d\lambda(t), \quad f \in \mathcal{H}_e(\mathbf{C}).$$

The function $F = h_\mu(T)$ is even and entire. Moreover, since $T = 0$ on $\mathcal{M}_V$, $F = 0$ on V. Hence $F = 0$ on $\mathbf{C}$.

Differentiating under the integral sign we obtain

$$\Delta_\mu^k F(z) = \int_{\mathbf{C}} \left(-t^2\right)^k (zt)^{-\mu} J_\mu(zt)\, d\lambda(t), \quad k \in \mathbf{N} \text{ and } z \in \mathbf{C},$$

where Δ_μ represents the Bessel operator $z^{-2\mu-1} D z^{2\mu+1} D$. Hence, for every $k \in \mathbf{N}$,

$$\begin{aligned}\Delta_\mu^k F(0) &= 2^\mu \Gamma(\mu+1) \int_{\mathbf{C}} (-1)^k t^{2k}\, d\lambda(t) \\ &= (-1)^k 2^\mu \Gamma(\mu+1) \left\langle T(t), t^{2k} \right\rangle = 0.\end{aligned}$$

Then $\langle T, f \rangle = 0$, for every $f \in \mathcal{H}_e(\mathbf{C})$.

Hahn–Banach theorem allows to conclude the desired result. □

3. Hankel translation and Hankel convolution on the spaces $\mathcal{H}_e(\mathbf{C})$ and $\mathcal{E}_*$ and their duals

We start this section by studying the Hankel translation operator on the space $\mathcal{H}_e(\mathbf{C})$.

In [14, p. 7] it was established that, for every $n \in \mathbf{N}$,

$$\tau_x\left(t^{2n}\right)(y) = \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)} x^{2(n-k)} y^{2k}, \quad x, y \in [0, \infty). \tag{3.1}$$

Recently, in [4] a new proof of (3.1) has been presented.

Let $f \in \mathcal{H}_e(\mathbf{C})$ and assume that $f(z) = \sum_{k=0}^\infty a_k z^{2k}$, $z \in \mathbf{C}$, where $a_k \in \mathbf{C}$, $k \in \mathbf{N}$. For every $x, y \in [0, \infty)$, we can write

$$\begin{aligned}(\tau_x f)(y) &= \int_{|x-y|}^{x+y} D(x,y,z) \left(\sum_{n=0}^\infty a_n z^{2n} \right) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dz \\ &= \sum_{n=0}^\infty a_n \int_{|x-y|}^{x+y} D(x,y,z) z^{2n} \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dz \\ &= \sum_{n=0}^\infty a_n \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)} x^{2(n-k)} y^{2k}.\end{aligned}$$

We now define the Hankel translate $\tau_z f$ of $f \in \mathcal{H}_e(\mathbf{C})$ by

$$(\tau_z f)(t) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)} z^{2(n-k)} t^{2k},$$
$$z, t \in \mathbf{C}. \tag{3.2}$$

Note that, for every $z, t \in \mathbf{C}$,

$$\sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)} |z|^{2(n-k)} |t|^{2k}$$
$$= \int_{||z|-|t||}^{|z|+|t|} D\big(|z|, |t|, x\big) \left(\sum_{n=0}^{\infty} |a_n| x^{2n} \right) \frac{x^{2\mu+1}}{2^{\mu}\Gamma(\mu+1)}\, dx.$$

Hence the series defining $\tau_z f$ converges uniformly on each compact subset of $\mathbf{C}$. We can interchange the order of summation to obtain that

$$(\tau_z f)(t) = \sum_{k=0}^{\infty} \frac{\Gamma(\mu+1)}{\Gamma(k+\mu+1)} t^{2k} \sum_{n=k}^{\infty} \binom{n}{k} z^{2(n-k)} a_n \frac{\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)},$$
$$z, t \in \mathbf{C}.$$

Thus we prove that $\tau_z f$ is in $\mathcal{H}_e(\mathbf{C})$, for every $z \in \mathbf{C}$.

Proposition 3.1. (i) *For every $z \in \mathbf{C}$, the Hankel translation τ_z defines a continuous linear mapping from $\mathcal{H}_e(\mathbf{C})$ into itself.*

(ii) *Let $f \in \mathcal{H}_e(\mathbf{C})$. Then the* (*nonlinear*) *mapping F_f defined by*

$$F_f : \mathbf{C} \to \mathcal{H}_e(\mathbf{C})$$
$$z \to \tau_z f,$$

is continuous from $\mathbf{C}$ into $\mathcal{H}_e(\mathbf{C})$.

Proof. (i) Let $z \in \mathbf{C}$. For every $f \in \mathcal{H}_e(\mathbf{C})$, $\tau_z f$ is also in $\mathcal{H}_e(\mathbf{C})$. Suppose now that $\{f_\nu\}_{\nu\in\mathbf{N}}$ is a sequence in $\mathcal{H}_e(\mathbf{C})$ such that $f_\nu \to f$, as $\nu \to \infty$, in $\mathcal{H}_e(\mathbf{C})$ and $\tau_z f_\nu \to g$, as $\nu \to \infty$, in $\mathcal{H}_e(\mathbf{C})$.

Since $\mathcal{H}_e(\mathbf{C})$ is continuously contained in $\mathcal{E}_*$, $f_\nu \to f$, as $\nu \to \infty$, in $\mathcal{E}_*$. Then, by [31, Proposition 8.3], $\tau_z f_\nu \to \tau_z f$, as $\nu \to \infty$, in $\mathcal{E}_*$. Hence, since convergence in $\mathcal{E}_*$ and convergence in $\mathcal{H}_e(\mathbf{C})$ imply pointwise convergence, $\tau_z f = g$.

Closed graph theorem allows to conclude that τ_z defines a continuous mapping from $\mathcal{H}_e(\mathbf{C})$ into itself.

(ii) Let $z_0 \in \mathbf{C}$. Assume that $\{z_\nu\}_{\nu\in\mathbf{N}\setminus\{0\}}$ is a sequence in $\mathbf{C}$ such that $z_\nu \to z_0$, as $\nu \to \infty$. We have to see that $\tau_{z_\nu} f \to \tau_{z_0} f$, as $\nu \to \infty$, uniformly in each compact subset of $\mathbf{C}$. Let $a > 0$. We choose $b > 0$ such that $|z_\nu| \leqslant b$, $\nu \in \mathbf{N}$. By [24, (2), 2] we can write

$$\begin{aligned}\left|\tau_{z_\nu}\left(y^{2n}\right)(t)-\tau_{z_0}\left(y^{2n}\right)(t)\right| &\leqslant \tau_{|z_\nu|}\left(y^{2n}\right)\left(|t|\right)+\tau_{|z_0|}\left(y^{2n}\right)\left(|t|\right)\\ &\leqslant \int_0^c D\left(|z_\nu|,|t|,y\right)y^{2n}\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy\\ &\quad+\int_0^c D\left(|z_0|,|t|,y\right)y^{2n}\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy\\ &\leqslant 2c^{2n},\quad \nu\in\mathbf{N}\setminus\{0\}\text{ and }|t|\leqslant a,\end{aligned}$$

where $c=a+b$.

Hence, if $f(z)=\sum_{n=0}^{\infty}a_nz^{2n}$, $z\in\mathbf{C}$, then for every $\varepsilon>0$, there exists $n_0\in\mathbf{N}$ such that

$$\left|\sum_{n=n_0}^{\infty}a_n\left(\tau_{z_\nu}\left(y^{2n}\right)(t)-\tau_{z_0}\left(y^{2n}\right)(t)\right)\right|\leqslant 2\sum_{n=n_0}^{\infty}|a_n|c^{2n}<\varepsilon,$$

for $\nu\in\mathbf{N}\setminus\{0\}$ and $|t|\leqslant a$.

Moreover, it is clear that

$$\begin{aligned}&\sum_{n=0}^{n_0-1}a_n\tau_{z_\nu}\left(y^{2n}\right)(t)\\ &\quad=\sum_{n=0}^{n_0-1}a_n\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)}z_\nu^{2(n-k)}t^{2k}\\ &\quad\to\sum_{n=0}^{n_0-1}a_n\tau_{z_0}\left(y^{2n}\right)(t),\quad\text{as }\nu\to\infty,\end{aligned}$$

uniformly in $|t|\leqslant a$.

Thus we can conclude that $\tau_{z_\nu}f\to\tau_{z_0}f$, as $\nu\to\infty$, uniformly in the disc $D(0,b)$ with center in the origin and radius b, and the proof is finished. □

Proposition 3.1, (i), allows to define the Hankel convolution $T\,\#\,f$ of $T\in\mathcal{H}_e(\mathbf{C})'$ and $f\in\mathcal{H}_e(\mathbf{C})$ as follows

$$(T\,\#\,f)(z)=\langle T,\tau_zf\rangle,\quad z\in\mathbf{C}.$$

Note that Proposition 3.1, (ii), implies that $T\,\#\,f$ is a continuous function on $\mathbf{C}$, for every $T\in\mathcal{H}_e(\mathbf{C})'$ and $f\in\mathcal{H}_e(\mathbf{C})$. Moreover, as we will prove in the following, $T\,\#\,f$ is in $\mathcal{H}_e(\mathbf{C})$, for each $T\in\mathcal{H}_e(\mathbf{C})'$ and $f\in\mathcal{H}_e(\mathbf{C})$.

Proposition 3.2. *Let $T\in\mathcal{H}_e(\mathbf{C})'$. Then the mapping F_T defined by*

$$F_T(f)=T\,\#\,f,\quad f\in\mathcal{H}_e(\mathbf{C}),$$

is a continuous linear mapping from $\mathcal{H}_e(\mathbf{C})$ into itself.

Proof. Let $f \in \mathcal{H}_e(\mathbf{C})$ and $z \in \mathbf{C}$. Assume that $f(t) = \sum_{n=0}^{\infty} a_n t^{2n}$, $t \in \mathbf{C}$. According to (3.2) and by taking into account that the series converges uniformly in every compact subset of $\mathbf{C}$, we can write

$$\begin{aligned}(T \,\#\, f)(z) &= \langle T, \tau_z f\rangle \\ &= \sum_{n=0}^{\infty} a_n \big\langle T(t), \tau_z\big(y^{2n}\big)(t)\big\rangle \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)} \\ &\quad \times z^{2(n-k)}\big\langle T(t), t^{2k}\big\rangle, \quad z \in \mathbf{C}.\end{aligned}$$

Hence $T \,\#\, f$ is an entire function.

To see that the mapping F_T is continuous we use the closed graph theorem. Assume that $\{f_\nu\}_{\nu\in\mathbf{N}}$ is a sequence in $\mathcal{H}_e(\mathbf{C})$ such that $f_\nu \to f$, as $\nu \to \infty$, in $\mathcal{H}_e(\mathbf{C})$, and $T \,\#\, f_\nu \to g$, as $\nu \to \infty$, in $\mathcal{H}_e(\mathbf{C})$. By Proposition 3.1, (i), for every $z \in \mathbf{C}$, $\tau_z f_\nu \to \tau_z f$, as $\nu \to \infty$, in $\mathcal{H}_e(\mathbf{C})$. Hence, since $T \in \mathcal{H}_e(\mathbf{C})'$, $(T \,\#\, f_\nu)(z) \to (T \,\#\, f)(z)$, as $\nu \to \infty$, for every $z \in \mathbf{C}$. Then $g = T \,\#\, f$. The closed graph theorem allows now to conclude that F_T is a continuous mapping. □

We define the Hankel convolution $S \,\#\, T$ of S and $T \in \mathcal{H}_e(\mathbf{C})'$ as the functional on $\mathcal{H}_e(\mathbf{C})$ given through

$$\langle S \,\#\, T, f\rangle = \langle S, T \,\#\, f\rangle, \quad f \in \mathcal{H}_e(\mathbf{C}).$$

Note that, according to Proposition 3.2, $S \,\#\, T \in \mathcal{H}_e(\mathbf{C})'$, for each $S, T \in \mathcal{H}_e(\mathbf{C})'$. By proceeding as in [17, Proposition 6] we can prove that the mapping defined by $(S, T) \to S \,\#\, T$ is bilinear and continuous from $\mathcal{H}_e(\mathbf{C})' \times \mathcal{H}_e(\mathbf{C})'$ into $\mathcal{H}_e(\mathbf{C})'$, when $\mathcal{H}_e(\mathbf{C})'$ has the strong topology.

We now establish the interchange formula involving distributional Hankel transformation and convolution.

Proposition 3.3. *If* $S, T \in \mathcal{H}_e(\mathbf{C})'$ *then*

$$h_\mu(S \,\#\, T) = h_\mu(S) h_\mu(T).$$

Proof. By [24, (1), Section 2], we can write

$$\begin{aligned}&h_\mu(S \,\#\, T)(z) \\ &\quad = 2^\mu \Gamma(\mu+1)\big\langle (S \,\#\, T)(t), (zt)^{-\mu} J_\mu(zt)\big\rangle \\ &\quad = 2^\mu \Gamma(\mu+1)\big\langle S(t), \big\langle T(y), \tau_t\big((z.)^{-\mu} J_\mu(z.)\big)(y)\big\rangle\big\rangle \\ &\quad = \big\langle S(t), \big\langle T(y), 2^\mu \Gamma(\mu+1)(zt)^{-\mu} J_\mu(zt) 2^\mu \Gamma(\mu+1)(zy)^{-\mu} J_\mu(zy)\big\rangle\big\rangle \\ &\quad = h_\mu(S)(z) h_\mu(T)(z), \quad z \in \mathbf{C}. \qquad \square\end{aligned}$$

The following algebraic properties for the Hankel convolution on $\mathcal{H}_e(\mathbf{C})'$ can be proved by using Propositions 2.2 and 3.3.

Proposition 3.4. *Let $T, R, S \in \mathcal{H}_e(\mathbf{C})'$. Then*

(i) $T \# R = R \# T$.
(ii) $T \# (R \# S) = (T \# R) \# S$.
(iii) $T \# \delta = T$, *where, as usual, δ denotes the Dirac functional.*

We now characterize the Hankel convolution operators in $\mathcal{H}_e(\mathbf{C})$ as those linear and continuous mappings from $\mathcal{H}_e(\mathbf{C})$ into itself which commute with Hankel translations and Bessel operators. Our result is inspired in [20, Proposition 5.2] concerning to the usual convolution operators on entire functions. Similar properties for Hankel convolution operators on Zemanian spaces can be found in [3,7].

Proposition 3.5. *Assume that L is a continuous linear mapping from $\mathcal{H}_e(\mathbf{C})$ into itself. The following assertions are equivalent.*

(i) *L commutes with τ_z, that is, $L\tau_z = \tau_z L$, on $\mathcal{H}_e(\mathbf{C})$, for every $z \in \mathbf{C}$.*
(ii) *L commutes with the Bessel operator $\Delta_\mu = z^{-2\mu-1}Dz^{2\mu+1}D$, that is, $L\Delta_\mu = \Delta_\mu L$ on $\mathcal{H}_e(\mathbf{C})$.*
(iii) *There exists a complex measure λ on $\mathbf{C}$ having compact support for which*

$$(Lf)(z) = \int_{\mathbf{C}} (\tau_z f)(t)\, d\lambda(t), \quad f \in \mathcal{H}_e(\mathbf{C}).$$

(*Note that the property says that there exists $T \in \mathcal{H}_e(\mathbf{C})'$ such that $Lf = T \# f$, $f \in \mathcal{H}_e(\mathbf{C})$.*)
(iv) *There exists an entire function Φ of exponential type such that $L = \Phi(\Delta_\mu)$ on $\mathcal{H}_e(\mathbf{C})$, that is, if $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbf{C}$, then*

$$Lf = \sum_{n=0}^{\infty} a_n \Delta_\mu^n f, \quad f \in \mathcal{H}_e(\mathbf{C}),$$

where the series converges in $\mathcal{H}_e(\mathbf{C})$.

Proof. (i) $\Rightarrow$ (ii). Let f be in $\mathcal{H}_e(\mathbf{C})$. Suppose that $f(t) = \sum_{n=0}^{\infty} a_n t^{2n}$, $t \in \mathbf{C}$.
If Δ_μ represents the Bessel operator $t^{-2\mu-1}Dt^{2\mu+1}D$, we can write

$$\Delta_\mu f(t) = \sum_{n=0}^{\infty} a_n 4(n+\mu)nt^{2(n-1)}, \quad z \in \mathbf{C}.$$

We are going to prove that

$$\lim_{z\to 0}\frac{\tau_z f - f}{c_\mu z^2} = \Delta_\mu f, \tag{3.3}$$

where $c_\mu = 1/4(\mu+1)$ and the convergence is understood in $\mathcal{H}_e(\mathbf{C})$.

A straightforward manipulation, by splitting the interior sum, allows us to write

$$\begin{aligned}
&\frac{(\tau_z f)(t) - f(t)}{c_\mu z^2}\\
&\quad = \frac{4(\mu+1)}{z^2}\sum_{n=1}^{\infty} a_n \sum_{k=0}^{n-1}\binom{n}{k}\frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)} z^{2(n-k)} t^{2k}\\
&\quad = 4(\mu+1)\sum_{n=1}^{\infty} a_n n \frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n+\mu)\Gamma(\mu+2)} t^{2(n-1)}\\
&\qquad + \frac{4(\mu+1)}{z^2}\sum_{n=2}^{\infty} a_n \sum_{k=0}^{n-2}\binom{n}{k}\frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)}\\
&\qquad\quad \times z^{2(n-k)} t^{2k}\\
&\quad = \Delta_\mu f(t) + 4(\mu+1) z^2 \sum_{n=2}^{\infty} a_n \sum_{k=0}^{n-2}\binom{n}{k}\frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)}\\
&\qquad\qquad \times z^{2(n-k-2)} t^{2k},
\end{aligned}$$

for each $t \in \mathbf{C}$ and $z \in \mathbf{C}\setminus\{0\}$.

Hence to see (3.3) we have to show that

$$\lim_{z\to 0} z^2 \sum_{n=2}^{\infty} a_n \sum_{k=0}^{n-2}\binom{n}{k}\frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)} z^{2(n-k-2)} t^{2k} = 0, \tag{3.4}$$

uniformly in every compact subset of $\mathbf{C}$.

Let $a > 0$. As it was mentioned above the series

$$\sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)} |z|^{2(n-k)} |t|^{2k}$$

converges uniformly in $|t| \leqslant a$, for every $z \in \mathbf{C}$. Moreover, it has

$$\begin{aligned}
&\sum_{n=2}^{\infty} |a_n| \sum_{k=0}^{n-2}\binom{n}{k}\frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)} |z|^{2(n-2-k)} |t|^{2k}\\
&\quad \leqslant \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(n+\mu+1)\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\Gamma(k+\mu+1)} a^{2k} < \infty,\\
&\quad |t| \leqslant a \text{ and } |z| \leqslant 1.
\end{aligned}$$

Hence (3.4) holds uniformly in $|t| \leqslant a$. Thus (3.3) is proved when the convergence is understood in $\mathcal{H}_e(\mathbf{C})$.

Then we can infer that, if (i) holds

$$\begin{aligned}\Delta_\mu Lf &= \lim_{z\to 0} \frac{\tau_z Lf - Lf}{c_\mu z^2} = \lim_{z\to 0} L\left(\frac{\tau_z f - f}{c_\mu z^2}\right) = L\left(\lim_{z\to 0} \frac{\tau_z f - f}{c_\mu z^2}\right)\\ &= L(\Delta_\mu f).\end{aligned}$$

Hence (i) implies (ii).

(ii) $\Rightarrow$ (i). Assume that $f \in \mathcal{H}_e(\mathbf{C})$ and it is given by $f(t) = \sum_{n=0}^{\infty} a_n t^{2n}$, $t \in \mathbf{C}$.

Let $z \in \mathbf{C}$. We can write

$$\begin{aligned}(\tau_z f)(t) &= \sum_{k=0}^{\infty} \frac{\Gamma(\mu+1)}{\Gamma(\mu+k+1)} z^{2k} \sum_{n=k}^{\infty} \binom{n}{k} a_n t^{2(n-k)} \frac{\Gamma(n+\mu+1)}{\Gamma(n-k+\mu+1)}\\ &= \Gamma(\mu+1) \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}\Gamma(\mu+k+1)k!} \big(\Delta_\mu^k f\big)(t), \quad z, t \in \mathbf{C}. \qquad (3.5)\end{aligned}$$

The last series is uniformly convergent in every compact subset of $\mathbf{C}$.

Then, from (ii) it follows that

$$\begin{aligned}L(\tau_z f) &= \Gamma(\mu+1) \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}\Gamma(\mu+k+1)k!} L\big(\Delta_\mu^k f\big)\\ &= \Gamma(\mu+1) \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}\Gamma(\mu+k+1)k!} \Delta_\mu^k L(f)\\ &= \tau_z(Lf).\end{aligned}$$

Hence, L commutes with Hankel translations.

(i) $\Rightarrow$ (iii). Assume that (i) holds. We define the functional T on $\mathcal{H}_e(\mathbf{C})$ as follows

$$\langle T, f\rangle = L(f)(0), \quad f \in \mathcal{H}_e(\mathbf{C}).$$

It is clear that T is in $\mathcal{H}_e(\mathbf{C})'$. Hence, there exists a complex measure λ on $\mathbf{C}$ having compact support [29, p. 231] such that

$$\langle T, f\rangle = \int_{\mathbf{C}} f(t)\, d\lambda(t), \quad f \in \mathcal{H}_e(\mathbf{C}). \qquad (3.6)$$

Then by using (3.6) it follows that

$$\begin{aligned}(Lf)(z) &= \tau_z(Lf)(0) = L(\tau_z f)(0)\\ &= \int_{\mathbf{C}} (\tau_z f)(t)\, d\lambda(t), \quad z \in \mathbf{C} \text{ and } f \in \mathcal{H}_e(\mathbf{C}).\end{aligned}$$

(iii) ⇒ (iv). Assume that

$$(Lf)(z) = \int_{\mathbf{C}} (\tau_z f)(t)\, d\lambda(t), \quad z \in \mathbf{C} \text{ and } f \in \mathcal{H}_e(\mathbf{C}),$$

for some complex measure λ on $\mathbf{C}$ having bounded support.

Let $f \in \mathcal{H}_e(\mathbf{C})$. According to (3.5), since $(\tau_z f)(t) = (\tau_t f)(z)$, $z, t \in \mathbf{C}$, it has

$$\begin{aligned}(Lf)(z) &= \int_{\mathbf{C}} \sum_{k=0}^{\infty} \frac{\Gamma(\mu+1)}{2^{2k}\Gamma(\mu+k+1)k!} t^{2k} \left(\Delta_\mu^k f\right)(z)\, d\lambda(t) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\mu+1)}{2^{2k}\Gamma(\mu+k+1)k!} \left(\Delta_\mu^k f\right)(z) \\ &\quad \times \int_{\mathbf{C}} t^{2k}\, d\lambda(t), \quad z \in \mathbf{C} \text{ and } f \in \mathcal{H}_e(\mathbf{C}). \end{aligned} \tag{3.7}$$

Here we have taken into account that the series is, for every $z \in \mathbf{C}$, uniformly convergent in the support of λ.

We denote, for every $k \in \mathbf{N}$, $\lambda_k = \int_{\mathbf{C}} t^{2k}\, d\lambda(t)$. We choose $M > 0$ such that $|t| \leqslant M$, for every t in the support of λ. Then, it follows

$$|\lambda_k| \leqslant \int_{\mathbf{C}} |t|^{2k}\, d|\lambda|(t) \leqslant M^{2k} |\lambda|(\mathbf{C}), \quad k \in \mathbf{N}, \tag{3.8}$$

where $|\lambda|$ represents the total variation measure of λ.

The function Φ is defined by

$$\Phi(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\mu+1)\lambda_k}{2^{2k}\Gamma(\mu+k+1)k!} z^k, \quad z \in \mathbf{C}.$$

From (3.8) it follows that

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{|\lambda_k|}{2^{2k}\Gamma(\mu+k+1)k!} |z|^k &\leqslant C \sum_{k=0}^{\infty} \frac{|zM^2|^k}{2^{2k}\Gamma(\mu+k+1)k!} \\ &\leqslant C \sum_{k=0}^{\infty} \frac{|zM^2|^k}{k!} = Ce^{M^2|z|}, \quad z \in \mathbf{C}.\end{aligned}$$

Hence Φ is an entire function of exponential type.

Moreover, (3.7) can be rewritten

$$(Lf)(z) = \left(\Phi(\Delta_\mu) f\right)(z), \quad z \in \mathbf{C} \text{ and } f \in \mathcal{H}_e(\mathbf{C}).$$

Note also that the series in (3.7) converges uniformly in every compact subset of $\mathbf{C}$.

(iv) $\Rightarrow$ (i). Suppose now that, for every $f \in \mathcal{H}_e(\mathbf{C})$,

$$(Lf)(z) = \sum_{k=0}^{\infty} a_k \big(\Delta_\mu^k f\big)(z), \quad z \in \mathbf{C},$$

for a certain $a_k \in \mathbf{C}$, $k \in \mathbf{N}$, where the series converges in $\mathcal{H}_e(\mathbf{C})$.

Hence, if $f \in \mathcal{H}_e(\mathbf{C})$, since $\tau_z \Delta_\mu f = \Delta_\mu \tau_z f$, $z \in \mathbf{C}$, according to Proposition 3.1, (i), it is concluded that

$$\begin{aligned}\tau_z(Lf)(t) &= \sum_{k=0}^{\infty} a_k \tau_z \big(\Delta_\mu^k f\big)(t) = \sum_{k=0}^{\infty} a_k \Delta_\mu^k (\tau_z f)(t) \\ &= L(\tau_z f)(t), \quad t, z \in \mathbf{C}. \qquad \Box\end{aligned}$$

Remark 3. Note that (3.5) can be rewritten as follows

$$\tau_z f = \Phi_z(\Delta_\mu) f, \quad f \in \mathcal{H}_e(\mathbf{C}) \text{ and } z \in \mathbf{C},$$

where Φ_z represents, for each $z \in \mathbf{C}$, the function defined by

$$\Phi_z(t) = \Gamma(\mu+1) \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}\Gamma(\mu+k+1)k!} t^k, \quad t \in \mathbf{C}.$$

Remark 4. The condition (iv) in Proposition 3.5 can be replaced by the following finer property:

(iv′) There exists an entire function Φ such that $L = \Phi(\Delta_\mu)$ on $\mathcal{H}_e(\mathbf{C})$ and that there exist $A, B > 0$ for which

$$\big|\Phi(z)\big| \leqslant A \mathbb{I}_\mu\big(B\sqrt{|z|}\big), \quad z \in \mathbf{C}.$$

Here $\mathbb{I}_\mu(z) = z^{-\mu} I_\mu(z)$, $z \in \mathbf{C}$, where I_μ denotes the modified Bessel function of the first kind and order μ [32, p. 77].

In the following we obtain a Hankel version of [20, Theorem 5]. We obtain a new class of hypercyclic operators in $\mathcal{H}_e(\mathbf{C})$.

Proposition 3.6. *Assume that L is a continuous linear mapping from $\mathcal{H}_e(\mathbf{C})$ into itself which commutes with the Hankel translation τ_z, for every $z \in \mathbf{C}$. Then L has an invariant and hypercyclic manifold that is dense in $\mathcal{H}_e(\mathbf{C})$ and L is a chaotic operator on $\mathcal{H}_e(\mathbf{C})$, provided that L is not a multiple of the identity operator.*

Proof. According to Proposition 3.5 there exists an entire function Φ of exponential type such that $L = \Phi(\Delta_\mu)$, that is, if $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbf{C}$, where $a_n \in \mathbf{C}$, $n \in \mathbf{N}$, then

$$Lf = \sum_{n=0}^{\infty} a_n \Delta_\mu^n f, \quad f \in \mathcal{H}_e(\mathbf{C}),$$

where the series converges in $\mathcal{H}_e(\mathbf{C})$.

For every $\alpha \in \mathbf{C}$, we define the function j_α by

$$j_\alpha(z) = (z\alpha)^{-\mu} J_\mu(z\alpha), \quad z \in \mathbf{C}.$$

We have that $\Delta_\mu j_\alpha(z) = -\alpha^2 j_\alpha(z)$, $\alpha, z \in \mathbf{C}$. Hence, for every $\alpha \in \mathbf{C}$,

$$Lj_\alpha = \sum_{n=0}^{\infty} a_n \left(-\alpha^2\right)^n j_\alpha = \Phi\left(-\alpha^2\right) j_\alpha. \tag{3.9}$$

To simplify we define $\Psi(z) = \Phi(-z^2)$, $z \in \mathbf{C}$.

From Proposition 2.2 it now follows that the range of L is dense in $\mathcal{H}_e(\mathbf{C})$, provided that $L \neq 0$. Indeed, suppose that Φ is not zero identically. Then the set

$$V = \left\{z \in \mathbf{C}\colon\ \Psi(z) \neq 0\right\}$$

is an open and non-empty subset of $\mathbf{C}$. Hence, according to Proposition 2.2, the linear space $\mathcal{M}_V$ generated by $\{j_\alpha\}_{\alpha \in V}$ is dense in $\mathcal{H}_e(\mathbf{C})$. Since $\mathcal{M}_V$ is contained in the range of L, it follows that the range of L is a dense subset of $\mathcal{H}_e(\mathbf{C})$.

Assume that L is not a multiple of the identity. Then Φ is not a constant function. The well known Liouville theorem implies that the sets W_1 and W_2 defined by

$$W_1 = \left\{z \in \mathbf{C}\colon\ \left|\Psi(z)\right| < 1\right\}$$

and

$$W_2 = \left\{z \in \mathbf{C}\colon\ \left|\Psi(z)\right| > 1\right\},$$

are not-empty open sets in $\mathbf{C}$. According to (3.9), it is clear that, for every $n \in \mathbf{N}$,

$$L^n j_\alpha = \Psi(\alpha)^n j_\alpha, \quad \alpha \in \mathbf{C}. \tag{3.10}$$

In particular, if $\alpha \in W_1$ then, from (3.10) we infer that $\lim_{n\to\infty} L^n j_\alpha = 0$, uniformly in every compact subset of $\mathbf{C}$. Hence $\lim_{n\to\infty} L^n f = 0$, in $\mathcal{H}_e(\mathbf{C})$, for every $f \in \mathcal{M}_{W_1}$.

We now define the mapping S on $\{j_\alpha\}_{\alpha \in W_2}$ by

$$Sj_\alpha = \frac{1}{\Psi(\alpha)} j_\alpha, \quad \alpha \in W_2,$$

and S is extended to the linear space $\mathcal{M}_{W_2}$ generates by $\{j_\alpha\}_{\alpha \in W_2}$ as a linear mapping. Thus S maps $\mathcal{M}_{W_2}$ into itself and

$$(LS) j_\alpha = L\left(\frac{1}{\Psi(\alpha)} j_\alpha\right) = j_\alpha, \quad \alpha \in W_2.$$

Hence, $(LS)f = f$, $f \in \mathcal{M}_{W_2}$. Moreover, by proceeding as above, we obtain that $\lim_{n\to\infty} S^n f = 0$, in $\mathcal{H}_e(\mathbf{C})$, for each $f \in \mathcal{M}_{W_2}$.

According to [20, Corollary 1.5], it follows that L has hypercyclic vectors. We denote by g a hypercyclic vector of L.

We are going to see that there exists an invariant and hypercyclic manifold with respect to L that is dense in $\mathcal{H}_e(\mathbf{C})$.

Let p be an holomorphic polynomial not identically zero. Then $p(L)$ is a continuous linear mapping from $\mathcal{H}_e(\mathbf{C})$ into itself and, as it is not hard to show, $p(L) = p(\Phi)(\Delta_\mu)$. Hence, $p(L)$ commutes with Hankel translation τ_z, for every $z \in \mathbf{C}$. Moreover, since Φ is not constant in $\mathbf{C}$, the range of $p(L)$ is dense in $\mathcal{H}_e(\mathbf{C})$. We now define the manifold $\mathcal{M}$ through

$$\mathcal{M} = \{p(L)g\colon\ p \text{ is a holomorphic polynomial}\}.$$

It is clear that $\mathcal{M}$ is invariant for L.

On the other hand, for every $n \in \mathbf{N}$ and every holomorphic polynomial p, it has

$$L^n p(L)g = p(L)L^n g.$$

Hence, if p is a holomorphic polynomial, since the set $\{L^n g\colon\ n \in \mathbf{N}\}$ and the range of $p(L)$ are dense in $\mathcal{H}_e(\mathbf{C})$, the set $\{L^n p(L)g\colon\ n \in \mathbf{N}\}$ is also dense in $\mathcal{H}_e(\mathbf{C})$.

Thus, we prove that $\mathcal{M}$ is a dense manifold of $\mathcal{H}_e(\mathbf{C})$ that is constituted by hypercyclic vectors.

We now prove that L is chaotic in $\mathcal{H}(\mathbf{C})$.

Since Ψ is entire and nonconstant, there exists $n \in \mathbf{N}$ such that $\Psi(G_n) \cap \partial D(0,1)$ contains a non-empty and open subset of the boundary $\partial D(0,1)$ of the unit disc $D(0,1)$. Here, for every $m \in \mathbf{N}$, G_m represents the closure of the disc $D(0,m)$ with center in the origin and radius m. The set E defined by

$$E = \{z \in G_n\colon\ \Psi(z)^l = 1,\ \text{for some } l \in \mathbf{N}\}$$

is infinity. Hence E has an adherence point in G_n. Then, by Proposition 2.2, we can prove that the linear space

$$\mathcal{M}_E = \operatorname{span}\{j_\alpha\colon\ \alpha \in E\}$$

generates by $\{j_\alpha\}_{\alpha\in E}$ is dense in $\mathcal{H}_e(\mathbf{C})$. Here, as above, $j_\alpha(z) = (\alpha z)^{-\mu} J_\mu(\alpha z)$, $z \in \mathbf{C}$ and $\alpha \in E$.

Assume that $\alpha \in E$. There exists $l \in \mathbf{N}$ such that $\Psi(\alpha)^l = 1$. Hence

$$L^l(j_\alpha) = \Psi(\alpha)^l j_\alpha = j_\alpha.$$

Thus, we see that j_α is a periodic point of L. Then $\mathcal{M}_E$ is constituted by periodic points of L and L is chaotic on $\mathcal{H}_e(\mathbf{C})$. $\square$

Remark 5. A continuous linear operator L on a topological linear space X is called cyclic if there exists a vector $x \in X$ for which the span of the orbit $\{L^n x\}_{n\in\mathbf{N}}$

is dense in X. In this case x is called a cyclic vector of L. It is obvious that if L is a hypercyclic operator on X then L is also a cyclic operator on X. Hence, according to Proposition 3.6 if L is a continuous linear mapping from $\mathcal{H}_e(\mathbf{C})$ into itself that commutes with Hankel translations τ_z, $z \in \mathbf{C}$, then L is a cyclic operator on $\mathcal{H}_e(\mathbf{C})$ provided that L is not a multiple of the identity operator. Moreover, by proceeding as in [12, p. 86], where the Bessel functions j_α, $\alpha \in \mathbf{C}$, replace the exponential functions, we can see that there exists a dense linear manifold $\mathcal{M}$ of $\mathcal{H}_e(\mathbf{C})$ such that each nonzero element of $\mathcal{M}$ is cyclic for every continuous linear mapping from $\mathcal{H}_e(\mathbf{C})$ into itself commuting with Hankel translations τ_z, $z \in \mathbf{C}$, that is not a multiple of the identity operator.

We now study the hypercyclicity and the chaoticity of the Hankel convolution operators on $\mathcal{E}_*$.

Proposition 3.7. *Suppose that $T \in \mathcal{E}'_*$. Then the Hankel convolution operator F_T defined on $\mathcal{E}_*$ by $F_T(\phi) = T \# \phi$, $\phi \in \mathcal{E}_*$, is hypercyclic and chaotic, provided that T is not a multiple of the Dirac δ-functional.*

Proof. Since the space $\mathcal{H}_e(\mathbf{C})$ is continuously contained in $\mathcal{E}(w)$ the restriction of T to $\mathcal{H}_e(\mathbf{C})$ is in $\mathcal{H}_e(\mathbf{C})'$. Also the restriction of the mapping F_T to $\mathcal{H}_e(\mathbf{C})$ defines a continuous linear mapping from $\mathcal{H}_e(\mathbf{C})$ into itself.

Suppose that $F_T(f) = \lambda f$, $f \in \mathcal{H}_e(\mathbf{C})$, for some $\lambda \in \mathbf{C}$. Then, for every $f \in \mathcal{H}_e(\mathbf{C})$

$$F_T(f)(0) = (T \# f)(0) = \langle T, \tau_0 f\rangle = \langle T, f\rangle = \lambda f(0).$$

Hence $T = \lambda\delta$.

Moreover, from (3.5) it follows that, for every $f \in \mathcal{H}_e(\mathbf{C})$,

$$F_T(f)(z) = \langle T, \tau_z f\rangle = \Gamma(\mu+1)\sum_{k=0}^{\infty}\frac{1}{2^{2k}\Gamma(\mu+k+1)k!}\big(\Delta_\mu^k f\big)(z) \times \big\langle T(t), t^{2k}\big\rangle, \quad z \in \mathbf{C}.$$

Thus Proposition 3.5 implies that F_T commutes with the Hankel translation τ_z, for every $z \in \mathbf{C}$.

Hence, if T is not a multiple of the Dirac δ-functional, from Proposition 3.6 it deduces that the mapping F_T is hypercyclic in $\mathcal{H}_e(\mathbf{C})$.

We will prove that $\mathcal{H}_e(\mathbf{C})$ is a dense subspace of $\mathcal{E}_*$. Then, according to [11, Lemma 1] and Proposition 3.6, we obtain that F_T is hypercyclic and chaotic in $\mathcal{E}_*$.

The density property of $\mathcal{H}_e(\mathbf{C})$ in $\mathcal{E}_*$ follows from Hahn–Banach theorem. Indeed, let $T \in \mathcal{E}'_*$ such that $\langle T, f\rangle = 0$, $f \in \mathcal{H}_e(\mathbf{C})$. In particular, for every $z \in \mathbf{C}$,

$$\big\langle T(t), (zt)^{-\mu} J_\mu(zt)\big\rangle = 0.$$

In other words, $h_\mu(T)(z) = 0$, $z \in \mathbf{C}$. Then, according to [5, Proposition 4.6], we obtain that

$$\langle T, \phi\rangle = \big\langle h_\mu(T), h_\mu(\phi)\big\rangle = 0, \quad \phi \in \mathcal{D}_*.$$

Hence, since $\mathcal{D}_*$ is a dense subspace of $\mathcal{E}_*$, it follows that $T = 0$ on $\mathcal{E}_*$. Then the Hahn–Banach theorem implies that $\mathcal{H}_e(\mathbf{C})$ is dense in $\mathcal{E}_*$.

Thus the proof is finished. □

As a consequence of Propositions 3.6 and 3.7 we obtain Hankel versions of celebrated results of Birkhoff [10], concerning the usual translation operators, and of MacLane [26], about the differentiation operators.

Corollary 3.8. (i) *For every $z \in \mathbf{C} \setminus \{0\}$, the Hankel translation operator τ_z is hypercyclic and chaotic on $\mathcal{H}_e(\mathbf{C})$ and on $\mathcal{E}_*$.*

(ii) *The operator Δ_μ is hypercyclic and chaotic on $\mathcal{H}_e(\mathbf{C})$ and on $\mathcal{E}_*$.*

4. Hankel convolution operators on the spaces $\mathcal{D}_*$ and its dual

In this section we study the Hankel convolution operators on the spaces $\mathcal{D}_*$ and $\mathcal{D}'_*$, the dual space of $\mathcal{D}_*$.

If $T \in \mathcal{E}'_*$, by using [5, Proposition 4.7, (3.1) and (3.2)], we can see that

$$\tau_x(T \# \phi) = T \# (\tau_x\phi), \qquad \Delta_\mu(T \# \phi) = T \# (\Delta_\mu\phi),$$
$$\psi \# (T \# \phi) = T \# (\psi \# \phi),$$

for every $\psi, \phi \in \mathcal{D}_*$ and $x \in (0,\infty)$.

In the following we characterize the Hankel convolution operators on $\mathcal{D}_*$ as those linear and continuous mappings on $\mathcal{D}_*$ into itself that commutes with Hankel translations, with Bessel operators or with Hankel convolutions. In Proposition 3.5 we established the corresponding result on the space $\mathcal{H}_e(\mathbf{C})$. Analogous properties on Zemanian spaces were shown in [3,7,8].

Proposition 4.1. *Let L be a continuous linear mapping from $\mathcal{D}_*$ into itself. The following assertions are equivalent.*

(i) *L commutes with Hankel translations, that is, for every $x \in (0,\infty)$, $L\tau_x = \tau_x L$ on $\mathcal{D}_*$.*
(ii) *There exists a (unique) $T \in \mathcal{E}'_*$ such that $L\phi = T \# \phi$, $\phi \in \mathcal{D}_*$.*
(iii) *L commutes with Hankel convolutions in the following sense, for each $\phi, \psi \in \mathcal{D}_*$, $L(\phi \# \psi) = \phi \# L(\psi)$.*
(iv) *L commutes with Hankel convolution in the following sense, for every $\phi \in \mathcal{D}_*$ and $T \in \mathcal{E}'_*$, $L(T \# \phi) = T \# L(\phi)$.*

Moreover (i) (*or, equivalently,* (ii), (iii) *and* (iv)) *implies that the following holds* (v) *L commutes with the Bessel operator Δ_μ, that is $L\Delta_\mu = \Delta_\mu L$, on $\mathcal{D}_*$.*

Proof. (i) $\Rightarrow$ (ii). We can proceed as in the proof of [7, Theorem 2.3].

(ii) $\Rightarrow$ (iii). It is sufficient to take into account [5, Proposition 4.1].

(iii) $\Rightarrow$ (iv). Let $T \in \mathcal{E}'_*$. We choose a function $\psi \in \mathcal{D}_1$ such that

$$\int_0^\infty \psi(x) x^{2\mu+1}\, dx = 2^\mu \Gamma(\mu+1).$$

For every $m \in \mathbf{N}$, we define

$$\psi_m(x) = m^{2\mu+2}\psi(mx), \quad x \in (0,\infty),$$

and

$$T_m = T \,\#\, \psi_m,$$

by invoking [5, Propositions 3.5 and 4.1] and by taking into account that T defines a continuous convolution operator from $\mathcal{D}_*$ into itself, we conclude that, for every $\phi \in \mathcal{D}_*$,

$$T_m \,\#\, \phi \to T \,\#\, \phi, \quad \text{as } m \to \infty,$$

in the sense of convergence in $\mathcal{D}_*$.

By (iii), since $T_m \in \mathcal{D}_*$, $m \in \mathbf{N}$ [5, Proposition 4.8], we can write

$$T \,\#\, (L\phi) = \lim_{m\to\infty} T_m \,\#\, L(\phi) = \lim_{m\to\infty} L(T_m \,\#\, \phi) = L(T \,\#\, \phi).$$

Thus (iv) is shown.

(iv) $\Rightarrow$ (i). Let $x \in (0,\infty)$. As usual, we define the Hankel translation operator τ_x on $\mathcal{D}'_*$ by transposition, that is, if $T \in \mathcal{D}'_*$ the functional $\tau_x T$ is defined by

$$\langle \tau_x T, \phi\rangle = \langle T, \tau_x\phi\rangle, \quad \phi \in \mathcal{D}_*.$$

Since τ_x is a continuous linear mapping from $\mathcal{D}_*$ into itself [5, Corollary 3.3], $\tau_x T \in \mathcal{D}'_*$, for each $T \in \mathcal{D}'_*$.

By denoting by δ the Dirac functional, we have that

$$\tau_x\phi = (\tau_x\delta) \,\#\, \phi, \quad \phi \in \mathcal{D}_*.$$

Indeed, if $\phi \in \mathcal{D}_*$, it follows

$$\begin{aligned}(\tau_x\delta) \,\#\, \phi(y) &= \langle \tau_x\delta, \tau_y\phi\rangle = \langle \delta, \tau_x\tau_y\phi\rangle = \big\langle \delta, \tau_y(\tau_x\phi)\big\rangle \\ &= (\tau_x\phi)(y), \quad y \in (0,\infty).\end{aligned}$$

Moreover, it is not hard to see, according to [5, Proposition 4.4], that $\tau_x\delta \in \mathcal{E}'_*$. Hence, from (iv) it follows that, for every $\phi \in \mathcal{D}_*$,

$$\tau_x(L\phi) = \tau_x\delta \,\#\, L\phi = L(\tau_x\delta \,\#\, \phi) = L(\tau_x\phi).$$

Hence L commutes with the Hankel translation operator τ_x.

Thus we have prove that the properties (i)–(iv) are equivalent.

To finish the proof of this proposition we are going to prove that (ii) $\Rightarrow$ (v).

Assume that there exists $T \in \mathcal{E}'_*$ such that

$$L\phi = T \# \phi, \quad \phi \in \mathcal{D}_*.$$

According to [1, Lemma 8, (b), (6)], we can write, for every $\phi \in \mathcal{D}_*$,

$$\begin{aligned} h_\mu\big(\Delta_\mu L(\phi)\big)(x) &= -x^2 h_\mu(T)(x) h_\mu(\phi)(x) \\ &= h_\mu\big(L(\Delta_\mu \phi)\big)(x), \quad x \in (0, \infty). \end{aligned}$$

Hence, from the uniqueness property of Hankel transformation on $\mathcal{D}_*$, it follows that

$$\Delta_\mu L\phi = L\Delta_\mu \phi, \quad \phi \in \mathcal{D}_*.$$

Thus we establish that L commutes with the Bessel operator Δ_μ. □

Remark 6. We do not know if condition (v) implies property (i) (and then (ii), (iii) and (iv)) in Proposition 4.1. The procedure developed in [3] does not work now because there is not any function $\phi \not\equiv 0$ in $\mathcal{D}_*$ having compactly supported h_μ transform.

Since $\mathcal{E}'_*$ is the space of convolution operators in $\mathcal{D}_*$, the elements of $\mathcal{E}'_*$ define Hankel convolution operators on $\mathcal{D}'_*$. If $S \in \mathcal{D}'_*$ and $T \in \mathcal{E}'_*$, the Hankel convolution $S \# T$ of S and T is the functional in $\mathcal{D}'_*$ defined by

$$\langle S \# T, \phi \rangle = \langle S, T \# \phi \rangle, \quad \phi \in \mathcal{D}_*.$$

Moreover, we can establish that the Hankel convolution operator associated to $T \in \mathcal{E}'_*$ is continuous on $\mathcal{D}'_*$.

Proposition 4.2. *Let $T \in \mathcal{E}'_*$. The mapping F_T defined by*

$$\begin{aligned} F_T : \mathcal{D}'_* &\to \mathcal{D}'_* \\ S &\to S \# T, \end{aligned}$$

is continuous from $\mathcal{D}'_*$ into itself, when on $\mathcal{D}'_*$ we consider the weak $*$ or the strong topology.

Proof. It is sufficient to take into account that the mapping $\phi \to T \# \phi$ is continuous from $\mathcal{D}_*$ into itself. □

Finally, it is shown that the Hankel convolution operator associated to every element of $\mathcal{E}'_*$ is hypercyclic and chaotic.

Proposition 4.3. *Let $T \in \mathcal{E}'_*$. Assume that T is not a multiple of the Dirac δ-functional. Then the Hankel convolution operator F_T defined as in Proposition* 4.2 *is hypercyclic and chaotic on $\mathcal{D}'_*$, when $\mathcal{D}'_*$ is equipped with the strong topology.*

Proof. According to Proposition 3.7 the functional $T \in \mathcal{E}'_*$ defines a Hankel convolution operator on $\mathcal{E}_*$ that is hypercyclic and chaotic. Since $\mathcal{E}_*$ is a dense subspace of $\mathcal{D}'_*$ when $\mathcal{D}'_*$ is endowed with the strong topology, by invoking [11, Lemma 1], we conclude that F_T is hypercyclic and chaotic on $\mathcal{D}'_*$, when on $\mathcal{D}'_*$ we consider the strong topology. □

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