JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 55 (1994) 99-124

Generalized incomplete gamma functions with applications

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Received 25 November 1992; revised 11 March 1993

Abstract

In this paper, we introduce new functions as generalizations of the incomplete gamma functions. The functions are found to be useful in heat conduction, probability theory and in the study of Fourier and Laplace transforms. Some important properties of the functions are studied. We have investigated the asymptotic behavior, Laplace transforms, special cases, decomposition formula, integral representations, convolutions, recurrence relations and differentiation formula of these functions. Applications of these functions in evaluation of certain inverse Laplace transforms to the definite integrals and to the infinite series of exponential functions are shown.

Keywords: Incomplete gamma functions; Convolutions; Laplace transforms

1. Introduction

The closed-form solutions to a considerable number of problems in applied mathematics, astrophysics, nuclear physics, statistics and engineering can be expressed in terms of incomplete gamma functions [2,9–12,16,20–23,27,32,34]

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t}\, dt, \quad \operatorname{Re}\alpha > 0, \tag{1}$$

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t}\, dt. \tag{2}$$

These functions were first investigated for real x by Legendre [24,25]. The functional behavior of these functions and the decomposition formula

$$\Gamma(\alpha) = \gamma(\alpha, x) + \Gamma(\alpha, x) \tag{3}$$

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SSDI 0377-0427(93)E0173-J

were studied by Prym [14]. The older theory of the incomplete gamma functions and references to the literature are given in [6,28,29].

Recently, the authors have shown [11,12,35–37] that the closed-form solutions to several problems in heat conduction can be expressed in terms of the generalized representation of the incomplete gamma functions

$$\gamma(\alpha, x; b) = \int_0^x t^{\alpha-1} e^{-t-bt^{-1}}\, dt, \quad \text{for } b = 0,\ \operatorname{Re}\alpha > 0, \tag{4}$$

$$\Gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1} e^{-t-bt^{-1}}\, dt. \tag{5}$$

They have shown that the functions $\Gamma(0, x^2/(4\alpha t); \lambda x^2/(4\alpha))$, $\Gamma(\pm\frac{1}{2}, x^2/(4\alpha t); \lambda x^2/(4\alpha))$ and $\Gamma(-\frac{3}{2}, x^2/(4\alpha t); \lambda x^2/(4\alpha))$ are solutions to the heat equation with time-dependent boundary conditions.

It should be noted that if we take $b = 0$ in (4) and (5), we get

$$\gamma(\alpha, x; 0) = \gamma(\alpha, x), \tag{6}$$

$$\Gamma(\alpha, x; 0) = \Gamma(\alpha, x). \tag{7}$$

Without loss of generality, we shall assume that $b \neq 0$.

Chaudhry and Ahmad [10] have introduced the probability density function

$$f(x) = 2\sqrt{\frac{\alpha}{\pi}} \exp(-(\sqrt{\alpha}x - \sqrt{\beta}x^{-1})^2), \quad \alpha > 0,\ \beta > 0,\ x > 0, \tag{8}$$

useful in size modeling. The cumulative probability function of the density (8) can be simplified in terms of the generalized incomplete gamma function:

$$F(x) = 1 - \frac{1}{\sqrt{\pi}}\Gamma(\tfrac{1}{2}, \alpha\sqrt{x}; \alpha\beta), \quad \alpha > 0,\ \beta > 0,\ x > 0. \tag{9}$$

Ahmad et al. [2] have also considered a class of three-parameter probability density functions

$$g(x) = cx^{\alpha-1}\exp(-ax - bx^{-1}), \quad a > 0,\ b > 0,\ c^{-1} = 2\left(\frac{b}{a}\right)^{\alpha/2} K_\alpha(2\sqrt{ab}). \tag{10}$$

The cumulative probability function of the density (10) can be expressed in terms of the generalized incomplete function to give

$$G(x) = 1 - ca^{-\alpha}\Gamma(\alpha, ax; ab), \quad a > 0,\ b > 0. \tag{11}$$

It was shown [2] that some of the well-known probability densities such as Weibull, log-normal, gamma, Chi-square, inverted Gaussian and their inverted densities are special cases of the density (10). Therefore, the cumulative density functions of all these classical probability densities can be expressed in terms of the function (5).

In this paper, we have studied important properties of these functions such as asymptotic behavior, Laplace transforms, special cases, decomposition formula, integral representations, convolutions, recurrence relations and differentiation formula. It is anticipated that the work presented in this paper will inspire scientists and engineers to find wide applications of these functions in several physical problems.

It should be noted that for the most part the expressions used are analytic and hence retain their validity for the complex case because of the principle of analytic continuation.

2. Main results and applications

Theorem 1 (Decomposition theorem).

$$\gamma(\alpha, x; b) + \Gamma(\alpha, x; b) = 2b^{\alpha/2}K_\alpha(2\sqrt{b}), \quad b > 0. \tag{12}$$

Proof. This follows from the definitions (4) and (5) of the generalized incomplete gamma functions and from [14, p.82 (23)], [15, p.146 (29)]. □

In view of the decomposition formula (12), it is sufficient to study the properties of the function $\Gamma(\alpha, x; b)$.

Theorem 2.

$$\int_x^\infty t^{\alpha-1}e^{-at-bt^{-1}}\,dt = a^{-\alpha}\Gamma(\alpha, ax; ab), \quad a > 0. \tag{13}$$

Proof. The left-hand side equals

$$\int_x^\infty t^{\alpha-1}e^{-at-bt^{-1}}\,dt.$$

Substituting $t = \xi/a$ and $dt = d\xi/a$, $a > 0$, we get that the left-hand side equals

$$a^{-\alpha}\int_{ax}^\infty \xi^{\alpha-1}e^{-\xi-ab\xi^{-1}}\,d\xi = a^{-\alpha}\Gamma(\alpha, ax; ab),$$

which is the right-hand side. □

Theorem 3 (Laplace transform representation). *Let L be the Laplace transform operator. Then,*

$$L\{(t+x)^{\alpha-1}e^{-b/(x+t)}; s\} = s^{-\alpha}e^{sx}\Gamma(\alpha, sx; bs), \quad s > 0. \tag{14}$$

Proof. This follows from (13) when we take $a = s$ and use the transformation $t = \tau + x$. □

Theorem 4 (Recurrence relation).

$$\Gamma(\alpha+1, x; b) = \alpha\Gamma(\alpha, x; b) + b\Gamma(\alpha-1, x; b) + x^\alpha e^{-x-bx^{-1}}. \tag{15}$$

Proof.

$$\frac{d}{dx}(x^\alpha e^{-x-bx^{-1}}) = (\alpha x^{\alpha-1} + bx^{\alpha-2} - x^\alpha)e^{-x-bx^{-1}}. \tag{16}$$

Integrating both sides in (16) with respect to x from x to ∞ and using (5), we get

$$-x^{\alpha}e^{-x-bx^{-1}} = \alpha\Gamma(\alpha, x; b) + b\Gamma(\alpha - 1, x; b) - \Gamma(\alpha + 1, x; b). \tag{17}$$

Rearranging the terms in (17) yields the proof. □

Corollary.

$$\Gamma(\alpha + 1, x) = \alpha\Gamma(\alpha, x) + x^{\alpha}e^{-x}. \tag{18}$$

Proof. This follows from (15) when we take $b = 0$. □

Theorem 5 (Differentiation formula).

$$\frac{\partial}{\partial x}\big(\Gamma(\alpha, x; b)\big) = -x^{\alpha-1}e^{-x-bx^{-1}}. \tag{19}$$

Proof. This follows from (5) when we differentiate with respect to x. □

Corollary.

$$\frac{\partial}{\partial x}\big(\Gamma(\alpha + 1, x)\big) = -x^{\alpha-1}e^{-x}. \tag{20}$$

Proof. This follows from (19) when we take $b = 0$. □

Theorem 6 (Parametric differentiation).

$$\frac{\partial}{\partial b}\big(\Gamma(\alpha, x; b)\big) = -\Gamma(\alpha - 1, x; b). \tag{21}$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial b}\big(\Gamma(\alpha, x; b)\big) &= \frac{\partial}{\partial b}\left(\int_x^{\infty} t^{\alpha-1}e^{-t-bt^{-1}}\,dt\right) = -\int_x^{\infty} t^{\alpha-2}e^{-t-bt^{-1}}\,dt \\ &= -\Gamma(\alpha - 1, x; b). \quad \square\end{aligned} \tag{22}$$

It should be noted that the process of differentiation with respect to the parameter b under the integral sign in (22) is justified [7, pp. 427–448].

Theorem 7 (Connection with other special functions).

$$\Gamma(\tfrac{1}{2}, x; b) = \tfrac{1}{2}\sqrt{\pi}\left[e^{-2\sqrt{b}}\,\mathrm{Erfc}\left(\sqrt{x} - \frac{\sqrt{b}}{\sqrt{x}}\right) + e^{2\sqrt{b}}\,\mathrm{Erfc}\left(\sqrt{x} + \frac{\sqrt{b}}{\sqrt{x}}\right)\right]. \tag{23}$$

Proof. It follows from (5) that

$$\Gamma\left(\frac{1}{2}, \frac{1}{x}; b\right) = \int_{1/x}^{\infty} \tau^{-1/2}e^{-\tau-b\tau^{-1}}\,d\tau. \tag{24}$$

Substituting $\tau = 1/t$ in (24), we get

$$\Gamma\left(\frac{1}{2}, \frac{1}{x}; b\right) = e^{-bx}(\{e^{bx}\} * \{x^{-3/2}e^{-1/x}\}), \tag{25}$$

where "*" is the convolution operator defined by

$$a(x) * b(x) = \int_0^x a(x-t)b(t)\,dt. \tag{26}$$

We can write (25) in the operational form [15, p.131 (20) and p.246 (10)]

$$\Gamma\left(\frac{1}{2}, \frac{1}{x}; b\right) = e^{-bx}L^{-1}\left\{\frac{1}{p-b}e^{-2\sqrt{p}}; x\right\}. \tag{27}$$

Replacing x by $1/x$ in (27) and using the identity [15, p.246 (10)], we get the proof of (23). □

Corollary.

$$\Gamma(-\tfrac{1}{2}, x; b) = \frac{\sqrt{\pi}}{2\sqrt{b}}\left[e^{-2\sqrt{b}}\,\mathrm{Erfc}\left(\sqrt{x} - \frac{\sqrt{b}}{\sqrt{x}}\right) - e^{2\sqrt{b}}\,\mathrm{Erfc}\left(\sqrt{x} + \frac{\sqrt{b}}{\sqrt{x}}\right)\right]. \tag{28}$$

Proof. This follows from (23) when we use the parametric differentiation formula (22). □

Remark. It should be noted that the functions $\Gamma(\frac{1}{2}, x; b)$ and $\Gamma(-\frac{1}{2}, x; b)$ can be expressed in terms of the incomplete gamma and confluent hypergeometric functions by using the relations [14, p.133 (42)] and [19, p.940 (8-351)(4)] and the equations (23) and (28). We have the following representations:

$$\Gamma(\tfrac{1}{2}, x; b) = e^{-x-bx^{-1}}[\psi(\tfrac{1}{2}, \tfrac{1}{2}; u) + \psi(\tfrac{1}{2}, \tfrac{1}{2}; v)], \tag{29}$$

$$\Gamma(\tfrac{1}{2}, x; b) = \tfrac{1}{2}[e^{-2\sqrt{b}}\Gamma(\tfrac{1}{2}, u) + e^{2\sqrt{b}}\Gamma(\tfrac{1}{2}, v)], \tag{30}$$

$$\Gamma(-\tfrac{1}{2}, x; b) = \frac{1}{2\sqrt{b}}e^{-x-bx^{-1}}[\psi(\tfrac{1}{2}, \tfrac{1}{2}; u) - \psi(\tfrac{1}{2}, \tfrac{1}{2}; v)], \tag{31}$$

$$\Gamma(-\tfrac{1}{2}, x; b) = \frac{1}{2\sqrt{b}}[e^{-2\sqrt{b}}\Gamma(\tfrac{1}{2}, u) - e^{2\sqrt{b}}\Gamma(\tfrac{1}{2}, v)], \tag{32}$$

where

$$u = x + \frac{b}{x} + 2\sqrt{b} \tag{33}$$

and

$$v = x + \frac{b}{x} - 2\sqrt{b}. \tag{34}$$

Remark. In view of formula (22) and the identities (23), (28)–(32), it follows that we can always express $\Gamma(n + \frac{1}{2}, x; b)$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$, in terms of the complementary error functions.

Theorem 8 (Laplace transformation). *Let L be the Laplace transform operator. Then,*

$$L\{\Gamma(\alpha,t;b);p\} = \frac{2}{p}b^{\alpha/2}[K_\alpha(2\sqrt{b}) - (p+1)^{-\alpha/2}K_\alpha(2\sqrt{b(p+1)})], \quad b>0, \ p>-1. \tag{35}$$

Proof. According to the decomposition theorem (12), we have

$$\Gamma(\alpha,t;b) = 2b^{\alpha/2}K_\alpha(2\sqrt{b}) - \int_0^t \tau^{\alpha-1}e^{-\tau-b\tau^{-1}}\,d\tau, \quad b>0. \tag{36}$$

We can write (36) in the convolution form as

$$\Gamma(\alpha,t;b) = 2b^{\alpha/2}K_\alpha(2\sqrt{b}) - \{1\} * \{t^{\alpha-1}e^{-t-bt^{-1}}\}. \tag{37}$$

Taking the Laplace transform of both sides in (37), and using the identities [15, p.131 (20) and p.146 (29)], we get the proof of (35). □

Corollary.

$$L\left\{e^{-2\sqrt{b}}\,\mathrm{Erfc}\left(\sqrt{t}-\frac{\sqrt{b}}{\sqrt{t}}\right) + e^{2\sqrt{b}}\,\mathrm{Erfc}\left(\sqrt{t}+\frac{\sqrt{b}}{\sqrt{t}}\right);p\right\} = \frac{2}{p}\left[e^{-2\sqrt{b}} - \frac{e^{-2\sqrt{b(p+1)}}}{\sqrt{p+1}}\right],$$
$$b>0, \quad p>-1, \tag{38}$$

and

$$L\left\{e^{-2\sqrt{b}}\,\mathrm{Erfc}\left(\sqrt{t}-\frac{\sqrt{b}}{\sqrt{t}}\right) - e^{2\sqrt{b}}\,\mathrm{Erfc}\left(\sqrt{t}+\frac{\sqrt{b}}{\sqrt{t}}\right);p\right\} = \frac{2}{p}[e^{-2\sqrt{b}} - e^{-2\sqrt{b(p+1)}}],$$
$$b>0, \quad p>-1. \tag{39}$$

Proof. This follows from (23), (28) and (32) when we use the fact [23, p.112 (5.8.5)]. □

Theorem 9 (Laplace transformation).

$$L\left\{\Gamma\left(\alpha,\frac{1}{t};b\right);p\right\} = \frac{2}{p}(p+b)^{\alpha/2}K_\alpha(2\sqrt{p+b}), \quad p>-b. \tag{40}$$

Proof. According to (5), we have

$$\Gamma\left(\alpha,\frac{1}{t};b\right) = \int_{1/t}^\infty \tau^{\alpha-1}e^{-\tau-b\tau^{-1}}\,d\tau. \tag{41}$$

Substituting $\tau = 1/\xi$ in (41), we get

$$\Gamma\left(\alpha,\frac{1}{t};b\right) = \int_0^t e^{-b\xi}e^{-1/\xi}\frac{d\xi}{\xi^{\alpha+1}}. \tag{42}$$

Multiplying both the sides in (42) by e^{bt}, we get

$$e^{bt}\Gamma\left(\alpha,\frac{1}{t};b\right) = \int_0^t e^{b(t-\xi)}e^{-1/\xi}\frac{d\xi}{\xi^{\alpha+1}}, \tag{43}$$

which can be written in the convolution form as

$$e^{bt}\Gamma\left(\alpha, \frac{1}{t}; b\right) = \{e^{bt}\} * \left\{\frac{1}{t^{\alpha+1}} e^{-1/t}\right\}. \tag{44}$$

Taking the Laplace transform of both sides in (44) and using [15, p.131 (20) and p.146 (29)], we get

$$L\left\{e^{bt}\Gamma\left(\alpha, \frac{1}{t}; b\right); p\right\} = \frac{2}{p-b} p^{\alpha/2} K_\alpha(2\sqrt{p}), \quad p > 0. \tag{45}$$

Using the shift property [15, p.129 (5)], we get the proof of (40). □

Corollary. (*See* [15, p.179 (32)].)

$$L\left\{\Gamma\left(\alpha, \frac{1}{t}\right); p\right\} = 2p^{\alpha/2-1} K_\alpha(2\sqrt{p}), \quad p > 0. \tag{46}$$

Proof. This follows from (40) when we take $b = 0$. □

Corollary.

$$L\left\{e^{-2\sqrt{b}} \operatorname{Erfc}\left(\frac{1}{\sqrt{t}} - \sqrt{bt}\right) + e^{2\sqrt{b}} \operatorname{Erfc}\left(\frac{1}{\sqrt{t}} + \sqrt{bt}\right); p\right\} = \frac{2}{p} e^{-2\sqrt{p+b}}, \quad p > -b, \tag{47}$$

$$L\left\{e^{-2\sqrt{b}} \operatorname{Erfc}\left(\frac{1}{\sqrt{t}} - \sqrt{bt}\right) - e^{2\sqrt{b}} \operatorname{Erfc}\left(\frac{1}{\sqrt{t}} + \sqrt{bt}\right); p\right\} = \frac{2}{p} b^{1/2}(p+b)^{-1/2} e^{-2\sqrt{p+b}},$$
$$p > -b. \tag{48}$$

Proof. This follows from (23), (28) and (39) when we use the fact [23, p.112 (5.8.5)]. In particular, when $b = 0$ in (47), we get

$$L\left\{\operatorname{Erfc}\left(\frac{1}{\sqrt{t}}\right); p\right\} = \frac{1}{p} e^{-2\sqrt{p}}, \quad p > 0. \qquad \square \tag{49}$$

Note. The entry [15, p.176 (6)] seems incorrect. This can also be seen from (45) and (47).

Theorem 10 (Definite integrals).

$$\int_0^t e^{(b-\lambda)x} \Gamma\left(\alpha, \frac{1}{x}; b\right) dx = \frac{e^{-\lambda t}}{b-\lambda}\left[e^{bt}\Gamma\left(\alpha, \frac{1}{t}; b\right) - e^{\lambda t}\Gamma\left(\alpha, \frac{1}{t}; \lambda\right)\right]. \tag{50}$$

Proof. According to (44), we have

$$e^{bt}\Gamma\left(\alpha, \frac{1}{t}; b\right) = \{e^{bt}\} * \left\{\frac{e^{-1/t}}{t^{\alpha+1}}\right\}. \tag{51}$$

Taking the convolution of both sides in (51) with $\{e^{\lambda t}\}$ and using the associative property of convolution, we get

$$\{e^{\lambda t}\} * \left\{e^{bt}\Gamma\left(\alpha, \frac{1}{t}; b\right)\right\} = (\{e^{\lambda t}\} * \{e^{bt}\}) * \left\{\frac{e^{-1/t}}{t^{\alpha+1}}\right\}. \tag{52}$$

However,

$$\{e^{\lambda t}\} * \{e^{bt}\} = \frac{1}{b-\lambda}[\{e^{bt}\} - \{e^{\lambda t}\}]. \tag{53}$$

Therefore, it follows from (52) and (53) that

$$\{e^{\lambda t}\} * \left\{e^{bt}\Gamma\left(\alpha, \frac{1}{t}; b\right)\right\} = \frac{1}{b-\lambda}\left[\{e^{bt}\} * \left\{\frac{e^{-1/t}}{t^{\alpha+1}}\right\} - \{e^{\lambda t}\} * \left\{\frac{e^{-1/t}}{t^{\alpha+1}}\right\}\right], \tag{54}$$

which can be simplified by using (51) to give

$$e^{\lambda t}\int_0^t e^{(b-\lambda)x}\Gamma\left(\alpha, \frac{1}{x}; b\right) dx = \frac{1}{b-\lambda}\left[e^{bt}\Gamma\left(\alpha, \frac{1}{t}; b\right) - e^{\lambda t}\Gamma\left(\alpha, \frac{1}{t}; \lambda\right)\right]. \tag{55}$$

The multiplication of both sides in (55) by $e^{-\lambda t}$ yields the proof of (50). □

Corollary.

$$\int_0^t e^{bx}\Gamma\left(\alpha, \frac{1}{x}; b\right) dx = \frac{1}{b}\left[e^{bt}\Gamma\left(\alpha, \frac{1}{t}; b\right) - \Gamma\left(\alpha, \frac{1}{t}\right)\right], \quad b > 0. \tag{56}$$

Proof. This follows from (50) when we take $\lambda = 0$ and use (7). In particular when we take $\alpha = 0$ in (56), we get

$$\int_0^t e^{bx}\Gamma\left(0, \frac{1}{x}; b\right) dx = \frac{1}{b}\left[e^{bt}\Gamma\left(0, \frac{1}{t}; b\right) + \mathrm{Ei}\left(-\frac{1}{t}\right)\right], \quad b > 0, \tag{57}$$

where $\mathrm{Ei}(x)$ is the exponential integral function [19, p.925 (8.211)(1)]. □

Corollary.

$$e^{bx}\Gamma\left(\alpha, \frac{1}{x}; b\right) \geqslant \Gamma\left(\alpha, \frac{1}{x}\right), \quad \textit{for all } b \geqslant 0, x > 0, \tag{58}$$

$$e^{bx}\Gamma\left(\alpha, \frac{1}{x}; b\right) \leqslant \Gamma\left(\alpha, \frac{1}{x}\right), \quad \textit{for all } b \leqslant 0, x > 0. \tag{59}$$

Proof. Since the integrand in (56) is positive for all b and α when $x > 0$, it follows that

$$\frac{1}{b}\left[e^{bt}\Gamma\left(\alpha, \frac{1}{t}; b\right) - \Gamma\left(\alpha, \frac{1}{t}\right)\right] \geqslant 0, \quad t > 0. \tag{60}$$

Therefore, if $b \geqslant 0$, we get (58) from (60) and if $b \leqslant 0$, we get (59) from (60). □

Corollary.

$$\int_0^t \Gamma\left(\alpha, \frac{1}{x}; \lambda\right) dx = t\Gamma\left(\alpha, \frac{1}{t}; \lambda\right) - \Gamma\left(\alpha - 1, \frac{1}{t}; \lambda\right). \tag{61}$$

Proof. Letting $b \to \lambda$ in (50) and using the l'Hôpital rule, we get

$$\int_0^t \Gamma\left(\alpha, \frac{1}{x}; \lambda\right) dx = \lim_{b \to \lambda} \frac{\partial}{\partial b}\left[e^{-\lambda t}\left\{e^{bt}\Gamma\left(\alpha, \frac{1}{t}; \lambda\right) - e^{\lambda t}\Gamma\left(\alpha, \frac{1}{t}; \lambda\right)\right\}\right]$$
$$= t\Gamma\left(\alpha, \frac{1}{t}; \lambda\right) + \frac{\partial}{\partial b}\left(\Gamma\left(\alpha, \frac{1}{t}; b\right)\right)\Big|_{b=\lambda}. \tag{62}$$

From (22) and (62) we get the proof. □

Remark. It should be noted that (61) is extremely important for practical purposes. Several of its special cases can be considered when we specialize the parameters α and λ. In particular, when we take $\lambda = 0$ in (61), we get the interesting identity

$$\int_0^t \Gamma\left(\alpha, \frac{1}{x}\right) dx = t\Gamma\left(\alpha, \frac{1}{t}\right) - \Gamma\left(\alpha - 1, \frac{1}{t}\right),$$

which does not seem to be available in the literature. Similarly, the cases when $\alpha = 0,\ n + \frac{1}{2}$, $n = 0, \pm 1, \pm 2, \ldots$, are also of interest to engineers and scientists.

Theorem 11. *For $\alpha \leqslant -1$ and $b < 0$,*

$$\Gamma(\alpha, x; b) < e^b \Gamma(\alpha, b + x), \quad x > 0. \tag{63}$$

Proof. Let $f(x) = e^x \Gamma(\alpha, x)$. Then, according to [14, p.135 (12)],

$$f^{(n)}(x) = (-1)^n (1 - \alpha)_n e^x \Gamma(\alpha - n, x), \quad n = 0, 1, 2, 3, \ldots. \tag{64}$$

Using the Taylor expansion theorem and (64), we get

$$e^{b+x}\Gamma(\alpha, b + x) = \sum_{n=0}^{\infty} \frac{(-b)^n}{n!}(1 - \alpha)_n e^x \Gamma(\alpha - n, x)$$

or

$$e^b \Gamma(\alpha, b + x) = \sum_{n=0}^{\infty} \frac{(-b)^n}{n!}(1 - \alpha)_n \Gamma(\alpha - n, x). \tag{65}$$

Using the series expansion of $e^{-b/t}$ in (5), we can write

$$\Gamma(\alpha, x; b) = \sum_{n=0}^{\infty} \frac{(-b)^n}{n!}\Gamma(\alpha - n, x). \tag{66}$$

Comparing (65) and (66), we get the proof of (63). □

Note. It is conjectured that the inequality (63) can be improved.

Theorem 12.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{b}{x}\right)^n E_n(x) = \frac{1}{x}\Gamma(1, x; b), \quad x > 0. \tag{67}$$

Proof. Substituting $\alpha = 1$ in (66), we get

$$\sum_{n=0}^{\infty} \frac{(-b)^n}{n!} \Gamma(1-n, x) = \Gamma(1, x; b). \tag{68}$$

However [14, p.136 (19)],

$$\Gamma(1-n, x) = x^{1-n} E_n(x). \tag{69}$$

From (68) and (69) we get the proof. □

Theorem 13. *Let* $f_n(x) = \sum_{m=0}^{n-1} m!\,(-x)^{m+1}$ *be the polynomial of degree n. Then,*

$$\Gamma(0, x; b) = {}_0F_1(-, 1; b) E_1(x) + \sum_{n=1}^{\infty} \frac{b^n}{(n!)^2} f_n\left(\frac{1}{x}\right). \tag{70}$$

Proof. This follows from (66) when we use the result [14, p.137 (20)]. □

Remark. It should be noted that (70) is extremely important to determine the asymptotic behavior of $\Gamma(0, x; b)$ and hence of $\Gamma(n, x; b)$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$.

Theorem 14 (Asymptotic formula). *For fixed* $b > 0$ *and* α,

$$\Gamma(\alpha, x; b) \sim 2b^{\alpha/2} K_\alpha(2\sqrt{b}) - x^\alpha e^{-x-bx^{-1}}, \quad \text{as } x \to 0^+. \tag{71}$$

Proof. It follows from (5) that for $b > 0$,

$$\Gamma(\alpha, x; b) = 2b^{\alpha/2} K_\alpha(2\sqrt{b}) - \int_0^x t^{\alpha-1} e^{-t-bt^{-1}}\, dt, \quad b > 0. \tag{72}$$

However, for fixed $b > 0$ and α,

$$\int_0^x t^{\alpha-1} e^{-t-bt^{-1}}\, dt \to x^\alpha e^{-x-bx^{-1}}, \quad \text{as } x \to 0^+. \tag{73}$$

Letting $x \to 0$ in (72) and using (73), we get the proof of (71). □

Theorem 15. (*See* [14, p.100 (19)].)

$$K_\alpha(2\sqrt{p+b}) = p^{\alpha/2}(p+b)^{-\alpha/2} \sum_{n=1}^{\infty} \frac{(-b)^n}{n!} p^{-n/2} K_{\alpha-n}(2\sqrt{p}), \quad p > 0, \quad b > 0.$$

Proof. Replacing x by $1/t$ in (66), we get

$$\Gamma\left(\alpha, \frac{1}{t}; b\right) = \sum_{n=0}^{\infty} \frac{(-b)^n}{n!} \Gamma\left(\alpha - n, \frac{1}{t}\right). \tag{74}$$

Taking the Laplace transform of both sides in (74) and using (40) and [15, p.179 (32)], we get the proof. □

Theorem 16. *Let F_c be the cosine transform operator. Then,*

$$F_c\left\{\frac{2\beta e^{-\alpha(x^2+\beta^2)}}{x^2+\beta^2}; y\right\} = \sqrt{\pi}\Gamma(\tfrac{1}{2}, \alpha\beta^2; \tfrac{1}{4}\beta^2 y^2), \quad \alpha > 0,\ \beta > 0,\ y > 0. \tag{75}$$

Proof. This follows from (23) and from [15, p.15 (15)]. □

Theorem 17. *Let F_s be the sine transform operator. Then,*

$$F_s\left\{\frac{4x e^{-\alpha(x^2+\beta^2)}}{\beta(x^2+\beta^2)}; y\right\} = \sqrt{\pi} y \Gamma(-\tfrac{1}{2}, \alpha\beta^2; \tfrac{1}{4}\beta^2 y^2), \quad \alpha > 0,\ \beta > 0,\ y > 0. \tag{76}$$

Proof. This follows from (28) and from [15, p.74 (26)]. □

Theorem 18 (Connection with trigonometric functions). *For $w > 0$ and $x \geqslant 0$,*

$$\gamma(\tfrac{1}{2}, x; iw) + \Gamma(\tfrac{1}{2}, x; iw) = \sqrt{\pi} e^{-\sqrt{2w}}[\cos(\sqrt{2w}) - i\sin(\sqrt{2w})]. \tag{77}$$

Proof. This follows from the decomposition theorem (12) when we take $\alpha = \frac{1}{2}$ and substitute $b = iw$, $w > 0$, and use the fact that $K_{1/2}(z) = \sqrt{\pi/(2z)}e^{-z}$. □

Remark. In view of (12) and (77), it follows that the sum $\gamma(n+\frac{1}{2}, x; iw) + \Gamma(n+\frac{1}{2}, x; iw)$ can always be expressed in terms of trigonometric functions for $w > 0$, $x \geqslant 0$ and $n = 0, \pm1, \pm2, \pm3, \ldots$. For other values of α the problem remains open. However, we have

$$|\Gamma(\alpha, x; iw)| \leqslant \Gamma(\alpha, x), \quad x \geqslant 0,\ w \geqslant 0, \tag{78}$$

and

$$|\gamma(\alpha, x; iw)| \leqslant \gamma(\alpha, x), \quad x \geqslant 0,\ w \geqslant 0. \tag{79}$$

3. Graphical and tabular representation of the function $\Gamma(\alpha, x; b)$

For numerical and scientific computations, the generalized representation of an incomplete gamma function can easily be tabulated by using IMSL FORTRAN subroutines for mathematical applications [38]. In this regard, the values of the function are calculated by using the numerical integration subroutine QDAGI. The subroutine uses a globally adaptive scheme in an attempt to reduce the absolute error. We emphasize that QDAGI is an implementation of the subroutine QAGI, which is fully documented in [31].

The modified Bessel functions of the third kind that are used for normalizing the function (cf. Theorem 1, Eq. (12)) are computed by the IMSL subroutine BSKS which is based on the work of Cody [13]. On the other hand, subroutine GAMIC is used for the incomplete gamma function, which is based on the computational procedure of Gautschi [18]. It should be added that $\Gamma(\alpha, x; 0)$ calculated by using the numerical integration subroutine QDAGI provides exactly the same results as that of the incomplete gamma function calculated by the subroutine GAMIC.

Table 1
Normalized representation of the generalized incomplete gamma function ($\alpha = -1.00$)

x	$\Gamma^*(\alpha,x;0.5)$	$\Gamma^*(\alpha,x;1.0)$	$\Gamma^*(\alpha,x;1.5)$	$\Gamma^*(\alpha,x;2.0)$	$\Gamma^*(\alpha,x;2.5)$
0.01000	1.00000	1.00000	1.00000	1.00000	1.00000
0.02000	1.00000	1.00000	1.00000	1.00000	1.00000
0.03000	1.00000	1.00000	1.00000	1.00000	1.00000
0.04000	0.99999	1.00000	1.00000	1.00000	1.00000
0.05000	0.99990	1.00000	1.00000	1.00000	1.00000
0.06000	0.99949	1.00000	1.00000	1.00000	1.00000
0.07000	0.99833	1.00000	1.00000	1.00000	1.00000
0.08000	0.99595	0.99999	1.00000	1.00000	1.00000
0.09000	0.99195	0.99995	1.00000	1.00000	1.00000
0.10000	0.98607	0.99985	1.00000	1.00000	1.00000
0.15000	0.92879	0.99601	0.99980	0.99999	1.00000
0.20000	0.84119	0.97968	0.99760	0.99973	0.99997
0.25000	0.74551	0.94670	0.98966	0.99808	0.99965
0.30000	0.65396	0.89962	0.97291	0.99300	0.99825
0.35000	0.57137	0.84345	0.94662	0.98255	0.99446
0.40000	0.49886	0.78290	0.91188	0.96565	0.98699
0.45000	0.43602	0.72148	0.87067	0.94225	0.97492
0.50000	0.38185	0.66152	0.82516	0.91301	0.95788
0.55000	0.33521	0.60445	0.77726	0.87901	0.93598
0.60000	0.29504	0.55102	0.72855	0.84146	0.90971
0.65000	0.26036	0.50157	0.68022	0.80151	0.87976
0.70000	0.23036	0.45617	0.63313	0.76021	0.84691
0.75000	0.20432	0.41469	0.58786	0.71842	0.81194
0.80000	0.18166	0.37696	0.54478	0.67686	0.77557
0.85000	0.16188	0.34270	0.50412	0.63606	0.73844
0.90000	0.14457	0.31165	0.46597	0.59642	0.70111
0.95000	0.12937	0.28354	0.43034	0.55825	0.66402
1.00000	0.11599	0.25810	0.39719	0.52172	0.62754
1.50000	0.04240	0.10448	0.17697	0.25423	0.33229
2.00000	0.01729	0.04504	0.08049	0.12181	0.16738
2.50000	0.00756	0.02040	0.03774	0.05906	0.08388
3.00000	0.00347	0.00960	0.01818	0.02914	0.04236
3.50000	0.00165	0.00465	0.00896	0.01462	0.02162
4.00000	0.00081	0.00230	0.00451	0.00745	0.01116
4.50000	0.00040	0.00116	0.00230	0.00384	0.00582
5.00000	0.00021	0.00060	0.00119	0.00201	0.00307
5.50000	0.00011	0.00031	0.00062	0.00106	0.00163
6.00000	0.00006	0.00016	0.00033	0.00056	0.00087
6.50000	0.00003	0.00009	0.00018	0.00030	0.00047
7.00000	0.00002	0.00005	0.00009	0.00016	0.00026
7.50000	0.00001	0.00003	0.00005	0.00009	0.00014
8.00000	0.00000	0.00001	0.00003	0.00005	0.00008
8.50000	0.00000	0.00001	0.00002	0.00003	0.00004
9.00000	0.00000	0.00000	0.00001	0.00001	0.00002
9.50000	0.00000	0.00000	0.00000	0.00001	0.00001
10.00000	0.00000	0.00000	0.00000	0.00000	0.00001
$C(\alpha,b)$	0.88869	0.27973	0.12851	0.06983	0.04186

$$\Gamma^*(\alpha,x;b) = \Gamma(\alpha,x;b)/C(\alpha,b),\ C(\alpha,b) = 2b^{\alpha/2}K_\alpha(2\sqrt{b})$$

Table 2
Normalized representation of the generalized incomplete gamma function ($\alpha = -0.75$)

x	$\Gamma^*(\alpha, x; 0.5)$	$\Gamma^*(\alpha, x; 1.0)$	$\Gamma^*(\alpha, x; 1.5)$	$\Gamma^*(\alpha, x; 2.0)$	$\Gamma^*(\alpha, x; 2.5)$
0.01000	1.00000	0.99999	1.00000	1.00000	1.00000
0.02000	1.00000	1.00000	1.00000	1.00000	1.00000
0.03000	1.00000	1.00001	1.00000	1.00000	1.00000
0.04000	1.00000	1.00000	1.00000	1.00000	1.00000
0.05000	0.99994	1.00000	1.00000	1.00000	1.00000
0.06000	0.99970	1.00000	1.00000	1.00000	1.00000
0.07000	0.99897	1.00000	1.00000	1.00000	1.00000
0.08000	0.99744	0.99999	1.00000	1.00000	1.00000
0.09000	0.99479	0.99997	1.00000	1.00000	1.00000
0.10000	0.99078	0.99991	1.00000	1.00000	1.00000
0.15000	0.94876	0.99738	0.99987	0.99999	1.00000
0.20000	0.87916	0.98577	0.99840	0.99983	0.99998
0.25000	0.79829	0.96088	0.99277	0.99871	0.99977
0.30000	0.71695	0.92357	0.98032	0.99509	0.99880
0.35000	0.64044	0.87719	0.95995	0.98734	0.99609
0.40000	0.57082	0.82537	0.93207	0.97437	0.99054
0.45000	0.50856	0.77115	0.89797	0.95584	0.98130
0.50000	0.45337	0.71673	0.85924	0.93204	0.96789
0.55000	0.40466	0.66361	0.81745	0.90367	0.95023
0.60000	0.36172	0.61274	0.77398	0.87162	0.92856
0.65000	0.32389	0.56467	0.72993	0.83680	0.90333
0.70000	0.29051	0.51967	0.68618	0.80011	0.87512
0.75000	0.26102	0.47783	0.64337	0.76232	0.84455
0.80000	0.23493	0.43912	0.60195	0.72410	0.81223
0.85000	0.21179	0.40342	0.56224	0.68599	0.77871
0.90000	0.19124	0.37059	0.52442	0.64842	0.74450
0.95000	0.17294	0.34045	0.48861	0.61173	0.71005
1.00000	0.15662	0.31281	0.45485	0.57617	0.67572
1.50000	0.06212	0.13685	0.21809	0.30080	0.38155
2.00000	0.02689	0.06253	0.10499	0.15227	0.20270
2.50000	0.01233	0.02968	0.05154	0.07724	0.10618
3.00000	0.00589	0.01452	0.02580	0.03958	0.05567
3.50000	0.00290	0.00727	0.01315	0.02052	0.02936
4.00000	0.00146	0.00371	0.00680	0.01076	0.01560
4.50000	0.00075	0.00192	0.00357	0.00570	0.00835
5.00000	0.00039	0.00101	0.00189	0.00305	0.00450
5.50000	0.00021	0.00054	0.00101	0.00164	0.00245
6.00000	0.00011	0.00029	0.00055	0.00089	0.00134
6.50000	0.00006	0.00016	0.00030	0.00049	0.00073
7.00000	0.00003	0.00008	0.00016	0.00027	0.00041
7.50000	0.00002	0.00005	0.00009	0.00015	0.00022
8.00000	0.00001	0.00003	0.00005	0.00008	0.00013
8.50000	0.00001	0.00001	0.00003	0.00005	0.00007
9.00000	0.00000	0.00001	0.00002	0.00003	0.00004
9.50000	0.00000	0.00000	0.00001	0.00001	0.00002
10.00000	0.00000	0.00000	0.00000	0.00001	0.00001
$C(\alpha, b)$	0.72390	0.25581	0.12537	0.07125	0.04419

$$\Gamma^*(\alpha, x; b) = \Gamma(\alpha, x; b)/C(\alpha, b),\ C(\alpha, b) = 2b^{\alpha/2}K_\alpha(2\sqrt{b})$$

Table 3
Normalized representation of the generalized incomplete gamma function ($\alpha = -0.50$)

x	$\Gamma^*(\alpha,x;0.5)$	$\Gamma^*(\alpha,x;1.0)$	$\Gamma^*(\alpha,x;1.5)$	$\Gamma^*(\alpha,x;2.0)$	$\Gamma^*(\alpha,x;2.5)$
0.01000	1.00000	1.00000	1.00000	1.00000	0.99999
0.02000	1.00000	1.00000	1.00000	1.00000	0.99999
0.03000	1.00000	1.00000	1.00000	1.00000	0.99999
0.04000	1.00000	1.00000	1.00000	1.00000	0.99999
0.05000	0.99997	1.00000	1.00000	1.00000	0.99999
0.06000	0.99983	1.00000	1.00000	1.00000	0.99999
0.07000	0.99940	1.00000	1.00000	1.00000	0.99999
0.08000	0.99844	1.00000	1.00000	1.00000	0.99999
0.09000	0.99674	0.99999	1.00000	1.00000	0.99999
0.10000	0.99409	0.99996	1.00000	1.00000	0.99999
0.15000	0.96424	0.99832	0.99992	0.99999	0.99999
0.20000	0.91072	0.99027	0.99896	0.99989	0.99998
0.25000	0.84456	0.97195	0.99505	0.99914	0.99985
0.30000	0.77462	0.94312	0.98598	0.99661	0.99919
0.35000	0.70603	0.90576	0.97054	0.99098	0.99728
0.40000	0.64134	0.86251	0.94863	0.98120	0.99323
0.45000	0.58165	0.81582	0.92099	0.96681	0.98628
0.50000	0.52724	0.76765	0.88873	0.94780	0.97591
0.55000	0.47800	0.71944	0.85303	0.92456	0.96190
0.60000	0.43361	0.67221	0.81505	0.89771	0.94432
0.65000	0.39365	0.62664	0.77576	0.86794	0.92343
0.70000	0.35772	0.58315	0.73597	0.83595	0.89962
0.75000	0.32541	0.54198	0.69633	0.80241	0.87335
0.80000	0.29633	0.50325	0.65734	0.76792	0.84511
0.85000	0.27014	0.46697	0.61936	0.73298	0.81536
0.90000	0.24653	0.43311	0.58267	0.69804	0.78455
0.95000	0.22522	0.40159	0.54743	0.66343	0.75309
1.00000	0.20596	0.37231	0.51377	0.62944	0.72132
1.50000	0.08844	0.17559	0.26438	0.35101	0.43296
2.00000	0.04061	0.08495	0.13452	0.18746	0.24219
2.50000	0.01951	0.04222	0.06908	0.09939	0.13248
3.00000	0.00969	0.02146	0.03593	0.05288	0.07208
3.50000	0.00493	0.01111	0.01892	0.02832	0.03926
4.00000	0.00256	0.00584	0.01008	0.01528	0.02146
4.50000	0.00135	0.00311	0.00542	0.00831	0.01179
5.00000	0.00072	0.00167	0.00294	0.00455	0.00651
5.50000	0.00039	0.00091	0.00161	0.00250	0.00361
6.00000	0.00021	0.00050	0.00088	0.00139	0.00201
6.50000	0.00012	0.00027	0.00049	0.00077	0.00112
7.00000	0.00006	0.00015	0.00027	0.00043	0.00063
7.50000	0.00004	0.00008	0.00015	0.00024	0.00036
8.00000	0.00002	0.00005	0.00009	0.00014	0.00020
8.50000	0.00001	0.00003	0.00005	0.00008	0.00011
9.00000	0.00001	0.00001	0.00003	0.00004	0.00006
9.50000	0.00000	0.00001	0.00002	0.00002	0.00004
10.00000	0.00000	0.00000	0.00001	0.00001	0.00002
$C(\alpha,b)$	0.60940	0.23987	0.12495	0.07408	0.04745

$\Gamma^*(\alpha,x;b) = \Gamma(\alpha,x;b)/C(\alpha,b),\ C(\alpha,b) = 2b^{\alpha/2}K_\alpha(2\sqrt{b})$

Table 4
Normalized representation of the generalized incomplete gamma function ($\alpha = -0.25$)

x	$\Gamma^*(\alpha, x; 0.5)$	$\Gamma^*(\alpha, x; 1.0)$	$\Gamma^*(\alpha, x; 1.5)$	$\Gamma^*(\alpha, x; 2.0)$	$\Gamma^*(\alpha, x; 2.5)$
0.01000	1.00000	1.00000	1.00000	0.99999	0.99999
0.02000	1.00000	1.00000	1.00000	0.99999	0.99999
0.03000	1.00000	1.00000	1.00000	0.99999	0.99999
0.04000	1.00000	1.00000	1.00000	0.99999	0.99999
0.05000	0.99999	1.00000	1.00000	0.99999	0.99999
0.06000	0.99991	1.00000	1.00000	0.99999	0.99999
0.07000	0.99965	1.00000	1.00000	0.99999	0.99999
0.08000	0.99908	1.00000	1.00000	0.99999	0.99999
0.09000	0.99802	1.00000	1.00000	0.99999	0.99999
0.10000	0.99634	0.99997	1.00000	0.99999	0.99999
0.15000	0.97582	0.99895	0.99995	0.99999	0.99999
0.20000	0.93601	0.99350	0.99933	0.99993	0.99998
0.25000	0.88368	0.98035	0.99668	0.99944	0.99990
0.30000	0.82553	0.95863	0.99022	0.99771	0.99946
0.35000	0.76607	0.92929	0.97875	0.99368	0.99814
0.40000	0.70796	0.89411	0.96190	0.98645	0.99523
0.45000	0.65263	0.85492	0.93998	0.97548	0.99009
0.50000	0.60078	0.81335	0.91367	0.96058	0.98220
0.55000	0.55266	0.77069	0.88383	0.94191	0.97128
0.60000	0.50828	0.72795	0.85135	0.91984	0.95727
0.65000	0.46749	0.68583	0.81704	0.89485	0.94027
0.70000	0.43010	0.64486	0.78162	0.86749	0.92051
0.75000	0.39586	0.60538	0.74571	0.83828	0.89832
0.80000	0.36454	0.56760	0.70978	0.80774	0.87406
0.85000	0.33588	0.53167	0.67424	0.77632	0.84812
0.90000	0.30966	0.49763	0.63939	0.74443	0.82085
0.95000	0.28567	0.46550	0.60545	0.71240	0.79260
1.00000	0.26369	0.43525	0.57261	0.68052	0.76371
1.50000	0.12230	0.22070	0.31529	0.40407	0.48564
2.00000	0.05947	0.11290	0.16930	0.22729	0.28556
2.50000	0.02992	0.05869	0.09086	0.12584	0.16295
3.00000	0.01544	0.03098	0.04906	0.06947	0.09194
3.50000	0.00812	0.01657	0.02669	0.03842	0.05169
4.00000	0.00433	0.00896	0.01463	0.02133	0.02907
4.50000	0.00234	0.00489	0.00807	0.01189	0.01637
5.00000	0.00128	0.00269	0.00448	0.00666	0.00925
5.50000	0.00070	0.00149	0.00250	0.00374	0.00524
6.00000	0.00039	0.00083	0.00140	0.00211	0.00297
6.50000	0.00022	0.00047	0.00079	0.00120	0.00169
7.00000	0.00012	0.00026	0.00045	0.00068	0.00097
7.50000	0.00007	0.00015	0.00025	0.00039	0.00055
8.00000	0.00004	0.00008	0.00014	0.00022	0.00032
8.50000	0.00002	0.00005	0.00008	0.00013	0.00018
9.00000	0.00001	0.00003	0.00005	0.00007	0.00011
9.50000	0.00001	0.00002	0.00003	0.00004	0.00006
10.00000	0.00000	0.00001	0.00002	0.00002	0.00004
$C(\alpha, b)$	0.53067	0.23076	0.12723	0.07850	0.05184

$$\Gamma^*(\alpha, x; b) = \Gamma(\alpha, x; b)/C(\alpha, b), \; C(\alpha, b) = 2b^{\alpha/2} K_\alpha(2\sqrt{b})$$

Table 5
Normalized representation of the generalized incomplete gamma function ($\alpha = 0.00$)

x	$-Ei(-x)$	$\Gamma^*(\alpha,x;0.5)$	$\Gamma^*(\alpha,x;1.0)$	$\Gamma^*(\alpha,x;1.5)$	$\Gamma^*(\alpha,x;2.0)$
0.01000	4.03793	0.99998	0.99999	0.99999	0.99998
0.02000	3.35471	0.99999	0.99999	0.99999	0.99998
0.03000	2.95912	1.00000	1.00000	0.99999	0.99998
0.04000	2.68126	1.00000	1.00000	0.99999	0.99998
0.05000	2.46790	0.99999	1.00000	0.99999	0.99998
0.06000	2.29531	0.99995	1.00000	0.99999	0.99998
0.07000	2.15084	0.99981	1.00000	0.99999	0.99999
0.08000	2.02694	0.99947	0.99999	0.99999	0.99999
0.09000	1.91874	0.99884	0.99999	0.99999	0.99999
0.10000	1.82292	0.99780	0.99998	0.99999	0.99999
0.15000	1.46446	0.98416	0.99935	0.99996	0.99998
0.20000	1.22265	0.95554	0.99576	0.99958	0.99995
0.25000	1.04428	0.91554	0.98656	0.99781	0.99963
0.30000	0.90568	0.86882	0.97060	0.99330	0.99847
0.35000	0.79422	0.81902	0.94815	0.98497	0.99565
0.40000	0.70238	0.76857	0.92025	0.97230	0.99041
0.45000	0.62533	0.71902	0.88820	0.95528	0.98220
0.50000	0.55977	0.67127	0.85324	0.93428	0.97074
0.55000	0.50336	0.62584	0.81646	0.90987	0.95602
0.60000	0.45438	0.58296	0.77875	0.88269	0.93822
0.65000	0.41152	0.54273	0.74082	0.85337	0.91765
0.70000	0.37377	0.50513	0.70320	0.82253	0.89469
0.75000	0.34034	0.47009	0.66630	0.79068	0.86974
0.80000	0.31060	0.43748	0.63040	0.75829	0.84321
0.85000	0.28402	0.40719	0.59571	0.72574	0.81548
0.90000	0.26018	0.37905	0.56237	0.69335	0.78692
0.95000	0.23874	0.35295	0.53045	0.66137	0.75784
1.00000	0.21938	0.32872	0.50000	0.63001	0.72852
1.50000	0.10002	0.16421	0.27177	0.36999	0.45901
2.00000	0.04890	0.08446	0.14676	0.20932	0.27147
2.50000	0.02491	0.04446	0.07974	0.11731	0.15679
3.00000	0.01305	0.02382	0.04369	0.06571	0.08975
3.50000	0.00697	0.01294	0.02414	0.03691	0.05123
4.00000	0.00378	0.00711	0.01343	0.02080	0.02925
4.50000	0.00207	0.00394	0.00752	0.01177	0.01672
5.00000	0.00115	0.00220	0.00424	0.00669	0.00958
5.50000	0.00064	0.00124	0.00240	0.00381	0.00550
6.00000	0.00036	0.00070	0.00136	0.00218	0.00316
6.50000	0.00020	0.00040	0.00078	0.00125	0.00183
7.00000	0.00012	0.00023	0.00045	0.00072	0.00105
7.50000	0.00007	0.00013	0.00026	0.00042	0.00061
8.00000	0.00004	0.00007	0.00015	0.00024	0.00035
8.50000	0.00002	0.00004	0.00009	0.00014	0.00021
9.00000	0.00001	0.00002	0.00005	0.00008	0.00012
9.50000	0.00001	0.00001	0.00003	0.00005	0.00007
10.00000	0.00000	0.00001	0.00002	0.00003	0.00004
$C(\alpha,b)$	-----	0.47828	0.22779	0.13240	0.08478

$\Gamma^*(\alpha,x;b) = \Gamma(\alpha,x;b)/C(\alpha,b)$, $C(\alpha,b) = 2b^{\alpha/2}K_\alpha(2\sqrt{b})$

Table 6
Normalized representation of the generalized incomplete gamma function ($\alpha = 0.25$)

x	$\Gamma^*(\alpha,x;0.0)$	$\Gamma^*(\alpha,x;0.5)$	$\Gamma^*(\alpha,x;1.0)$	$\Gamma^*(\alpha,x;1.5)$	$\Gamma^*(\alpha,x;2.0)$
0.01000	0.65181	1.00000	1.00000	0.99998	0.99998
0.02000	0.58676	1.00000	1.00000	0.99998	0.99998
0.03000	0.54358	1.00000	1.00000	0.99998	0.99998
0.04000	0.51051	1.00000	1.00000	0.99998	0.99998
0.05000	0.48344	0.99999	1.00000	0.99998	0.99998
0.06000	0.46041	0.99997	1.00000	0.99998	0.99998
0.07000	0.44031	0.99990	1.00000	0.99998	0.99998
0.08000	0.42244	0.99971	1.00000	0.99998	0.99998
0.09000	0.40633	0.99934	0.99999	0.99998	0.99998
0.10000	0.39166	0.99872	0.99999	0.99998	0.99998
0.15000	0.33317	0.98996	0.99961	0.99997	0.99998
0.20000	0.29015	0.97008	0.99730	0.99973	0.99995
0.25000	0.25632	0.94053	0.99103	0.99858	0.99976
0.30000	0.22867	0.90426	0.97961	0.99551	0.99899
0.35000	0.20548	0.86396	0.96287	0.98958	0.99705
0.40000	0.18567	0.82166	0.94131	0.98025	0.99332
0.45000	0.16852	0.77879	0.91576	0.96732	0.98730
0.50000	0.15351	0.73631	0.88710	0.95093	0.97865
0.55000	0.14027	0.69486	0.85620	0.93139	0.96727
0.60000	0.12851	0.65485	0.82379	0.90913	0.95319
0.65000	0.11800	0.61652	0.79050	0.88462	0.93658
0.70000	0.10857	0.58000	0.75684	0.85833	0.91767
0.75000	0.10006	0.54534	0.72323	0.83069	0.89676
0.80000	0.09237	0.51256	0.68998	0.80212	0.87415
0.85000	0.08538	0.48161	0.65735	0.77295	0.85015
0.90000	0.07903	0.45245	0.62551	0.74349	0.82506
0.95000	0.07323	0.42502	0.59462	0.71401	0.79916
1.00000	0.06792	0.39922	0.56475	0.68470	0.77270
1.50000	0.03342	0.21416	0.32798	0.42739	0.51472
2.00000	0.01729	0.11632	0.18665	0.25429	0.31947
2.50000	0.00921	0.06400	0.10590	0.14865	0.19225
3.00000	0.00501	0.03559	0.06019	0.08633	0.11403
3.50000	0.00277	0.01997	0.03433	0.05004	0.06715
4.00000	0.00154	0.01128	0.01965	0.02900	0.03942
4.50000	0.00087	0.00642	0.01129	0.01683	0.02310
5.00000	0.00049	0.00367	0.00651	0.00978	0.01354
5.50000	0.00028	0.00210	0.00376	0.00569	0.00794
6.00000	0.00016	0.00121	0.00218	0.00332	0.00465
6.50000	0.00009	0.00070	0.00127	0.00194	0.00273
7.00000	0.00005	0.00041	0.00074	0.00113	0.00161
7.50000	0.00003	0.00024	0.00043	0.00066	0.00094
8.00000	0.00002	0.00014	0.00025	0.00039	0.00056
8.50000	0.00001	0.00008	0.00015	0.00023	0.00033
9.00000	0.00001	0.00005	0.00009	0.00013	0.00019
9.50000	0.00000	0.00003	0.00005	0.00008	0.00011
10.00000	0.00000	0.00002	0.00003	0.00005	0.00007
$C(\alpha,b)$	3.62561	0.44624	0.23076	0.14081	0.09335

$\Gamma^*(\alpha,x;b) = \Gamma(\alpha,x;b)/C(\alpha,b)$, $C(\alpha,b) = 2b^{\alpha/2}K_\alpha(2\sqrt{b})$

Table 7
Normalized representation of the generalized incomplete gamma function ($\alpha = 0.50$)

x	$\Gamma^*(\alpha, x; 0.0)$	$\Gamma^*(\alpha, x; 0.5)$	$\Gamma^*(\alpha, x; 1.0)$	$\Gamma^*(\alpha, x; 1.5)$	$\Gamma^*(\alpha, x; 2.0)$
0.01000	0.88754	0.99999	0.99999	0.99997	0.99996
0.02000	0.84148	0.99999	0.99999	0.99998	0.99996
0.03000	0.80650	0.99999	0.99999	0.99998	0.99996
0.04000	0.77730	0.99999	0.99999	0.99998	0.99996
0.05000	0.75183	0.99999	0.99999	0.99998	0.99996
0.06000	0.72903	0.99998	0.99999	0.99998	0.99996
0.07000	0.70828	0.99994	0.99999	0.99998	0.99996
0.08000	0.68916	0.99984	0.99999	0.99998	0.99996
0.09000	0.67137	0.99963	0.99999	0.99998	0.99997
0.10000	0.65472	0.99928	0.99999	0.99998	0.99997
0.15000	0.58388	0.99383	0.99977	0.99997	0.99997
0.20000	0.52709	0.98048	0.99832	0.99983	0.99995
0.25000	0.47950	0.95939	0.99416	0.99910	0.99983
0.30000	0.43858	0.93218	0.98619	0.99704	0.99934
0.35000	0.40278	0.90067	0.97402	0.99292	0.99803
0.40000	0.37109	0.86640	0.95778	0.98619	0.99543
0.45000	0.34278	0.83055	0.93793	0.97659	0.99109
0.50000	0.31731	0.79404	0.91504	0.96406	0.98469
0.55000	0.29427	0.75751	0.88974	0.94875	0.97606
0.60000	0.27332	0.72143	0.86259	0.93091	0.96513
0.65000	0.25421	0.68614	0.83411	0.91086	0.95197
0.70000	0.23672	0.65186	0.80477	0.88893	0.93669
0.75000	0.22067	0.61875	0.77493	0.86547	0.91950
0.80000	0.20590	0.58690	0.74492	0.84080	0.90059
0.85000	0.19229	0.55636	0.71501	0.81522	0.88021
0.90000	0.17971	0.52716	0.68539	0.78901	0.85859
0.95000	0.16808	0.49930	0.65624	0.76240	0.83596
1.00000	0.15730	0.47276	0.62770	0.73561	0.81253
1.50000	0.08326	0.27146	0.38811	0.48622	0.57007
2.00000	0.04550	0.15544	0.23236	0.30366	0.37056
2.50000	0.02535	0.08929	0.13750	0.18495	0.23208
3.00000	0.01431	0.05150	0.08101	0.11127	0.14251
3.50000	0.00815	0.02982	0.04767	0.06652	0.08654
4.00000	0.00468	0.01733	0.02806	0.03964	0.05220
4.50000	0.00270	0.01011	0.01652	0.02358	0.03136
5.00000	0.00157	0.00591	0.00974	0.01401	0.01879
5.50000	0.00091	0.00346	0.00575	0.00833	0.01124
6.00000	0.00053	0.00203	0.00340	0.00495	0.00672
6.50000	0.00031	0.00120	0.00201	0.00294	0.00402
7.00000	0.00018	0.00071	0.00119	0.00175	0.00240
7.50000	0.00011	0.00042	0.00070	0.00104	0.00143
8.00000	0.00006	0.00025	0.00042	0.00062	0.00086
8.50000	0.00004	0.00015	0.00025	0.00037	0.00051
9.00000	0.00002	0.00009	0.00015	0.00022	0.00031
9.50000	0.00001	0.00005	0.00009	0.00013	0.00018
10.00000	0.00001	0.00003	0.00005	0.00008	0.00011
$C(\alpha, b)$	1.77245	0.43091	0.23987	0.15303	0.10476

$$\Gamma^*(\alpha, x; b) = \Gamma(\alpha, x; b)/C(\alpha, b), \quad C(\alpha, b) = 2b^{\alpha/2}K_\alpha(2\sqrt{b})$$

Table 8
Normalized representation of the generalized incomplete gamma function ($\alpha = 0.75$)

x	$\Gamma^*(\alpha,x;0.0)$	$\Gamma^*(\alpha,x;0.5)$	$\Gamma^*(\alpha,x;1.0)$	$\Gamma^*(\alpha,x;1.5)$	$\Gamma^*(\alpha,x;2.0)$
0.01000	0.96574	0.99998	0.99998	0.99996	0.99995
0.02000	0.94263	0.99998	0.99998	0.99996	0.99995
0.03000	0.92257	0.99998	0.99998	0.99996	0.99995
0.04000	0.90433	0.99998	0.99998	0.99996	0.99995
0.05000	0.88738	0.99998	0.99998	0.99996	0.99995
0.06000	0.87142	0.99998	0.99998	0.99996	0.99995
0.07000	0.85627	0.99996	0.99998	0.99996	0.99995
0.08000	0.84180	0.99990	0.99998	0.99996	0.99995
0.09000	0.82792	0.99979	0.99998	0.99996	0.99995
0.10000	0.81455	0.99959	0.99998	0.99996	0.99995
0.15000	0.75383	0.99632	0.99985	0.99996	0.99995
0.20000	0.70079	0.98765	0.99897	0.99988	0.99995
0.25000	0.65344	0.97309	0.99627	0.99942	0.99988
0.30000	0.61061	0.95336	0.99085	0.99808	0.99956
0.35000	0.57154	0.92954	0.98222	0.99527	0.99870
0.40000	0.53569	0.90268	0.97030	0.99053	0.99692
0.45000	0.50264	0.87371	0.95527	0.98354	0.99385
0.50000	0.47206	0.84337	0.93745	0.97417	0.98921
0.55000	0.44370	0.81224	0.91725	0.96243	0.98278
0.60000	0.41732	0.78079	0.89508	0.94844	0.97446
0.65000	0.39276	0.74938	0.87134	0.93239	0.96422
0.70000	0.36983	0.71828	0.84640	0.91449	0.95212
0.75000	0.34841	0.68770	0.82058	0.89499	0.93824
0.80000	0.32837	0.65778	0.79418	0.87415	0.92274
0.85000	0.30960	0.62865	0.76745	0.85219	0.90576
0.90000	0.29201	0.60038	0.74059	0.82936	0.88748
0.95000	0.27551	0.57302	0.71379	0.80587	0.86807
1.00000	0.26002	0.54662	0.68719	0.78190	0.84771
1.50000	0.14760	0.33476	0.45067	0.54514	0.62395
2.00000	0.08506	0.20170	0.28327	0.35662	0.42383
2.50000	0.04947	0.12083	0.17463	0.22602	0.27589
3.00000	0.02895	0.07223	0.10658	0.14076	0.17526
3.50000	0.01703	0.04316	0.06468	0.08675	0.10967
4.00000	0.01005	0.02578	0.03912	0.05311	0.06795
4.50000	0.00595	0.01541	0.02361	0.03238	0.04183
5.00000	0.00353	0.00922	0.01424	0.01968	0.02563
5.50000	0.00210	0.00551	0.00858	0.01193	0.01565
6.00000	0.00125	0.00330	0.00516	0.00723	0.00953
6.50000	0.00074	0.00198	0.00311	0.00437	0.00580
7.00000	0.00044	0.00118	0.00187	0.00264	0.00352
7.50000	0.00026	0.00071	0.00113	0.00160	0.00213
8.00000	0.00016	0.00043	0.00068	0.00096	0.00129
8.50000	0.00009	0.00026	0.00041	0.00058	0.00078
9.00000	0.00006	0.00015	0.00025	0.00035	0.00047
9.50000	0.00003	0.00009	0.00015	0.00021	0.00029
10.00000	0.00002	0.00006	0.00009	0.00013	0.00017
$C(\alpha,b)$	1.22542	0.43043	0.25581	0.16993	0.11983

$\Gamma^*(\alpha,x;b) = \Gamma(\alpha,x;b)/C(\alpha,b)$, $C(\alpha,b) = 2b^{\alpha/2}K_\alpha(2\sqrt{b})$

Table 9
Normalized representation of the generalized incomplete gamma function ($\alpha = 1.00$)

x	$\Gamma^*(\alpha, x; 0.0)$	$\Gamma^*(\alpha, x; 0.5)$	$\Gamma^*(\alpha, x; 1.0)$	$\Gamma^*(\alpha, x; 1.5)$	$\Gamma^*(\alpha, x; 2.0)$
0.01000	0.99005	0.99998	0.99996	1.00000	1.00000
0.02000	0.98020	0.99998	0.99996	1.00000	1.00000
0.03000	0.97045	0.99998	0.99996	1.00000	1.00000
0.04000	0.96079	0.99998	0.99996	1.00000	1.00000
0.05000	0.95123	0.99998	0.99996	1.00000	1.00000
0.06000	0.94176	0.99997	0.99996	1.00000	1.00000
0.07000	0.93239	0.99996	0.99996	1.00000	1.00000
0.08000	0.92312	0.99994	0.99996	1.00000	1.00000
0.09000	0.91393	0.99988	0.99996	1.00000	1.00000
0.10000	0.90484	0.99977	0.99996	1.00000	1.00000
0.15000	0.86071	0.99787	0.99990	1.00000	1.00000
0.20000	0.81873	0.99242	0.99937	0.99995	0.99999
0.25000	0.77880	0.98269	0.99767	0.99967	0.99995
0.30000	0.74082	0.96884	0.99407	0.99881	0.99975
0.35000	0.70469	0.95141	0.98811	0.99693	0.99919
0.40000	0.67032	0.93105	0.97957	0.99366	0.99799
0.45000	0.63763	0.90839	0.96847	0.98868	0.99586
0.50000	0.60653	0.88399	0.95494	0.98182	0.99255
0.55000	0.57695	0.85833	0.93921	0.97300	0.98785
0.60000	0.54881	0.83180	0.92156	0.96226	0.98163
0.65000	0.52205	0.80475	0.90225	0.94967	0.97382
0.70000	0.49659	0.77744	0.88158	0.93537	0.96441
0.75000	0.47237	0.75011	0.85980	0.91950	0.95342
0.80000	0.44933	0.72292	0.83715	0.90226	0.94093
0.85000	0.42741	0.69603	0.81385	0.88382	0.92705
0.90000	0.40657	0.66954	0.79009	0.86435	0.91188
0.95000	0.38674	0.64356	0.76606	0.84403	0.89555
1.00000	0.36788	0.61814	0.74189	0.82303	0.87819
1.50000	0.22313	0.40221	0.51400	0.60280	0.67535
2.00000	0.13534	0.25449	0.33847	0.41214	0.47827
2.50000	0.08209	0.15881	0.21710	0.27144	0.32314
3.00000	0.04979	0.09832	0.13712	0.17484	0.21215
3.50000	0.03020	0.06056	0.08577	0.11102	0.13671
4.00000	0.01832	0.03718	0.05330	0.06981	0.08698
4.50000	0.01111	0.02278	0.03296	0.04360	0.05484
5.00000	0.00674	0.01393	0.02032	0.02709	0.03435
5.50000	0.00409	0.00850	0.01249	0.01676	0.02140
6.00000	0.00248	0.00519	0.00766	0.01034	0.01328
6.50000	0.00150	0.00316	0.00469	0.00637	0.00822
7.00000	0.00091	0.00193	0.00287	0.00391	0.00507
7.50000	0.00055	0.00117	0.00176	0.00240	0.00312
8.00000	0.00034	0.00071	0.00107	0.00147	0.00192
8.50000	0.00020	0.00043	0.00065	0.00090	0.00118
9.00000	0.00012	0.00026	0.00040	0.00055	0.00072
9.50000	0.00007	0.00016	0.00024	0.00034	0.00044
10.00000	0.00005	0.00010	0.00015	0.00021	0.00027
$C(\alpha, b)$	1.00000	0.44434	0.27973	0.19276	0.13967

$$\Gamma^*(\alpha, x; b) = \Gamma(\alpha, x; b)/C(\alpha, b), \quad C(\alpha, b) = 2b^{\alpha/2}K_\alpha(2\sqrt{b})$$

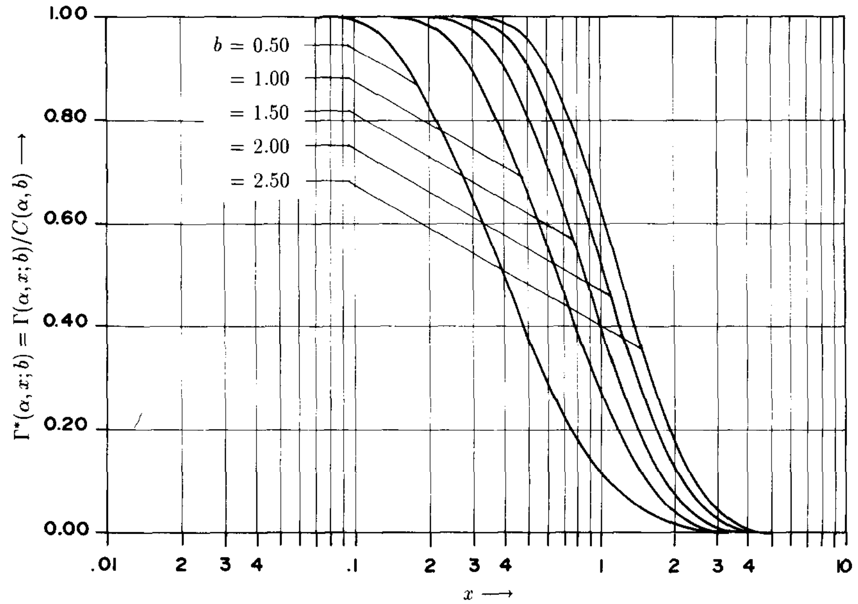

Fig. 1. Normalized representation of the generalized incomplete gamma function ($\alpha = -1.00$).

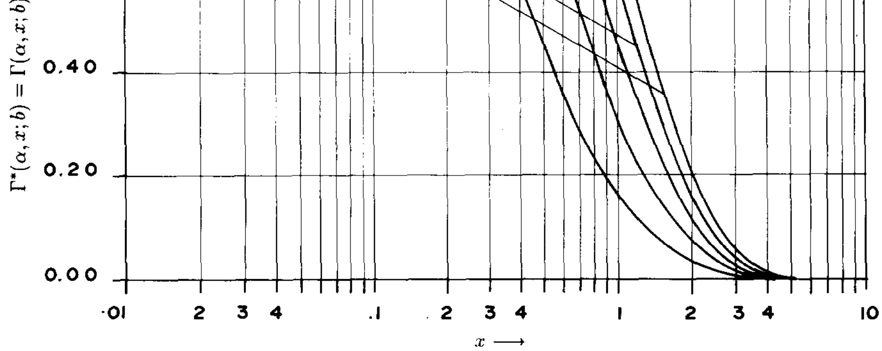

Fig. 2. Normalized representation of the generalized incomplete gamma function ($\alpha = -0.75$).

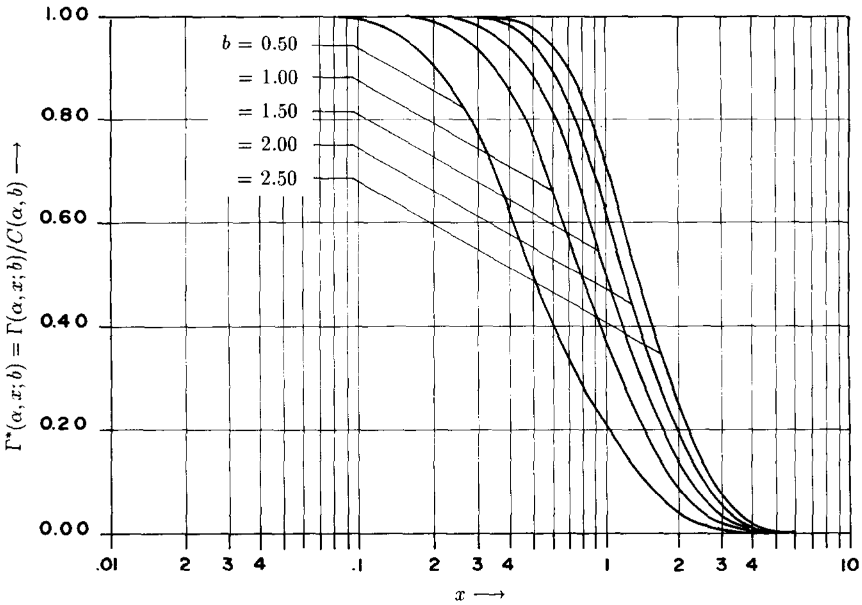

Fig. 3. Normalized representation of the generalized incomplete gamma function ($\alpha = -0.50$).

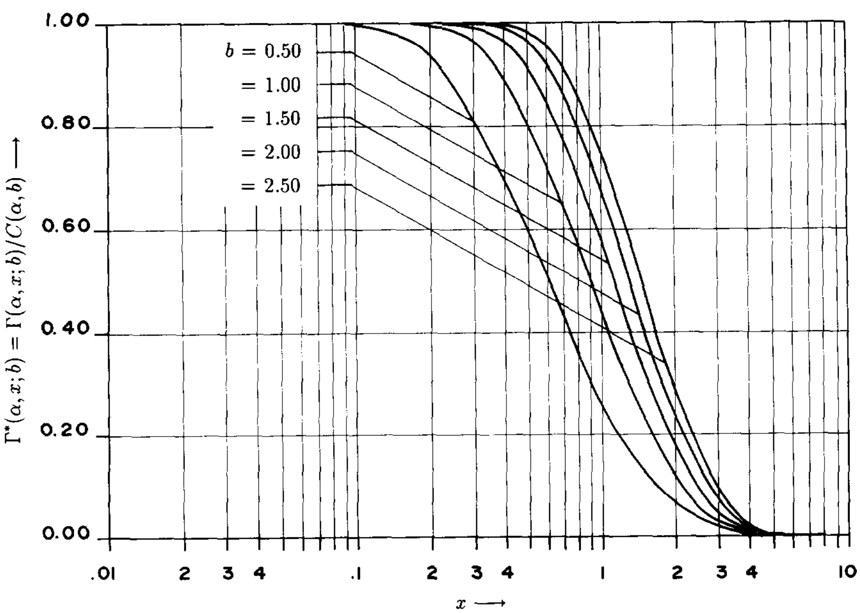

Fig. 4. Normalized representation of the generalized incomplete gamma function ($\alpha = -0.25$).

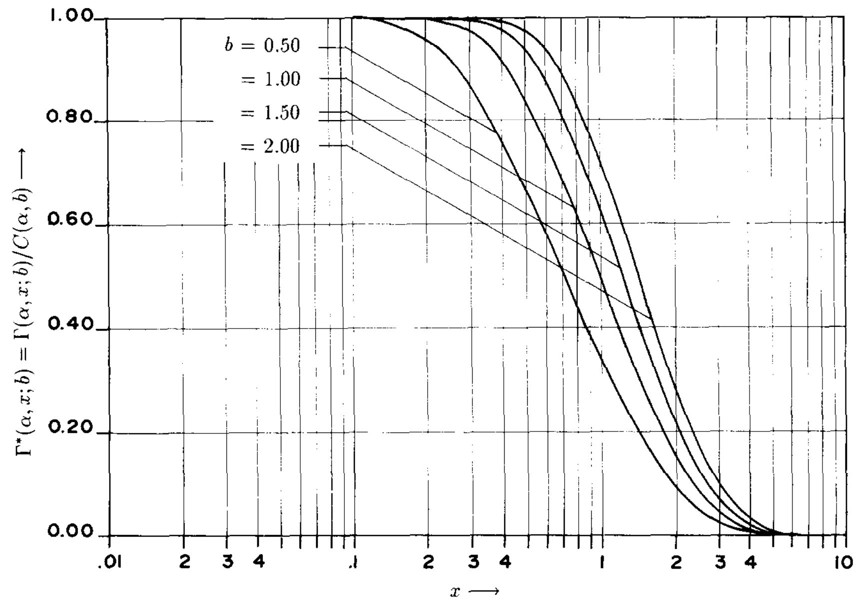

Fig. 5. Normalized representation of the generalized incomplete gamma function ($\alpha = 0.00$).

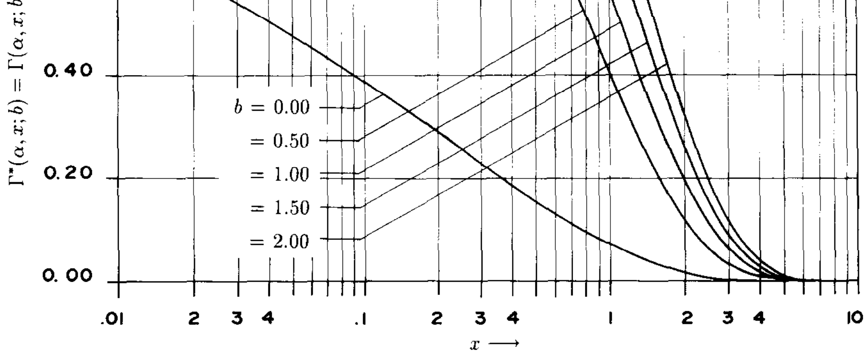

Fig. 6. Normalized representation of the generalized incomplete gamma function ($\alpha = 0.25$).

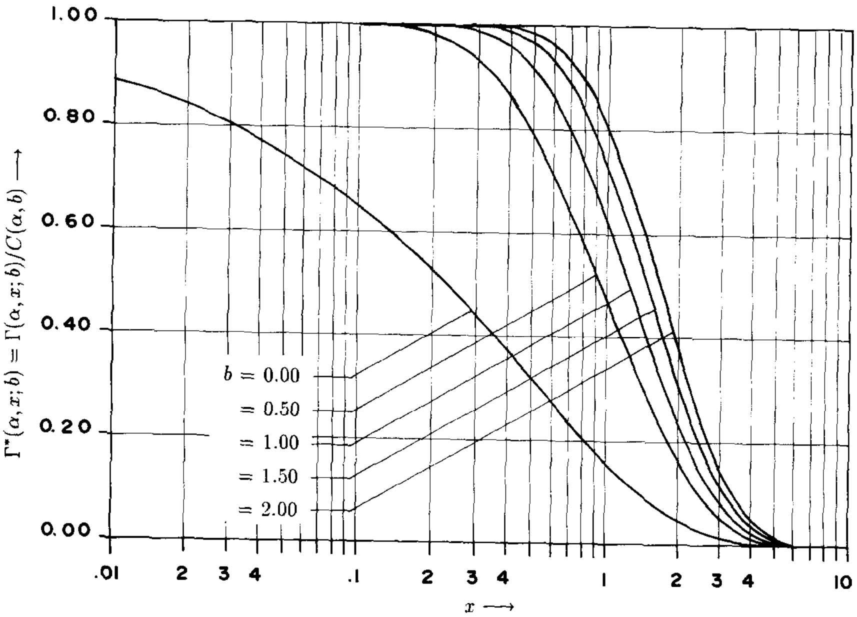

Fig. 7. Normalized representation of the generalized incomplete gamma function ($\alpha = 0.50$).

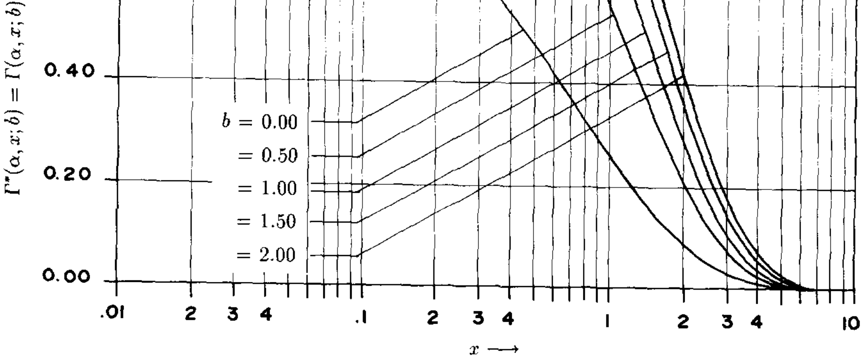

Fig. 8. Normalized representation of the generalized incomplete gamma function ($\alpha = 0.75$).

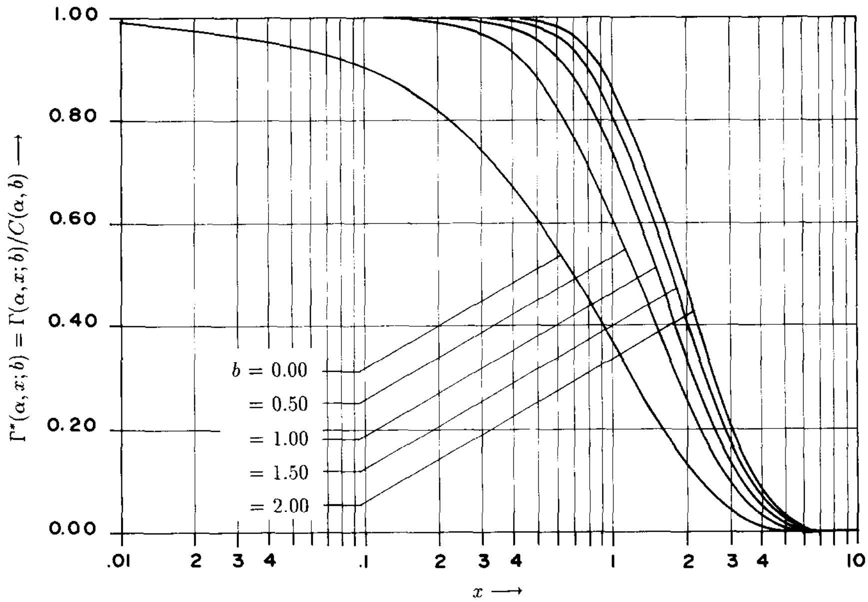

Fig. 9. Normalized representation of the generalized incomplete gamma function ($\alpha = 1.00$).

The normalized representation of the generalized incomplete gamma function for various values of the parameters α and b is presented in Tables 1–9 and Figs. 1–9. It should be noted that the values of the normalizing constant $C(\alpha, b)$, for a given α and various values of b are given in the last row of the respective tables. Notice that the second column of Table 5 represents the value of the exponential integral function, which corresponds to the value of the generalized incomplete gamma function when both α and b are zero. It should be added that the second column of Tables 6–9 gives the normalized representation of the incomplete gamma function for $\alpha = 0.25, 0.50, 0.75$ and 1.00, respectively; so an easy comparison with existing tables (or approximations) can be made. Furthermore, the second column of Table 7 represents (scaled) complementary error function (cf. Theorem 7, Eq. (23)).

Acknowledgements

The authors acknowledge the support provided by the King Fahd University of Petroleum and Minerals. The useful comments made by the referee are appreciated.

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