ON A CONJECTURE OF TURÁN AND ERDŐS

BY

R. TIJDEMAN

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1. Introduction

One of the fundamental results of Turán in his book „Eine neue Methode in der Analysis und deren Anwendungen“, [3], Satz VII, p. 38, is the following theorem:

Let $b_1, b_2, \ldots, b_n$ and $z_1, z_2, \ldots, z_n$ be two sets of $n$ complex numbers. Then for every non-negative integer $m$ there exists an integer $v$ with $m < v < m + n$ such that

$$\left| \frac{b_1 z_1^v + b_2 z_2^v + \ldots + b_n z_n^v}{\min_{j=1,2,\ldots,n} |z_j^v|} \right| \geq \left( \frac{n}{2e(m+n)} \right)^n |b_1 + b_2 + \ldots + b_n|$$

This result has found many applications in analysis and number theory. Sometimes however one finds it desirable to have an estimation of this type where $v$ runs through a shorter interval. In order to come to a non-trivial result in this direction it seems obvious to try first to single out the systems $z_1, z_2, \ldots, z_n$ satisfying

$$\max_{v=m+1, m+2, \ldots, m+n-1} \left| \sum_{j=1}^{n} z_j^v \right| = 0.$$  

Vera Sós and Turán started such investigations in [2] and gave a complete characterization for the cases $m=0, 1, 2$. See also Uchiyama, [4], [5].

For the case $m=0$ the authors of [2] found that (1) can only be satisfied when $z_1, z_2, \ldots, z_n$ are the solutions of

$$z^n + a = 0$$

where $a$ is a certain complex number. Now Turán made the following conjecture:

If for a system of $n$ distinct, non-zero complex numbers, $z_1, z_2, \ldots, z_n, n > 2$, equality (1) holds for infinitely many values of $m$, then $z_1, z_2, \ldots, z_n$ must be the solutions of the equation (2) for a certain complex number $a$. 

$$z^n + a = 0$$
It was conjectured by Erdös that one may weaken the conditions in Turán’s conjecture and still get the same result. More precisely one should have, according to this conjecture of Erdös’:

\[ \text{If for a system of } n \text{ distinct, non-zero complex numbers, } z_1, z_2, \ldots, z_n, n > 2, \text{ equality (1) holds for two different values of } m, \text{ then } z_1, z_2, \ldots, z_n \text{ are the solutions of the equation (2) for a certain complex number } a. \]

(4)

Until now the best result with respect to Turán’s conjecture proved, is that under the conditions in (3) the numbers \( z_1, z_2, \ldots, z_n \) must be solutions of an equation

\[ z^A + b = 0, \]

with \( b \) a complex number, \( A \) an integer with either \( A = n \) or \( A > 2n \). This is a special case of a result of Jager, [1]. As to Erdös’ conjecture no partial result was known at all.

In this paper we first show that the two conjectures are equivalent, i.e. we prove that if (1) is satisfied for two different values of \( m \), it is satisfied for infinitely many values of \( m \). Then we shall prove that the conjectures are true for odd \( n \) and false for even \( n \). Moreover we are able to indicate all the counterexamples in the case of an even \( n \).

From a correspondence with Professor Turán we learnt that a special case of our theorem 2 has been proved independently by Dr. D. G. Cantor of Los Angeles, who also has given some counterexamples in the case of even \( n \).

I want to express my thanks to Dr. H. Jager for the assistance I received during the preparation of this paper.

2. The formulae of Newton-Girard.

Let \( b_1, b_2, \ldots, b_n \) be a set of \( n \) non-zero complex numbers, \( z_1, z_2, \ldots, z_n \) a set of \( n \) distinct, non-zero complex numbers and let \( \{s_k\}_{k=1}^{\infty} \) denote their sequence of generalized power sums, i.e.

\[ s_k = \sum_{j=1}^{n} b_j z_j^k, \quad k = 1, 2, \ldots. \]

Putting \( G(t) = \prod_{j=1}^{n} (1-z_j t)^{b_j} \) one has for \( |t| < (\max_{j=1,2,\ldots,n} |z_j|)^{-1} \):

\[ -G'(t)/G(t) = \sum_{j=1}^{n} b_j z_j/(1-z_j t) = \sum_{k=1}^{\infty} s_k t^{k-1}. \]

Further, if \( G^*(t) = \prod_{j=1}^{n} (1-z_j t) = 1+a_1 t + a_2 t^2 + \ldots + a_n t^n, \) the product \( P(t) \) of \( G^*(t) \) and \( \sum_{k=1}^{\infty} s_k t^{k-1} \) reduces to a polynomial of degree at most \( n-1 \).
From this it follows that

\[(5) \quad s_k + a_1 s_{k-1} + \ldots + a_n s_{k-n} = 0, \quad k > n.\]

In the case of ordinary power sums, i.e. \(b_j = 1, j = 1, 2, \ldots, n\), we have \(G^*(t) = G(t)\), hence

\[P(t) = -G'(t) = -a_1 - 2a_2 t - \ldots - na_n t^{n-1},\]

so that one has moreover (NEWTON-GIRARD)

\[(6) \quad s_k + a_1 s_{k-1} + \ldots + a_{k-1} s_1 + ka_k = 0, \quad 1 < k < n.\]

3. **Rational functions**

**Theorem 1:** Let \(F(t)\) be a rational function over the complex field,

\[F(t) = \frac{P(t)}{Q(t)},\]

where \(P(t)\) and \(Q(t)\) have no common divisors and with degr. \(P(t) <\) degr. \(Q(t) = n, Q(0) \neq 0\). Let \(\sum_{k=1}^{\infty} S_k t^{k-1}\) be the Taylor series of \(F(t)\) in a neighbourhood of \(t=0\). Suppose that there exist two positive integers \(p, q, p < q\), and a complex number \(c\) such that

\[(7) \quad s_{q+k} = cs_{p+k}, \quad k = 1, 2, \ldots, n.\]

Then the poles of \(F(t)\) are single and lying on the vertices of a regular \((q-p)\)-gon with its centre in the origin.

**Proof.** Let \(Q(t) = a_0 + a_1 t + \ldots + a_n t^n\). Then since degr. \(P(t) = n\), we have

\[(8) \quad s_k a_0 + s_{k-1} a_1 + \ldots + s_{k-n} a_n = 0, \quad k > n.\]

Now we prove by induction that \(s_{q+k} = cs_{p+k}\) holds for every positive integer \(k\). Suppose \(s_{q+k} = cs_{p+k}\) has already been proved for all values of \(k\) with \(1 < k < k_0, k_0 > n\). From (8) with \(k = q + k_0\) and this hypothesis it follows that

\[s_{q+k_0} a_0 = -ca_1 s_{p+k_0-1} - ca_2 s_{p+k_0-2} - \ldots - ca_n s_{p+k_0-n} = cs_{p+k_0} a_0.\]

But \(a_0 = Q(0) \neq 0\), hence \(s_{q+k_0} = cs_{p+k_0}\) and therefore

\[(9) \quad s_{q+k} = cs_{p+k}, \quad k = 1, 2, \ldots.\]

We define \(S_r(t)\) by \(S_r(t) = \sum_{k=1}^{r} s_k t^{k-1}, r = 1, 2, \ldots\). Then putting \(A = q - p\) and using (9) we can write for sufficiently small \(|t|\)

\[\frac{P(t)}{Q(t)} = \sum_{k=1}^{\infty} s_k t^{k-1} = S_p(t) + \sum_{k=p+1}^{p+A} s_k t^{k-1}(1 + ctA + c^2 t^2 A + \ldots) = \frac{S_{p+A}(t) - ctA S_p(t)}{1 - ctA}.\]
Hence $Q(t)$ divides $P(t)(1-ct^A)$; but $Q(t)$ and $P(t)$ have no common divisors, therefore $Q(t)$ divides $(1-ct^A)$ and the theorem follows.

4. Generalized power sums

Let $b_1, b_2, \ldots, b_n$ and $z_1, z_2, \ldots, z_n$ be two sets of $n$ complex numbers. It is our purpose to study the occurrence of zeros in the sequence of generalized power sums $\{s_k\}_k=1^n$, where $s_k = \sum_{j=1}^{n} b_jz_j^k$, $k=1, 2, \ldots$. Clearly it is no loss of generality if we suppose that the $n$ numbers $z_1, z_2, \ldots, z_n$ are distinct and non-zero; moreover that the $b_1, b_2, \ldots, b_n$ do not vanish either. Under these conditions it is easily seen that the sequence $\{s_k\}_k=1^n$ cannot contain $n$ (or more) consecutive zeros, since $s_k = s_k+1 = \ldots = s_k+n-1 = 0$ would lead to

$$
\begin{vmatrix}
  z_1^{k_0} & z_2^{k_0} & \cdots & z_n^{k_0} \\
  z_1^{k_0+1} & z_2^{k_0+1} & \cdots & z_n^{k_0+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  z_1^{k_0+n-1} & z_2^{k_0+n-1} & \cdots & z_n^{k_0+n-1}
\end{vmatrix} = 0.
$$

Multiplication of the numbers $z_1, z_2, \ldots, z_n$ with a non-zero complex constant is not a restriction of our problem either. Therefore we may suppose that $z_1 = 1$.

Now suppose that for $n < p < q$ we have

$$s_p - s_{p-1} = s_{p-2} = \ldots = s_{p-n+1} = 0, \quad s_q - s_{q-1} = s_{q-2} = \ldots = s_{q-n+1} = 0.$$

We put $c = s_q/s_p$ (observe that $s_p \neq 0$), then clearly

$$s_q = cs_q, \quad s_{q-1} = cs_{p-1}, \ldots, \quad s_{q-n+1} = cs_{p-n+1}.$$

Put $G(t) = \prod_{j=1}^{n} (1-z_j t)^{b_j}$. Then by section 2 we have

$$-G'(t)/G(t) = \sum_{j=1}^{n} b_jz_j/(1-z_j t) = \sum_{k=1}^{n} s_k t^{k-1}.$$

If we put the rational function $F(t) = -G'(t)/G(t)$ in its irreducible form $P(t)/Q(t)$, then the degree of its denominator $Q(t)$ is exactly $n$ and $Q(0) \neq 0$, since $G(0) = 1$. We can now apply theorem 1 to the function

$$F(t) = -G'(t)/G(t)$$

since condition (7) is satisfied with $q - n$ instead of $q$ and $p - n$ instead of $p$. It follows that the poles $z_1^{-1} = 1, z_2^{-1}, \ldots, z_n^{-1}$ of this function are $(q-p)^{th}$ roots of unity. In this way we have obtained:

**Theorem 2:** Let $b_1, b_2, \ldots, b_n$ be a set of $n$ non-zero complex numbers, let $z_1 = 1, z_2, \ldots, z_n$ be a set of $n$ distinct non-zero complex numbers and let
\[\{s_k\}_{k=1}^{\infty}\] be the sequence of generalized power sums. If there exist two positive integers \(p\) and \(q\), \(n < p < q\), such that
\[s_{p-1} = s_{p-2} = \ldots = s_{p-n+1} = 0, s_{q-1} = s_{q-2} = \ldots = s_{q-n+1} = 0,\]
then the numbers \(z_1, z_2, \ldots, z_n\) are \((q-p)\)th roots of unity.

**Remark:** If we put
\[A = q - p,\]
then from the assertion of theorem 2 it follows
\[z_1^A = z_2^A = \ldots = z_n^A = 1,\]
hence \(s_{k+A} = s_k\), so that the sequence \(\{s_k\}_{k=1}^{\infty}\) is periodic mod. \(A\). This again implies that there exist in the sequence infinitely many gaps of \(n-1\) consecutive zeros. Therefore the two conjectures (3) and (4) are equivalent.

In the sequel we take \(A\) minimal, so that \(A\) is the “distance” between two consecutive gaps of \(n-1\) zeros.

From now on we suppose that the numbers \(b_1, b_2, \ldots, b_n\) are real and positive. Very important for the next section is that in this special case we can say something about the location of the gaps of \(n-1\) consecutive zeros.

**Theorem 3:** Suppose that the conditions of theorem 2 hold and that moreover the numbers \(b_1, b_2, \ldots, b_n\) are positive. Let in particular
\[s_{p-1} = s_{p-2} = \ldots = s_{p-n+1} = 0\]
denote the first gap of \(n-1\) consecutive zeros in the sequence \(\{s_k\}_{k=1}^{\infty}\) and let \(A\) be its minimal period. Then \(s_{p-1}, s_{p-2}, \ldots, s_{p-n+1}\) are contained in the sequence \(\{s_1, s_2, \ldots, s_{A-1}\}\); moreover there are as many terms \(s_1, s_2, \ldots, s_{p-n}\) preceding the gap as there are terms \(s_p, s_{p+1}, \ldots, s_{A-1}\) following the gap; i.e.
\[A + n = 2p.\]

**Remark:** From this it follows that
\[s_{(A-n+2)/2} = s_{(A-n+4)/2} = \ldots = s_{(A+n-2)/2} = 0.\]

**Proof.** We have \(s_A = \sum_{j=1}^{n} b_j > 0\), so that \(A\) cannot occur in the sequence \(p-1, p-2, \ldots, p-n+1\). It follows
\[p < A,\]
for otherwise even \(p-1 > p-2 > \ldots > p-n+1 > A\), so that the sequence \(\{s_1, s_2, \ldots, s_A\}\) would not contain a gap of \(n-1\) zeros, but this is not possible on account of the periodicity mod. \(A\) of \(\{s_k\}_{k=1}^{\infty}\). Hence there is at least one gap of \(n-1\) zeros contained in the sequence \(\{s_1, s_2, \ldots, s_{A-1}\}\). However this sequence can contain only one of these gaps. This follows from the remark following theorem 2 and from the minimality condition for \(A\).
We have $A > p > n$; further

$$s_{A-k} = \sum_{j=1}^{n} b_j z_j^{A-k} = \sum_{j=1}^{n} b_j \bar{z}_j^k = \bar{s}_k, \quad k = 1, 2, \ldots, n - 1.$$  

hence

$$(12) \quad s_{A-k} = 0 \iff s_k = 0, \quad k = 1, 2, \ldots, n - 1.$$  

We have $s_{p-1} = s_{p-2} = \ldots = s_{p-n+1} = 0$ and hence by (12) we obtain $s_{A-p+1} = s_{A-p+2} = \ldots = s_{A-p+n-1} = 0$. Since there exists only one gap of $n-1$ zeros, this implies e.g. $p-1 = A - p + n - 1$, or $A + n = 2p$.

**Corollary:** $A + n$ is an even number.

This corollary will play an important part in our further investigations.

5. The case $n$ is odd.

Now we turn to the study of (3) and (4). Hence the numbers $b_1, b_2, \ldots, b_n$ will be all equal to 1 in the sequel. In all our next theorems the numbers $z_1, z_2, \ldots, z_n$ are supposed to be non-zero and distinct. This is no restriction since otherwise the numbers $z_1, z_2, \ldots, z_n$ would all be equal to zero. This follows by considering Vandermonde’s determinant, compare the first paragraph of section 4.

We start by showing that Turán’s and Erdős’ conjectures are true for odd $n$.

**Theorem 4:** Let $z_1, z_2, \ldots, z_n$ be $n$ distinct, non-zero complex numbers, $z_1 = 1$, $n$ odd. Let $\{s_k\}_{k=1}^{\infty}$ denote their sequence of power sums, $s_k = \sum_{j=1}^{n} z_j^k$, and suppose there exist two positive integers $p$ and $q$ such that

$$s_{p-1} = s_{p-2} = \ldots = s_{p-n+1} = 0, \quad s_{q-1} = s_{q-2} = \ldots = s_{q-n+1} = 0.$$  

Then the numbers $z_1, z_2, \ldots, z_n$ are the $n$ $n^{th}$ roots of unity.

**Proof.** By theorem 2 the numbers $z_1, z_2, \ldots, z_n$ are $A^{th}$ roots of unity. We take, as before, $p$ and $A$ minimal. According to theorem 3: $A + n = 2p$ and $n$ being odd, $A$ is odd. Therefore there exists for each $j$, $j = 1, 2, \ldots, n$, just one complex number $v_j$ satisfying

$$v_j^A = 1, \quad v_j^2 = z_j.$$  

Let $\{\sigma_k\}_{k=1}^{\infty}$ be the sequence of power sums of these $n$ distinct, non-zero complex numbers $v_1, v_2, \ldots, v_n$, $v_1 = 1$.

We have $\sigma_{2k} = s_k$, $k = 1, 2, \ldots$. It follows from (11) that $s_k = 0$ for

$$k = \frac{A + 1}{2}, \quad \frac{A + 3}{2}, \quad \frac{A + 5}{2}, \quad \ldots, \quad \frac{A + n - 2}{2}.$$
Hence $\sigma_k = 0$ for $k = A + 1, A + 3, A + 5, \ldots, A + n - 2$ and since the numbers $v_1, v_2, \ldots, v_n$ are $A$th roots of unity we have

(13) \[ \sigma_k = 0 \text{ for } k = 1, 3, 5, \ldots, n - 2 \]

Let the numbers $a_1, a_2, \ldots, a_n$ be defined by

\[ \prod_{j=1}^{n} (t - v_j) = t^n + a_1 t^{n-1} + \ldots + a_{n-1} t + a_n. \]

According to the formulae of Newton-Girard, (6), we have

(14) \[ \sigma_k + a_1 \sigma_{k-1} + a_2 \sigma_{k-2} + \ldots + a_{k-1} \sigma_1 + k \ a_k = 0, \ 1 \leq k \leq n. \]

From (13) and (14) it follows by induction that

(15) \[ a_1 = a_3 = a_5 = \ldots = a_{n-2} = 0. \]

From (14) applied for $k = n$ and (15) we derive that

\[ |\sigma_n| = |n \ a_n|. \]

But $|a_n| = \prod_{j=1}^{n} |v_j| = 1$, hence $|\sigma_n| = n$.

Recalling that $\sigma_n = v_1^n + v_2^n + \ldots + v_n^n$ and that $v_1 = 1$ we see that we must have

\[ v_1^n = v_2^n = \ldots = v_n^n = 1. \]

Hence the numbers $v_1, v_2, \ldots, v_n$ are the $n$ $n$th roots of unity and the original numbers $z_1, z_2, \ldots, z_n$ are also the $n$ $n$th roots of unity.

6. The case $n$ is even.

As already has been mentioned in the introduction it will turn out that Turán’s and Erdős’ conjectures in this case are false. Indeed in theorem 5 we give for each even $n$ an infinity of counterexamples (compare the remark following this theorem).

However all these counterexamples have a very simple structure: The $z_1, z_2, \ldots, z_n$ in these cases are provided by the vertices of two regular $(n/2)$-gons with the same circumscribed circle with centre in the origin. The proof is easy and computational.

Finally in theorem 6 we state that all counterexamples are given by the preceding theorem. The proof is the deepest part of this section; it proceeds by a construction of all sets $z_1, z_2, \ldots, z_n$ with the property that there are at least two gaps of length $n - 1$ in the sequence of power sums.

Theorem 5: Let $n$ and $A$ be two even positive integers, such that $n|A$, and let $r$ be any odd positive integer. Let the $n$ complex numbers $z_1, z_2, \ldots, z_n$ be given by

\[ z_j = \begin{cases} \exp \left( \frac{4\pi ij}{n} \right), & j = 1, 2, \ldots, \frac{n}{2} \\ \exp \left( \frac{4\pi ij}{n} + \frac{2\pi ir}{A} \right), & j = \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n. \end{cases} \]
Then the sequence \( \{s_k\}_{k=1}^{\infty} \) of power sums of these numbers contains infinitely many gaps of \( n-1 \) consecutive zeros.

Remark: For \( r\cdot n/A = 1, 3, 5, \ldots \) the set \( z_1, z_2, \ldots, z_n \) coincides with the set of the \( n \) \( A \)th roots of unity (e.g. for \( n = A \)); otherwise this is no longer true.

Proof. From the choice of the \( z_j \) it follows that for \( (n/2) \not\equiv k \):

\[
 z_1^k + z_2^k + \ldots + z_n^k = 0 \quad \text{and} \quad z_{(n/2)+1}^k + z_{(n/2)+2}^k + \ldots + z_n^k = 0.
\]

Hence \( s_k = 0 \) for \( (n/2) \not\equiv k \). Since \( n/A \) we have \( \{\exp(4\pi ij/n)\}^{A/2} = 1 \); since \( r \) is odd: \( \{\exp(2\pi ir/A)\}^{A/2} = -1 \). Hence \( s_{A/2} = (n/2) - (n/2) = 0 \). Therefore \( s_k = 0 \) for

\[
 k = \frac{A}{2} - \frac{n}{2} + 1, \frac{A}{2} - \frac{n}{2} + 2, \ldots, \frac{A}{2} + \frac{n}{2} - 1
\]

and this is a gap of \( n-1 \) consecutive zeros in the sequence of power sums. Since clearly \( \{s_k\}_{k=1}^{\infty} \) is periodic mod. \( A \) we obtain thus an infinity of gaps of the desired length.

Lemma: Let \( z_1, z_2, \ldots, z_n \) be \( n \) complex numbers with \( |z_j| = 1 \), \( j = 1, 2, \ldots, n \). Let the numbers \( a_1, a_2, \ldots, a_n \) be defined by

\[
 \prod_{j=1}^{n} (t - z_j) = t^n + a_1 t^{n-1} + \ldots + a_n - t + a_n.
\]

Then

\[
 a_{n-k} = \overline{a_k} \cdot a_n, \quad k = 1, 2, \ldots, n-1.
\]

Proof. Put \( f(t) = \prod_{j=1}^{n} (t - z_j) \). Then

\[
 f(t) = t^n \prod_{j=1}^{n} z_j (\bar{z}_j - 1/t) = t^n a_n \prod_{j=1}^{n} (1/t - \bar{z}_j) = a_n t^n f(1/t) = a_n a_n t^n + a_n a_{n-1} t^{n-1} + \ldots + a_n a_1 + a_n
\]

and the lemma follows by comparing the coefficients of \( f(t) \) and \( a_n t^n f(1/t) \).

Theorem 6: Let \( z_1, z_2, \ldots, z_n \) be a set of \( n \) distinct, non-zero complex numbers, \( n \) even, and let \( \{s_k\}_{k=1}^{\infty} \) be the sequence of power sums of these numbers. Suppose that there exist two positive integers \( p \) and \( q \) such that

\[
 s_{p-1} = s_{p-2} = \ldots = s_{p-n+1} = 0, \quad s_{q-1} = s_{q-2} = \ldots = s_{q-n+1} = 0.
\]

Then there exist an odd positive integer \( r \) and a positive integer \( A \) divisible by \( n \) (so that \( A \) is even) such that after an appropriate relabelling of the numbers \( z_1, z_2, \ldots, z_n \) one has

\[
 z_j = \begin{cases} 
 \exp \left( \frac{4\pi ij}{n} \right), & j = 1, 2, \ldots, n \\
 \exp \left( \frac{4\pi ij}{n} + \frac{2\pi ir}{A} \right), & j = \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n.
\end{cases}
\]
Proof. As we showed in section 4 it follows from theorem 1 that the numbers \( z_1, z_2, \ldots, z_n \) are \( A \)th roots of unity. By the corollary of theorem 3 this \( A \) is even.

Now it is convenient for our purpose to introduce for the moment the notions of an even and of an odd \( A \)th root of unity, \( (A \) even). We call \( z_j \) an even \( A \)th root of unity when \( z_j^{A/2} = 1 \), we call it odd when \( z_j^{A/2} = -1 \). From theorem 3, formula (11), we know that

\[
\begin{align*}
S_{A/2} = S_{(A+2)/2} = \ldots = S_{(A+n-2)/2} = 0.
\end{align*}
\]

In particular

\[
\begin{align*}
z_1^{A/2} + z_2^{A/2} + \ldots + z_n^{A/2} = 0,
\end{align*}
\]

in other words: there are as many even as odd unit roots among the numbers \( z_1, z_2, \ldots, z_n \). Let after an appropriate permutation \( z_1 = 1, z_2, \ldots, z_{n/2} \) be even, \( z_{(n/2)+1}, z_{(n/2)+2}, \ldots, z_n \) be odd. We put

\[
\begin{align*}
\Gamma^*(t) &= \prod_{j=1}^{n/2} (t - z_j) = t^{n/2} + \alpha_1^* t^{(n/2)-1} + \ldots + \alpha_{n/2}^* t + 1, \\
\Gamma(t) &= \prod_{j=(n/2)+1}^{n} (t - z_j) = t^{n/2} + \alpha_1 t^{(n/2)-1} + \ldots + \alpha_{n/2} t + 1,
\end{align*}
\]

\[
\begin{align*}
\sigma_k^* &= z_1^k + z_2^k + \ldots + z_{n/2}^k, \\
\sigma_k &= z_{(n/2)+1}^k + z_{(n/2)+2}^k + \ldots + z_n^k, \quad k = 1, 2, \ldots
\end{align*}
\]

By (18) we have

\[
\begin{align*}
\sigma_k^* + \sigma_k = 0, \quad k = \frac{A}{2} + 1, \frac{A}{2} + 2, \ldots, \frac{A}{2} + \frac{n}{2} - 1.
\end{align*}
\]

Further, since the \( \sigma_k^* \) are power sums of even roots of unity and the \( \sigma_k \) of odd ones we have

\[
\begin{align*}
\begin{cases}
\sigma_{(A/2)+k}^* = \sigma_k^* \\
\sigma_{(A/2)+k} = -\sigma_k
\end{cases}
\end{align*}
\]

Hence

\[
\sigma_k^* = \sigma_{(A/2)+k} = -\sigma_{(A/2)+k} = \sigma_k \quad \text{for} \quad k = 1, 2, \ldots, \frac{n}{2} - 1.
\]

From this it follows by induction, using (6), that

\[
\begin{align*}
\alpha_k^* = \alpha_k, \quad k = 1, 2, \ldots, \frac{n}{2} - 1.
\end{align*}
\]

Therefore \( \Gamma^*(t) - \Gamma(t) = \alpha_{n/2}^* - \alpha_{n/2} \), but \( \Gamma^*(1) \neq \Gamma(t) \) since \( \Gamma^*(1) = 0 \) and \( \Gamma(1) \neq 0 \), hence

\[
\alpha_{n/2}^* \neq \alpha_{n/2}.
\]
Suppose that there exists an index \( k, 1 \leq k < (n/2) - 1 \), such that \( \alpha_k^* = \alpha_k \neq 0 \). By the lemma, formula (17) with \( n/2 \) instead of \( n \), we then should have

\[
\alpha_{n/2}^* = \alpha_{n/2}^{(n/2)-k} / \alpha_k^*, \quad \alpha_{n/2} = \alpha_{(n/2)-k} / \alpha_k.
\]

This however would give us, in view of (19)

\[
\alpha_{n/2}^* = \alpha_{n/2}^*.
\]

in contradiction with (20), so that \( \alpha_k^* = \alpha_k = 0 \) for \( k = 1, 2, \ldots, (n/2) - 1 \), therefore

\[
\Gamma(t) = t^{n/2} + \alpha_{n/2}^* \quad \text{and} \quad \Gamma^*(t) = t^{n/2} + \alpha_{n/2}^*.
\]

Hence the points \( z_1, z_2, \ldots, z_n \) are lying on the vertices of two regular \((n/2)\)-gons with radius 1 and centre at 0. Since \( z_1 = 1 \), the numbers \( z_1, z_2, \ldots, z_{n/2} \) coincide with the \((n/2)\)th roots of unity. But they are also even \( A \)th roots, hence \((A/2)\)th roots of unity. Therefore \( \sigma_{n/2}^* = n/2 \neq 0 \). But \( \sigma_k^* = 0 \) for \((n/2) + k\) in view of (16). Hence it follows that \((n/2)/(A/2)\) or \(n/A\). Finally, since the \( z_{(n/2)+1}, z_{(n/2)+2}, \ldots, z_n \) are odd \( A \)th roots of unity, there exists an odd integer \( r \) such that after relabelling:

\[
z_j = \exp \left( \frac{4\pi ij}{n} + \frac{2\pi vr}{A} \right), \quad j = \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n.
\]

Mathematisch Instituut
University of Amsterdam.

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