Weighted Poincaré-Type Inequalities for Differential Forms in $L^r(\mu)$-Averaging Domains

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We first introduce a new class of weighted functions and obtain some basic properties of this class. Then, as applications of this class, we prove local weighted integral inequalities for differential forms. Finally, we obtain the global weighted integral inequalities for differential forms in $L^r(\mu)$-averaging domains which can be considered as generalizations of the Poincaré inequality for Sobolev functions.

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1. INTRODUCTION

In recent years there has been new interest developed in the study of differential forms, largely pertaining to applications in quasi-conformal analysis and nonlinear elasticity. The objective of this paper is to introduce $A_r(\lambda)$-weights and to prove the local weighted Poincaré-type inequalities for differential forms in any kind of domains and the global weighted Poincaré-type inequalities for differential forms in $L^r(\mu)$-averaging domains, where $\mu$ is a measure defined by $d\mu = w(x)\,dx$ and $w \in A_r(\lambda)$. As we know, $A$-harmonic tensors are the special differential forms which are solutions to the $A$-harmonic equation for differential forms: $d^*A(x, du) = 0$. 

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0, where \( A : \Omega \times \wedge^l(R^n) \to \wedge^l(R^n) \) is an operator satisfying some conditions (see [7, 8, 10]). So that all of the results about differential forms in this paper remain true for \( A \)-harmonic tensors. Therefore, our new results concerning differential forms are of interest in partial differential equations, tensor analysis, nonlinear potential theory, quasi-regular mappings, and the theory of \( H^p \)-space (see [1–3, 6–9]).

We now establish some notation and definitions. Throughout this paper, we always assume \( \Omega \) is a connected open subset of \( R^n \). Let \( e_1, e_2, \ldots, e_n \) denote the standard unit basis of \( R^n \). For \( l = 0, 1, \ldots, n \), the linear space of \( l \)-vectors, spanned by the exterior products \( e_t = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l} \), corresponding to all ordered \( l \)-tuples \( I = (i_{j_1}, i_{j_2}, \ldots, i_{j_l}), 1 \leq i_{j_1} < i_{j_2} < \cdots < i_{j_l} \leq n \), is denoted by \( \wedge^l = \wedge^l(R^n) \). The Grassman algebra \( \wedge = \oplus \wedge^l \) is a graded algebra with respect to the exterior products. For \( \alpha = \Sigma \alpha^t e_t \in \wedge \) and \( \beta = \Sigma \beta^t e_t \in \wedge \), the inner product in \( \wedge \) is given by \( \langle \alpha, \beta \rangle = \Sigma \alpha^t \beta^t \) with summation over all \( l \)-tuples \( I = (i_{j_1}, i_{j_2}, \ldots, i_{j_l}) \) and all integers \( l = 0, 1, \ldots, n \). We define the Hodge star operator \( \star : \wedge \to \wedge \) by the rule \( \star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n \) and \( \alpha \wedge \star \beta = \beta \wedge \star \alpha = \langle \alpha, \beta \rangle (\star 1) \) for all \( \alpha, \beta \in \wedge \). Hence the norm of \( \alpha \in \wedge \) is given by the formula \( |\alpha|^2 = \langle \alpha, \alpha \rangle = \langle \star \alpha, \alpha \rangle \in \wedge^0 = R \). The Hodge star is an isometric isomorphism on \( \wedge \) with \( \star : \wedge^l \to \wedge^{n-l} \) and \( \star \star (1) = (1) : \wedge^l \to \wedge^l \). Let \( 0 < p < \infty \), we denote the weighted \( L^p \)-norm of a measurable function \( f \) over \( E \) by

\[
\|f\|_{p,E,w} = \left( \int_E |f(x)|^p w(x) \, dx \right)^{1/p}.
\]

As we know, a differential \( l \)-form \( \omega \) on \( \Omega \) is a Schwartz distribution on \( \Omega \) with values in \( \wedge^l(R^n) \). In particular, for \( l = 0 \), \( \omega \) is a real function or a distribution. We denote the space of differential \( l \)-forms by \( D'(\Omega, \wedge^l) \). We write \( L^p(\Omega, \wedge^l) \) for the \( l \)-forms \( \omega(x) = \sum_i \omega_i(x) \, dx_i = \sum_i \omega_i \wedge dx_i \wedge \cdots \wedge dx_{i_l} \) with \( \omega_i \in L^p(\Omega, R) \) for all ordered \( l \)-tuples \( I \). Thus \( L^p(\Omega, \wedge^l) \) is a Banach space with norm

\[
\|\omega\|_{p,\Omega} = \left( \int_\Omega |\omega(x)|^p \, dx \right)^{1/p} = \left( \int_\Omega \left( \sum_i |\omega_i(x)|^2 \right)^{p/2} \, dx \right)^{1/p}.
\]

Similarly, \( W^p_2(\Omega, \wedge^l) \) are those differential \( l \)-forms on \( \Omega \) whose coefficients are in \( W^p_2(\Omega, R) \). The notation \( W^p_{2,\text{loc}}(\Omega, R) \) and \( W^p_{2,\text{loc}}(\Omega, \wedge^l) \) are self-explanatory. We denote the exterior derivative by \( d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1}) \) for \( l = 0, 1, \ldots, n \). Its formal adjoint operator \( d^* : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l) \) is given by \( d^* = (-1)^l d \star d \star \) on \( D'(\Omega, \wedge^{l+1}) \), \( l = 0, 1, \ldots, n \).
We write $\mathbb{R} = \mathbb{R}^1$. Balls are denoted by $B$, and $\sigma B$ is the ball with the same center as $B$ and with $\text{diam}(\sigma B) = \sigma \text{diam}(B)$. The $n$-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is denoted by $|E|$. We call $w$ a weight if $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $w > 0$ a.e. Also in general $d\mu = w \, dx$, where $w$ is a weight. The following result appears in [8].

**Lemma 1.1.** Let $Q \subset \mathbb{R}^n$ be a cube or a ball. To each $y \in Q$ there corresponds a linear operator $K_y: C^2(Q, \wedge^l) \to C^2(Q, \wedge^{l-1})$ defined by

$$(K_y \omega)(x; \xi_1, \ldots, \xi_l) = \int_0^1 t^{l-1} \omega(tx + ty; x - y, \xi_1, \ldots, \xi_{l-1}) \, dt$$

and the decomposition

$$\omega = d(K_y \omega) + K_y(d\omega).$$

We define another linear operator $T_Q: C^2(Q, \wedge^l) \to C^2(Q, \wedge^{l-1})$ by averaging $K_y$ over points $y$ in $Q$

$$T_Q \omega = \int_Q \phi(y) K_y \omega \, dy,$$  

(1.2)

where $\phi \in C_0^\infty(Q)$ is normalized by $\int_Q \phi(y) \, dy = 1$. We define the $l$-form $\omega_Q \in \Lambda^l(Q, \wedge^l)$ by

$$\omega_Q = |Q|^{-1} \int_Q \omega(y) \, dy, \quad l = 0, \text{ and}$$

$$\omega_Q = d(T_Q \omega), \quad l = 1, 2, \ldots, n,$$  

(1.3)

for all $\omega \in L^p(Q, \wedge^l)$, $1 \leq p < \infty$.

Different versions of the classical Poincaré inequality have been established in the study of the Sobolev space and differential forms (see [2, 8, 10]). Susan G. Staples proves the Poincaré inequality for Sobolev functions in $L^2$-averaging domains in [11]. Tadeusz Iwaniec and Adam Lutoborski prove the following Poincaré-type inequality (Lemma 1.4) in [8] which plays a crucial role in generalizing the theory of Sobolev functions to differential forms.
**Lemma 1.4.** Let $u \in D'(Q, \Lambda^l)$ and $du \in L^p(Q, \Lambda^{l+1})$. Then $u - u_Q$ is in $L^{np/(n-p)}(Q, \Lambda^l)$ and
\[
\left( \int_Q |u - u_Q|^{np/(n-p)} \, dx \right)^{(n-p)/np} \leq C_p(n) \left( \int_Q |du|^p \, dx \right)^{1/p} \tag{1.5}
\]
for $Q$ a cube or a ball in $\mathbb{R}^n$, $l = 0, 1, \ldots, n$ and $1 < p < n$.

The following version of the Poincaré inequality appears in [10].

**Lemma 1.6.** Let $u \in D'(Q, \Lambda^l)$ and $du \in L^p(Q, \Lambda^{l+1})$. Then $u - u_Q$ is in $W^1_p(Q, \Lambda^l)$ with $1 < p < \infty$ and
\[
\|u - u_Q\|_{p, Q} \leq C(n, p)|Q|^{1/n}\|du\|_{p, Q} \tag{1.7}
\]
for $Q$ a cube or a ball in $\mathbb{R}^n$, $l = 0, 1, \ldots, n$.

The following generalized Hölder inequality will be used repeatedly.

**Lemma 1.8.** Let $0 < \alpha < \infty$, $0 < \beta < \infty$, and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If $f$ and $g$ are measurable functions on $\mathbb{R}^n$, then
\[
\|fg\|_{s, \Omega} \leq \|f\|_{\alpha, \Omega} \cdot \|g\|_{\beta, \Omega}
\]
for any $\Omega \subset \mathbb{R}^n$.

## 2. $A_r(\lambda)$-Weights

We now introduce a new class of weights as follows.

**Definition 2.1.** Let $w$ be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < w < \infty$ almost everywhere. We say that $w$ belongs to the $A_r(\lambda)$ class, $1 < r < \infty$ and $1 \leq \lambda < \infty$, or that $w$ is an $A_r(\lambda)$-weight, write $w \in A_r(\lambda)$, if
\[
\sup_B \left( \frac{1}{|B|} \int_B w^\lambda \, dx \right)^{(1/r - 1)} < \infty
\]
for all balls $B \subset \mathbb{R}^n$.

It is clear that $A_r(1)$ is the usual $A_r$ class (see [5] or [6]). The following theorem says that $A_r(\lambda)$ is an increasing class with respect to $r$. 
Theorem 2.2. If $1 < r < s < \infty$, then $A_r(\lambda) \subset A_s(\lambda)$.

Proof. Let $w \in A_r(\lambda)$. Since $1 < r < s < \infty$, by Hölder's inequality

$$\left( \int_B \left( \frac{1}{w} \right)^{1/(s-1)} \, dx \right)^{s-1} \leq \left( \int_B \left( \frac{1}{w} \right)^{1/(r-1)} \, dx \right)^{r-1} \left( \int_B \left( \frac{1}{w} \right)^{1/(s-r)} \, dx \right)^{s-r}$$

$$= |B|^{r-s} \left( \int_B \left( \frac{1}{w} \right)^{1/(r-1)} \, dx \right)^{r-1}$$

$$= \frac{|B|^{r-1}}{|B|^{r-1}} \left( \int_B \left( \frac{1}{w} \right)^{1/(r-1)} \, dx \right)^{r-1}$$

so that

$$\left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{1/(s-1)} \, dx \right)^{s-1} \leq \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{1/(r-1)} \, dx \right)^{r-1}.$$ 

Therefore, we find that

$$\sup_B \left( \frac{1}{|B|} \int_B w^\lambda \, dx \right) \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{1/(r-1)} \, dx \right)^{(r-1)}$$

$$\leq \sup_B \left( \frac{1}{|B|} \int_B w^\lambda \, dx \right) \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{1/(r-1)} \, dx \right)^{(r-1)} < \infty$$

for all balls $B \subset \mathbb{R}^n$ since $w \in A_r(\lambda)$. Therefore $w \in A_r(\lambda)$ and $A_s(\lambda) \subset A_r(\lambda)$. We have completed the proof of Theorem 2.2.

As we can see in the following Theorem 2.3, $A_r(\lambda)$-weights have a property similar to the strong doubling property of $A_r$-weights.

Theorem 2.3. If $w \in A_r(\lambda)$, then

$$\frac{|E|'}{|B|^{s+r-1}} \leq C_{r, \lambda, w} \frac{\mu(E)}{\mu(B)^{\lambda}}$$

whenever $B$ is a ball in $\mathbb{R}^n$ and $E$ is a measurable subset of $B$. 

(2.4)
Proof. By Hölder’s inequality, we have
\[
|E| = \int_E dx = \int_E w^{1/r} w^{-1/r} dx \\
\leq \left( \int_E w^r dx \right)^{1/r} \left( \int_E w^{1/(1-r)} dx \right)^{(r-1)/r} \\
= (\mu(E))^{1/r} \left( \int_E w^{1/(1-r)} dx \right)^{(r-1)/r}.
\]
This implies
\[
|E|^r = \mu(E) \left( \int_E w^{1/(1-r)} dx \right)^{(r-1)}.
\] (2.5)
Note \(\lambda \geq 1\), by Hölder’s inequality again, we have
\[
\frac{1}{|B|} \int_B w^\lambda dx \leq \left( \frac{1}{|B|} \int_B w^\lambda dx \right)^{1/\lambda},
\]
so that
\[
1 = \frac{1}{\mu(B)} \int_B w^\lambda dx \leq \frac{|B|}{\mu(B)} \left( \frac{1}{|B|} \int_B w^\lambda dx \right)^{1/\lambda}.
\]
Hence, we obtain
\[
\mu(B)^\lambda \leq |B|^\lambda-1 \int_B w^\lambda dx. \tag{2.6}
\]
Since \(w \in A_r(\lambda)\), there exists a constant \(C_{r,\lambda,w}\) such that
\[
\left( \frac{1}{|B|} \int_B w^\lambda dx \right)^{1/(r-1)} \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{1/(r-1)} dx \right)^{(r-1)} \leq C_{r,\lambda,w}, \tag{2.7}
\]
Combining (2.5), (2.6), and (2.7), we deduce that
\[
|E|^r \mu(B)^\lambda \leq \mu(E)|B|^\lambda-1 \int_B w^\lambda dx \left( \int_E w^{1/(1-r)} dx \right)^{(r-1)} \\
\leq \mu(E)|B|^\lambda r-1 \left( \frac{1}{|B|} \int_B w^\lambda dx \right)^{(1/(r-1))} \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{1/(r-1)} dx \right)^{(r-2)} \\
\leq C_{r,\lambda,w} \mu(E)|B|^\lambda r-1.
\]
Hence
\[
\frac{|E'|}{|B|^r} \leq C_{r,\lambda,w} \frac{\mu(E)}{\mu(B)^r}.
\]
This ends the proof of Theorem 2.3.

If we put \(\lambda = 1\) in Theorem 2.3, then (2.4) becomes
\[
\frac{|E'|}{|B|^r} \leq C_{r,w} \frac{\mu(E)}{\mu(B)^r},
\]
which is called the strong doubling property of \(A_r\)-weights (see [6]).

We also need the following reverse Hölder inequality [5].

**Lemma 2.8.** If \(w \in A_1(1)\), then there exist constants \(\alpha > 1\) and \(C\), independent of \(w\), such that
\[
\|w\|_{\alpha,B} \leq C|B|^{(1-\alpha)/\alpha}\|w\|_1,B
\]
for all balls \(B \subset \mathbb{R}^n\).

3. POINCARÉ-TYPE INEQUALITIES

Susan G. Staples introduces the following \(L^r\)-averaging domains [11]: a proper subdomain \(\Omega \subset \mathbb{R}^n\) is called an \(L^r\)-averaging domain, \(s \geq 1\), if there exists a constant \(C\) such that
\[
\left(\frac{1}{|\Omega|} \int_{\Omega} |u - u_\Omega| \, dm \right)^{1/s} \leq C \sup_{B \subset \Omega} \left(\frac{1}{|B|} \int_{B} |u - u_B| \, dm \right)^{1/s}
\]
for all \(u \in L^r_{loc}(\Omega)\). Here \(|\Omega|\) is the \(n\)-dimensional Lebesgue measure of \(\Omega\). Susan G. Staples proves the Poincaré inequality for Sobolev functions in \(L^r\)-averaging domains in [11]. In [4] we introduce \(L^r(\mu)\)-averaging domains. We call a proper subdomain \(\Omega \subset \mathbb{R}^n\) an \(L^r(\mu)\)-averaging domain, \(s \geq 1\), if \(\mu(\Omega) < \infty\) and there exists a constant \(C\) such that
\[
\left(\frac{1}{\mu(B_0)} \int_{\Omega} |u - u_{B_0}| \, d\mu \right)^{1/s} \leq C \sup_{2B \subset \Omega} \left(\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \right)^{1/s}
\]
for some ball \(B_0 \subset \Omega\) and all \(u \in L^r_{loc}(\Omega; \cup^t)\). Here the measure \(\mu\) is defined by \(d\mu = w(x) \, dx\), where \(w(x)\) is a weight and \(w(x) > 0\) a.e., and the supremum is over all balls \(B \subset \Omega\).
In this section, we first prove some local results, and then we prove the global results in $L^s(\mu)$-averaging domains.

**Theorem 3.3.** Let $u \in D'(B, \Lambda^I)$ and $du \in L^s(B, \Lambda^{I+1})$, $l = 0, 1, \ldots, n$. If $1 < s < n$ and $w \in A_{n/s}(n/s)$, then there exists a constant $C$, independent of $u$ and $du$, such that

$$
\left( \int_B |u - u_B|^s \, d\mu \right)^{1/s} \leq C|B|^{1/s} \left( \int_B |du|^n \, d\mu \right)^{1/n}
$$

(3.4)

for all balls $B \subset \mathbb{R}^n$. Here $d\mu = w(x) \, dx$.

Note that (3.4) can be written as

$$
\|u - u_B\|_{s, B, w} \leq C|B|^{1/s}\|du\|_{n, B, w}
$$

or

$$
\left( \frac{1}{|B|} \int_B |u - u_B|^s \, w(x) \, dx \right)^{1/s} \leq C \left( \int_B |du|^n \, w(x) \, dx \right)^{1/n}.
$$

**Proof.** Since $1/s = 1/n + (n-s)/ns$, by Hölder’s inequality and Lemma 1.4, we have

$$
\int_B |u - u_B|^s w \, dx = \left( \int_B (|u - u_B|^{s/n})^s \, dx \right)^{1/s} \leq \left( \int_B w^{s/n} \, dx \right)^{1/n} \left( \int_B |u - u_B|^{s(n-s)/n} \, dx \right)^{(n-s)/ns} \leq C_1\|w^{1/s}\|_{n, B} \cdot \|du\|_{s, B}.
$$

(3.5)

Applying Hölder’s inequality again, we find that

$$
\|du\|_{s, B} = \left( \int_B |du|^s \, dx \right)^{1/s} \leq \left( \int_B (|du| w^{1/n} w^{1/s}) \, dx \right)^{1/s} \leq \left( \int_B (|du| w^{1/n}) \, dx \right)^{1/n} \left( \int_B \left( \frac{1}{w} \right)^{s(n-s)/ns} \, dx \right)^{(n-s)/ns} \leq \left( \int_B |du|^n \, w \, dx \right)^{1/n} \cdot \left( \int_B \left( \frac{1}{w} \right)^{s(n-s)/ns} \, dx \right)^{(n-s)/ns} \leq \left( \int_B |du|^n \, w \, dx \right)^{1/n} \cdot \left\| \frac{1}{w} \right\|_{s/(n-s), B}.
$$

(3.6)
Combining (3.5) and (3.6) yields
\[
\left( \int_B |u - u_B|^s w \, dx \right)^{1/s} \leq C_1 \|w^{1/s}\|_{n,B} \cdot \|1/w\|^{1/n}_{1/(n-s),B} \cdot \left( \int_B |Du|^n w \, dx \right)^{1/n}.
\] (3.7)

Since \( w \in A_{n/3}(n/s) \), we obtain that
\[
\|w^{1/s}\|_{n,B} \cdot \|1/w\|^{1/n}_{1/(n-s),B} \\
= \left( \int_B w^{n/s} \, dx \right)^{1/n} \left( \int_B \frac{1}{w} \, dx \right)^{(n-s)/ns} \\
= \left( \int_B w^{n/s} \, dx \right)^{1/(ns)} \left( \int_B \frac{1}{w} \, dx \right)^{1/(n-s-1)} \cdot \frac{1}{n-s-1} \cdot \frac{1}{n} \\
= |B|^{1/s} \left( \frac{1}{|B|} \int_B w^{n/s} \, dx \right)^{1/(ns-1)} \left( \frac{1}{|B|} \int_B \frac{1}{w} \, dx \right)^{1/(n-s-1)} \cdot \frac{1}{n-s-1} \cdot \frac{1}{n} \\
\leq C_2 |B|^{1/s}.
\] (3.8)

From (3.7) and (3.8), we find that
\[
\left( \int_B |u - u_B|^s w(x) \, dx \right)^{1/s} \leq C_3 |B|^{1/s} \left( \int_B |Du|^n w(x) \, dx \right)^{1/n}.
\] (3.9)

It is easy to see that (3.9) is equivalent to (3.4) since \( d\mu = w(x) \, dx \). We have completed the proof of Theorem 3.3.

**Theorem 3.10.** Let \( u \in D'(B, \wedge^l) \) and \( du \in L^p(B, \wedge^{l+1}) \), \( l = 0, 1, \ldots, n \), and \( 1 < p < \infty \). If \( w \in A_r(t/(t-s)) \), where \( r = (p-s)/p(t-s) \) and \( 1 < s < t < p \), then there exists a constant \( C \), independent of \( u \) and \( du \), such that
\[
\left( \frac{1}{|B|} \int_B |u - u_B|^s w \, dx \right)^{1/s} \leq C |B|^{1/s} \left( \frac{1}{|B|} \int_B |Du|^n w^{p(t-s)/st} \, dx \right)^{1/p}
\] (3.11)
for all balls \( B \subset \mathbb{R}^n \).
Proof. Note $1/s = 1/t + (s - t)/st$, by Lemma 1.6 and Hölder’s inequality, we obtain

$$ \left( \int_B |u - u_B|^s w \, dx \right)^{1/s} = \left( \int_B (|u - u_B| w^{1/s})^s \, dx \right)^{1/s} \leq \left( \int_B |u - u_B| \, dx \right)^{1/t} \left( \int_B w^{s/(s-t)} \, dx \right)^{(s-t)/st} = \|u - u_B\|_{1,B} \cdot \|w\|^{1/t}_{1/(t-s),B} \leq C_1 |B|^{1/n} \|u\|_{1,B} \cdot \|w\|^{1/t}_{1/(t-s),B}. \quad (3.12) $$

Since $1/t = 1/p + (p - t)/pt$, from Hölder’s inequality again, we deduce that

$$ \|du\|_{1,B} = \left( \int_B (|du| w^{s/(s-t)/st})^p \, dx \right)^{1/p} \leq \left( \int_B (|du| w^{s/(st - t)})^p \, dx \right)^{1/p} \left( \int_B \left( \frac{1}{w} \right)^{p(t-s)/(p-t)} \, dx \right)^{(p-t)/pt} \leq \left( \int_B (|du| w^{s/(st - t)})^p \, dx \right)^{1/p} \left( \frac{1}{w} \left( \frac{1}{w} \right)^{t-st/(t-s)} \right)^{(p-t)/pt} \cdot \quad (3.13) $$

Combining (3.12) and (3.13) yields

$$ \left( \int_B |u - u_B|^s w \, dx \right)^{1/s} \leq C_1 |B|^{1/n} \|w\|^{1/t}_{1/(t-s),B} \left( \frac{1}{w} \right)^{t-st/(t-s)} \left( \frac{1}{w} \right)^{(p-t)/pt} \times \left( \int_B |du|^{p(t-st)/pt} \, dx \right)^{1/p}. \quad (3.14) $$

Since

$$ w \in A_{s'(p-s)/p(t-s)} \left( \frac{t}{t-s} \right) $$

and

$$ \frac{t(p-s)}{p(t-s)} = \frac{s(p-t)}{p(t-s)} + 1 > 1, $$
then we have

\[ \|w\|_{l,(s-1)/s}^{1/s} \left\| \left( \frac{1}{w} \right)^{(s-1)/st} \right\|_{p/(p-1), B} = \left( \int_B w^{t/(s-1)} \, dx \right)^{(s-1)/st} \left( \int_B \left( \frac{1}{w} \right)^{p(s-1)/(p-1)} \, dx \right)^{(p-1)/pt} \]

\[ = \left[ \left( \int_B w^{t/(s-1)} \, dx \right) \left( \int_B \left( \frac{1}{w} \right)^{p(s-1)/(p-1)} \, dx \right) \right] \left[ \left( \int_B w^{t/(s-1)} \, dx \right) \left( \int_B \left( \frac{1}{w} \right)^{p(s-1)/(p-1)} \, dx \right) \right] \]

\[ = |B|^{(p-1)/ps} \left[ \left( \int_B w^{t/(s-1)} \, dx \right) \right] \]

\[ \times \left( \int_B \frac{1}{w} \right)^{1/(p-1)} \left( \int_B \frac{1}{w} \right)^{(p-1)/pt} \]

\[ \leq C_2 |B|^{(p-1)/ps}. \quad (3.15) \]

From (3.14) and (3.15), we find that

\[ \left( \int_B |u - u_B|^s w \, dx \right)^{1/s} \leq C_3 |B|^{1/n} |B|^{(p-1)/ps} \left( \int_B |u|^p w^{p(s-1)/st} \, dx \right)^{1/p} \]

\[ \leq C |B|^{1/n} \left( \int_B |u|^p w^{p(s-1)/st} \, dx \right)^{1/p}. \quad (3.11) \]

which is equivalent to

\[ \left( \frac{1}{|B|} \int_B |u - u_B|^s w(x) \, dx \right)^{1/s} \leq C |B|^{1/n} \left( \frac{1}{|B|} \int_B |u|^p w^{p(s-1)/st} \, dx \right)^{1/p}. \]

We have completed the proof of Theorem 3.10.

**Theorem 3.16.** Let \( u \in D'(\Omega, \Lambda^*) \) and \( du \in L^\alpha(\Omega, \Lambda^{l+1}) \), \( l = 0, 1, \ldots, n \). If \( 1 < s < n \) and \( w \in A_n^{n/(s)}(n/s) \) with \( w(x) \geq \alpha > 0 \), then there exists a constant \( C \), independent of \( u \) and \( du \), such that

\[ \left( \frac{1}{\mu(\Omega)} \int \Omega |u - u_{B_0}|^s d\mu \right)^{1/s} \leq C \left( \int \Omega |du|^n d\mu \right)^{1/n} \quad (3.17) \]

for any \( L^\alpha(\mu) \)-averaging domain \( \Omega \) and some ball \( B_0 \) with \( 2B_0 \subset \Omega \). Here \( d\mu = w(x) \, dx \).
Proof. For any ball $B \subset \Omega$, 

$$
\mu(B) = \int_B w(x) \, dx \geq \int_B \alpha \, dx = \alpha |B|,
$$

so that 

$$
\frac{1}{\mu(B)} \leq \frac{C_1}{|B|^s},
$$

(3.18)

where $C_1 = \frac{1}{\alpha}$. By (3.18) and Theorem 3.3, we obtain 

$$
\left( \frac{1}{\mu(B)} \int_B |u - u_B|^s \, d\mu \right)^{1/s} \leq \mu(B)^{-1/s} \left( \int_B |u - u_B|^s \, d\mu \right)^{1/s} 
\leq C_2 |B|^{-1/s} \left( \int_B |u - u_B|^s \, d\mu \right)^{1/s} 
\leq C_2 |B|^{-1/s} \cdot C_3 |B|^{1/s} \left( \int_B |du|^n \, d\mu \right)^{1/n} 
= C_4 \left( \int_B |du|^n \, d\mu \right)^{1/n},
$$

(3.19)

Thus, by (3.19) and the definition of $L^s(\mu)$-averaging domains, we deduce that 

$$
\left( \frac{1}{\mu(\Omega)} \int_\Omega |u - u_{B_0}|^s \, d\mu \right)^{1/s} \leq \left( \frac{1}{\mu(B_0)} \int_{B_0} |u - u_{B_0}|^s \, d\mu \right)^{1/s} 
\leq C_5 \sup_{2B \subset \Omega} \left( \frac{1}{\mu(B)} \int_B |u - u_B|^s \, d\mu \right)^{1/s} 
\leq C_5 \sup_{2B \subset \Omega} \left( C_4 \left( \int_B |du|^n \, d\mu \right)^{1/n} \right) 
\leq C_6 \sup_{2B \subset \Omega} \left( \int_\Omega |du|^n \, d\mu \right)^{1/n} 
= C_6 \left( \int_\Omega |du|^n \, d\mu \right)^{1/n}.
$$

We have completed the proof of Theorem 3.16.
THEOREM 3.20. Let \( u \in D'(\Omega, \Lambda^l) \) and \( du \in L^p(\Omega, \Lambda^{l+1}) \), \( l = 0, 1, \ldots, n \). Let \( 1 < s < t < p < \alpha s \) and \( 1/p = 1/n + 1/\alpha s \), where \( \alpha \) is the exponent in the reverse Hölder inequality (Lemma 2.8). If \( w \in A_\alpha(t) \cap A_s(t/(t-s)) \) with \( r = \gamma(p-s)/p(t-s) \), then there exists a constant \( C \), independent of \( u \) and \( du \), such that

\[
\left( \frac{1}{\mu(\Omega)} \int_\Omega \left| u - u_{B_2} \right|^s dx \right)^{1/s} \leq C \left( \int_\Omega |du|^n w^{-n/r} dx \right)^{1/n} \tag{3.21}
\]

for any \( L^r(\mu) \)-averaging domain \( \Omega \) and some ball \( B_2 \) with \( 2B_2 \subset \Omega \).

Proof. Since \( 1/p = 1/n + 1/\alpha s \), by Lemma 1.8 and (2.9), we have

\[
\left( \int_B |du|^r w^{p(u-s)/s} dx \right)^{1/p} = \left( \int_B \left( |du| w^{-1/s} w^{1/s} \right)^p dx \right)^{1/p} \leq \|w\|_{1/s,B}^{1/s} \left( \int_B |du|^n w^{-n/r} dx \right)^{1/n} \leq C_1 |B|^{(1-a)/\alpha s} \cdot \|w\|_{1/s,B}^{1/s} \cdot \left( \int_B |du|^n w^{-n/r} dx \right)^{1/n} = C_1 |B|^{(1-a)/\alpha s} \cdot \mu(B)^{1/s} \cdot \left( \int_B |du|^n w^{-n/r} dx \right)^{1/n}. \tag{3.22}
\]

Combining (3.11) and (3.22) and applying \( 1/p = 1/n + 1/\alpha s \), we obtain

\[
\left( \frac{1}{\mu(B)} \int_B \left| u - u_B \right|^r d\mu \right)^{1/s} = \left( \frac{1}{\mu(B)} \int_B \left| u - u_B \right|^s w dx \right)^{1/s} \leq C_2 |B|^{1/s + 1/n - 1/p \mu(B)^{-1/r}} \cdot \left( \int_B |du|^p w^{p(u-s)/s} dx \right)^{1/p} \leq C_3 \left( \int_B |du|^n w^{-n/r} dx \right)^{1/n}. \tag{3.23}
\]
Using (3.23) and the definition of $L^s(\mu)$-averaging domains, we have

$$
\left( \frac{1}{\mu(\Omega)} \int_{\Omega} |u - u_{B_0}|^s \, d\mu \right)^{1/s} \leq \left( \frac{1}{\mu(B_0)} \int_{\Omega} |u - u_{B_0}|^s \, d\mu \right)^{1/s}
$$

$$
\leq C_4 \sup_{2B \subset \Omega} \left( \frac{1}{\mu(B)} \int_B |u - u_B|^s \, d\mu \right)^{1/s}
$$

$$
\leq C_5 \sup_{2B \subset \Omega} \left( \int_B |u|^n w^{-n/s} \, dx \right)^{1/n}
$$

$$
\leq C_5 \sup_{2B \subset \Omega} \left( \int_{\Omega} |u|^n w^{-n/s} \, dx \right)^{1/n}
$$

$$
= C_5 \left( \int_{\Omega} |u|^n w^{-n/s} \, dx \right)^{1/n}.
$$

We have completed the proof of Theorem 3.20.

**Theorem 3.24.** Let $u \in D'(\Omega, \mathbb{R}^l)$ and $\rho \in L^s(\Omega, \mathbb{R}^{l+1})$, $l = 0, 1, \ldots, n$. If $1 < s < n < \alpha s$ and $w \in A_{\alpha/s}(1) \cap A_{n/s}(n/s)$, then there exists a constant $C$, independent of $u$ and $\rho$, such that

$$
\left( \frac{1}{\mu(\Omega)} \int_{\Omega} |u - u_{B_0}|^s \, dx \right)^{1/s} \leq C |\Omega|^{1/\alpha s} \left( \int_{\Omega} |u|^s w^{\kappa s(3-n)/ns} \, dx \right)^{1/\kappa}
$$

(3.25)

for any $L^s(\mu)$-averaging domain $\Omega$ and some ball $B_0$ with $2B_0 \subset \Omega$. Here $\kappa = \alpha s/(\alpha s - n)$ and $\alpha$ is the exponent in the reverse Hölder inequality (Lemma 2.8).

**Proof.** Note $1/n = 1/\alpha s + 1/\kappa$. By the same method used in the proof of Theorem 3.20, we have

$$
\left( \int_B |u|^n w \, dx \right)^{1/n} \leq C_1 |B|^{(1-\alpha)/\alpha s} \cdot \mu(B)^{1/s} \cdot \left( \int_B |u|^s w^{\kappa s(3-n)/ns} \, dx \right)^{1/\kappa}.
$$

(3.26)

Applying (3.4) and (3.26) yields

$$
\left( \frac{1}{\mu(B)} \int_B |u - u_B|^s \, d\mu \right)^{1/s} \leq C_2 |B|^{1/s} \cdot \mu(B)^{-1/s} \cdot \left( \int_B |u|^n w \, dx \right)^{1/n}
$$

$$
\leq C_3 |B|^{1/\alpha s} \cdot \left( \int_B |u|^s w^{\kappa s(3-n)/ns} \, dx \right)^{1/\kappa}.
$$

(3.27)
Using (3.27) and the definition of $L^p(\mu)$-averaging domains, we obtain that

$$
\left( \frac{1}{\mu(\Omega)} \int_{\Omega} |u - u_{B_2}|^p \, d\mu \right)^{1/p} \\
\leq \left( \frac{1}{\mu(B_0)} \int_{B_0} |u - u_{B_2}|^p \, d\mu \right)^{1/p} \\
\leq C_4 \sup_{2B \subseteq \Omega} \left( \frac{1}{\mu(B)} \int_B |u - u_B|^p \, d\mu \right)^{1/p} \\
\leq C_5 \sup_{2B \subseteq \Omega} \left( |B|^{1/ax} \left( \int_B |du|^\infty w^{k(s-n)/ns} \, dx \right)^{1/x} \right) \\
\leq C_5 \sup_{2B \subseteq \Omega} \left( |\Omega|^{1/ax} \left( \int_{\Omega} |du|^\infty w^{k(s-n)/ns} \, dx \right)^{1/x} \right) \\
= C_5 |\Omega|^{1/ax} \left( \int_{\Omega} |du|^\infty w^{k(s-n)/ns} \, dx \right)^{1/x}.
$$

We have completed the proof of Theorem 3.24.

**Remark.** In Theorem 3.10, we may choose $t$ to be some special numbers; then we will have different versions of the local results. By using different local results, we may obtain some other versions of the global results. Considering the length of this paper, we do not include these particular cases.

**REFERENCES**