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The perturbed compound Poisson risk model with linear dividend barrier

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ABSTRACT

In this paper, we consider a diffusion perturbed classical compound Poisson risk model in the presence of a linear dividend barrier. Partial integro-differential equations for the moment generating function and the nth moment of the present value of all dividends until ruin are derived. Moreover, explicit solutions for the nth moment of the present value of dividend payments are obtained when the individual claim size distribution is exponential. We also provided some numerical examples to illustrate the applications of the explicit solutions. Finally we derive partial integro-differential equations with boundary conditions for the Gerber–Shiu function.

1. Introduction

De Finetti [1] first proposed dividend strategies for an insurance risk model, and he found that the optimal dividend strategy is a barrier strategy under some conditions. Since then, the risk model in the presence of dividend payments has become a more and more popular topic in risk theory. For the classical risk model and a constant dividend barrier, Lin et al. [2] have studied the discounted penalty function at ruin, which is an important tool to quantify the riskiness of the barrier strategy. A second important quantity in assessing the quality of a dividend barrier strategy is the distribution of the discounted sum of dividend payments until ruin. For the classical risk model and constant barrier strategy, Dickson and Waters [3] studied arbitrary moments.

The classical risk model perturbed by a diffusion was first introduced in [4] and has been further studied by many authors during the last few years, see [5–11]. For the risk model perturbed by a Brownian motion, Gerber and Shiu [12] give some very explicit calculations on the moments and distribution of the discounted dividends paid until ruin. Li [8] studied the expected discounted dividend payments prior to ruin and the Gerber–Shiu expected discounted penalty function. Under the barrier strategy, ultimate ruin of the company is certain. To overcome the deficiency of horizontal barrier models that they lead to ruin with probability 1, the linear barrier model was introduced in [4]. Albrecher et al. [13] studied the distribution of dividend payments and the discounted penalty function in the compound poisson risk model with linear dividend barrier. However, there is no work that deals with the perturbed compound Poisson risk model with linear dividend barrier. This motivates us to investigate such a risk model in this work.

The purpose of this paper is to present some results on the distribution of dividend payments until ruin and the Gerber–Shiu function under the perturbed compound Poisson risk model with linear dividend barrier. In Section 2 we describe the model, in Section 3, partial integro-differential equations for the moment-generation function and the nth moment of the sum of the discounted dividend payments until ruin are derived, moreover, explicit solutions and some

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numerical examples for the nth moment of the present value of dividend payments are obtained in Section 4. We then derive integro-differential equations for the Gerber–Shiu function in Section 5.

2. The model

Let (Ω, F, F_t, P) be a field probability space satisfying the usual conditions, containing all objects defined in the following. In the perturbed compound Poisson risk model, the surplus of an insurer has the form

$$U(t) = u + ct + \sigma W(t) - \sum_{i=1}^{N(t)} Z_i, \quad t \geq 0, \tag{2.1}$$

where $\{N(t), t \geq 0\}$ is a Poisson process with parameter λ, denoting the total number of claims from an insurance portfolio. $\{Z_i, i = 1, 2, \ldots\}$ are positive i.i.d. random variables with distribution function $F(z) = P(Z \leq z)$ and density function $f(z)$. $\{W(t), t \geq 0\}$ is a standard Brownian motion with $W(0) = 0$ and $\sigma > 0$ is a constant, representing the diffusion volatility parameter. $S(t) = \sum_{i=1}^{N(t)} Z_i$ is the aggregate claims process. In the above model, $0 \leq u \leq b$ is the initial surplus, $c = \lambda(1+\theta)EZ$ is the premium rate per unit time, and $\theta > 0$ is the relative security loading factor. In addition, $\{Z_i, i = 1, 2, \ldots\}$, $\{N(t), t \geq 0\}$ and $\{W(t), t \geq 0\}$ are mutually independent.

We consider the following extension of model (2.1), whenever the surplus $U(t)$ reaches a time-dependent barrier of type

$$b_t = b_0 + at, \quad (0 \leq a < c)$$

dividends are paid out to the shareholders with intensity $c - a$ and the surplus remains on the barrier until the next claim occurs. The dynamics of $U(t)$ are thus given by

$$\begin{cases} \mathrm{d}U(t) = c\mathrm{d}t - \mathrm{d}S(t) + \sigma \mathrm{d}W(t), & \text{if } U(t) < b_0 + at, \\ \mathrm{d}U(t) = a\mathrm{d}t - \mathrm{d}S(t) + \sigma \mathrm{d}W(t), & \text{if } U(t) = b_0 + at. \end{cases} \tag{2.2}$$

For $0 \leq u \leq b$, define $T_{u,b} = \inf\{t : U(t) < 0 | U(0) = u, b_0 = b\}$; For $t \geq 0$, let $\delta > 0$ be the force of interest valuation, where $D(t)$ denotes the aggregate dividends paid by time t, and let $D_{u,b}$ denote the present value of all dividends until time of ruin $T_{u,b}$,

$$D_{u,b} = \int_0^{T_{u,b}} \mathrm{e}^{-\delta t} \mathrm{d}D(t).$$

Define the moment generating function of $D_{u,b}$ by

$$M(u, y, b) = E[\mathrm{e}^{yD_{u,b}}], \quad 0 \leq u \leq b,$$

where y is such that $M\{u, y, b\}$ exists, and the nth moment by

$$V_n(u, b) = E[D_{u,b}^n], \quad 0 \leq u \leq b, n \in N,$$

and the expected discounted penalty function $m(u, b)$ by

$$m(u, b) = E[\omega(U(T_{u,b}^-)), |U(T_{u,b}|)\mathrm{e}^{-\delta T_{u,b}} I_{T_{u,b}} < \infty],$$

where $U(T_{u,b}^-)$ is the surplus immediately before ruin, $|U(T_{u,b})|$ is the deficit at ruin and the penalty $\omega(x_1, x_2)$ is an arbitrary non-negative function on $[0, \infty) \times [0, \infty) \cdot \delta \geq 0$ may be interpreted as a force of interest.

3. The moments of the discounted dividends

In this paper, we only consider dividend payments stopped at ruin.

Theorem 3.1. *For $0 < u < b$, $M(u, y, b)$ satisfies the following partial integro-differential equation:*

$$\frac{\sigma^2}{2} \frac{\partial^2 M}{\partial u^2}(u, y, b) + c \frac{\partial M}{\partial u}(u, y, b) + a \frac{\partial M}{\partial b}(u, y, b) - \lambda M(u, y, b) - \delta y \frac{\partial M}{\partial y}(u, y, b)$$
$$+ \lambda \int_0^u M(u - z, y, b) \mathrm{d}F(z) + \lambda(1 - F(u)) = 0 \tag{3.1}$$

with boundary conditions:

$$M(0, y, b) = 1, \tag{3.2}$$
$$\lim_{b \to \infty} M(u, y, b) = 1, \tag{3.3}$$
$$\left. \frac{\partial M}{\partial u} \right|_{u=b} = yM(b, y, b). \tag{3.4}$$

Proof. Consider the infinitesimal time from 0 to t, By conditioning on the time and amount of the first claim, we obtain that:

$$M(u,y,b) = (1-\lambda t)M(u+ct+\sigma W(t), ye^{-\delta t}, b+at) + \lambda t\int_{u+ct+\sigma W(t)}^{\infty} dF(z) + \lambda t\int_0^{u+ct+\sigma W(t)} M(u+ct+\sigma W(t)-z, ye^{-\delta t}, b+at)dF(z) + o(t). \tag{3.5}$$

By Taylor's expansion

$$M(u+ct+\sigma W(t), ye^{-\delta t}, b+at) = M(u,y,b) + \left[\frac{\sigma^2}{2}\frac{\partial^2 M}{\partial u^2}(u,y,b) + c\frac{\partial M}{\partial u}(u,y,b) - \delta y\frac{\partial M}{\partial y}(u,y,b)\right]t + a\frac{\partial M}{\partial b}(u,y,b)t + o(t). \tag{3.6}$$

Substituting the above expression into (3.5), dividing both sides of (3.5) by t and letting $t \to 0$, we can get Eq. (3.1).

The boundary condition (3.2) is obvious: if $u = 0$, ruin is immediate and no dividend is paid.

Similarly, when $b \to \infty$, no dividend is paid, so the condition (3.3) is correct.

When $u = b$

$$M(b,y,b) = (1-\lambda t)e^{y(c-a)t}M(b+at+\sigma W(t), ye^{-\delta t}, b+at) + \lambda te^{y(c-a)t}\int_{b+at+\sigma W(t)}^{\infty} dF(z) + \lambda t\int_0^{b+at+\sigma W(t)} M(b+at+\sigma W(t)-z, ye^{-\delta t}, b+at)dF(z) + o(t), \tag{3.7}$$

which implies

$$\frac{\sigma^2}{2}\frac{\partial^2 M}{\partial u^2}(u,y,b)\Big|_{u=b} + a\frac{\partial M}{\partial u}(u,y,b)\Big|_{u=b} - \delta y\frac{\partial M}{\partial y}(u,y,b)\Big|_{u=b} + a\frac{\partial M}{\partial b}(u,y,b)\Big|_{u=b} + [y(c-a)-\lambda]M(b,y,b) + \lambda\int_0^b M(b-z,y,b)dF(z) + \lambda[1-F(b)] = 0. \tag{3.8}$$

Setting $u = b$ in (3.1), we can obtain the boundary condition (3.4). □

Remark 3.1. In the case of $a = 0$, (3.1), (3.2) and (3.4) reduce, respectively, to the Eqs. (19), (20) and (21) of [8].

Using the representation

$$M(u,y,b) = 1 + \sum_{n=1}^{\infty}\frac{y^n}{n!}V_n(u,b)$$

and equating the coefficients of $y^n (n \in N)$ in (3.1), we have

Theorem 3.2. *When $0 < u < b$, $V_n(u,b)$ satisfies the following partial integro-differential equation:*

$$\frac{\sigma^2}{2}\frac{\partial^2 V_n}{\partial u^2}(u,b) + c\frac{\partial V_n}{\partial u}(u,b) + a\frac{\partial V_n}{\partial b}(u,b) - (\lambda+n\delta)V_n(u,b) + \lambda\int_0^u V_n(u-z,b)dF(z) = 0, \tag{3.9}$$

with boundary conditions:

$$V(0,b) = 0, \tag{3.10}$$

$$\lim_{b\to\infty} V_n(u,b) = 0, \tag{3.11}$$

$$\frac{\partial V_n}{\partial u}\Big|_{u=b} = nV_{n-1}(b,b). \tag{3.12}$$

Remark 3.2. For $n = 1$, we retain the risk process and indeed (3.9) simplifies in this case to

$$\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial u^2}(u,b) + c\frac{\partial V}{\partial u}(u,b) + a\frac{\partial V}{\partial b}(u,b) - (\lambda+\delta)V(u,b) + \lambda\int_0^u V(u-z,b)dF(z) = 0. \tag{3.13}$$

Correspondingly, the boundary condition (3.12) simplifies to $\frac{\partial V}{\partial u}|_{u=b} = 1$.

4. Closed form expression for the expression of $D_{u,b}$

In this section, we will derive the explicit formulae for $V_n(u, b)$ when the claim size is exponentially distributed $F(z) = 1 - e^{-z}$ and $n = 1$, i.e. the expression for the expectation of $D_{u,b} = V(u, b)$.

In this case, (3.9) can be rewritten as a partial differential equation:

$$\frac{\sigma^2}{2}\frac{\partial^3 V_n}{\partial u^3}(u,b) + \left(\frac{\sigma^2}{2} + c\right)\frac{\partial^2 V_n}{\partial u^2}(u,b) + a\frac{\partial^2 V_n}{\partial u \partial b}(u,b) + (c - \lambda - n\delta)\frac{\partial V_n}{\partial u}(u,b) - n\delta V_n(u,b) = 0 \tag{4.1}$$

with boundary conditions

$$\frac{\sigma^2}{2}\frac{\partial^2 V_n}{\partial u^2}(u,b)\bigg|_{u=0} + c\frac{\partial V_n}{\partial u}(u,b)\bigg|_{u=0} + a\frac{\partial V_n}{\partial b}(u,b)\bigg|_{u=0} - (\lambda + n\delta)V_n(0,b) = 0, \tag{4.2}$$

and (3.10)–(3.12).

Its special solutions are of the form

$$\{c_1 e^{r_1 u} + c_2 e^{r_2 u} + c_3 e^{r_3 u}\}, \tag{4.3}$$

where c_1, c_2, c_3 are arbitrary coefficients, r_1, r_2, r_3 are the solutions of the equation:

$$\frac{\sigma^2}{2}R^3 + \left(\frac{\sigma^2}{2} + c\right)R^2 + (as + c - \lambda - n\delta)R + as - n\delta = 0. \tag{4.4}$$

From (3.10), we obtain $c_1 + c_2 + c_3 = 0$.

With the substitution $x = u - v$, (3.9) can be

$$\frac{\sigma^2}{2}\frac{\partial^2 V_n}{\partial u^2}(u,b) + c\frac{\partial V_n}{\partial u}(u,b) + a\frac{\partial V_n}{\partial b}(u,b) - (\lambda + n\delta)V_n(u,b) + \lambda\int_0^u V_n(x,b)e^{-(u-x)}dx = 0. \tag{4.5}$$

Substituting (4.3) into (4.5), we have

$$\begin{aligned}&\frac{\sigma^2}{2}[c_1 r_1^2 e^{r_1 u} + c_2 r_2^2 e^{r_2 u} + c_3 r_3^2 e^{r_3 u}] + c[c_1 r_1 e^{r_1 u} + c_2 r_2 e^{r_2 u} + c_3 r_3 e^{r_3 u}] + as[c_1 e^{r_1 u} + c_2 e^{r_2 u} + c_3 e^{r_3 u}] \\ &\quad - [\lambda + n\delta][c_1 e^{r_1 u} + c_2 e^{r_2 u} + c_3 e^{r_3 u}] + \lambda c_1 e^{-u}\frac{e^{(r_1+1)u} - 1}{r_1 + 1} + \lambda c_2 e^{-u}\frac{e^{(r_2+1)u} - 1}{r_2 + 1} + \lambda c_3 e^{-u}\frac{e^{(r_3+1)u} - 1}{r_3 + 1} = 0,\end{aligned} \tag{4.6}$$

i.e.

$$\begin{aligned}&c_1 e^{r_1 u}\left[\frac{\sigma^2}{2}r_1^2 + cr_1 + as - \lambda - n\delta\right] + \lambda c_1\frac{e^{r_1 u} - e^{-u}}{r_1 + 1} + c_2 e^{r_2 u}\left[\frac{\sigma^2}{2}r_2^2 + cr_2 + as - \lambda - n\delta\right] + \lambda c_2\frac{e^{r_2 u} - e^{-u}}{r_2 + 1} \\ &\quad + c_3 e^{r_3 u}\left[\frac{\sigma^2}{2}r_3^2 + cr_3 + as - \lambda - n\delta\right] + \lambda c_3\frac{e^{r_3 u} - e^{-u}}{r_3 + 1} = 0.\end{aligned} \tag{4.7}$$

Since r_1, r_2, r_3 is the solution of (4.4), we can get

$$\frac{-\lambda c_1 e^{-u}}{r_1 + 1} + \frac{-\lambda c_2 e^{-u}}{r_2 + 1} + \frac{-\lambda c_3 e^{-u}}{r_3 + 1} = 0. \tag{4.8}$$

We see that (4.6) satisfies (4.8) only if

$$\frac{c_1}{r_1 + 1} + \frac{c_2}{r_2 + 1} + \frac{c_3}{r_3 + 1} = 0. \tag{4.9}$$

Consequently, the challenge is to find a combination of functions of type:

$$V_n(u,b) = c\left(e^{r_1 u} - \frac{(1 + r_2)(r_1 - r_3)}{(1 + r_1)(r_2 - r_3)}e^{r_2 u} - \frac{(1 + r_3)(r_1 - r_2)}{(1 + r_1)(r_3 - r_2)}e^{r_3 u}\right)e^{sb} \tag{4.10}$$

that satisfies the boundary conditions (3.10)–(3.12).

For the case $n = 1$

$$V_1(u,b) = \sum_{k=0}^{\infty} c_k e^{s_k b}\left(e^{r_{1,k} u} - \frac{(1 + r_{2,k})(r_{1,k} - r_{3,k})}{(1 + r_{1,k})(r_{2,k} - r_{3,k})}e^{r_{2,k} u} - \frac{(1 + r_{3,k})(r_{1,k} - r_{2,k})}{(1 + r_{1,k})(r_{3,k} - r_{2,k})}e^{r_{3,k} u}\right). \tag{4.11}$$

Table 1
The expected sum of discounted dividend $V_1(u, b)$ when $a = 1.1$, $c = 1.5$.

$b \setminus u$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
0.1	4.02e−017	0.161											
0.2	3.89e−017	0.138	0.273										
0.3	3.77e−017	0.118	0.234	0.353									
0.4	3.65e−017	0.101	0.199	0.301	0.410								
0.5	3.53e−017	0.086	0.170	0.257	0.349	0.452							
0.6	3.41e−017	0.073	0.145	0.219	0.298	0.385	0.484						
0.7	3.30e−017	0.062	0.123	0.189	0.253	0.328	0.412	0.508					
0.8	3.20e−017	0.053	0.105	0.158	0.215	0.279	0.350	0.432	0.527				
0.9	3.09e−017	0.045	0.089	0.134	0.183	0.237	0.297	0.367	0.448	0.541			
1.0	2.99e−017	0.038	0.075	0.114	0.155	0.201	0.252	0.312	0.380	0.460	0.553		
1.1	2.90e−017	0.032	0.064	0.097	0.131	0.170	0.214	0.264	0.323	0.390	0.470	0.563	
1.2	2.80e−017	0.027	0.054	0.082	0.111	0.144	0.181	0.224	0.273	0.331	0.398	0.477	0.570

Table 2
The expected sum of discounted dividend $V_1(u, b)$ when $a = 1.3$, $c = 1.5$.

$b \setminus u$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
0.1	7.73e−017	0.152											
0.2	7.43e−017	0.130	0.259										
0.3	7.15e−017	0.112	0.223	0.338									
0.4	6.88e−017	0.096	0.192	0.291	0.398								
0.5	6.61e−017	0.083	0.165	0.251	0.343	0.445							
0.6	6.36e−017	0.072	0.143	0.217	0.297	0.384	0.483						
0.7	6.12e−017	0.062	0.124	0.188	0.257	0.333	0.418	0.515					
0.8	5.89e−017	0.054	0.108	0.164	0.224	0.290	0.363	0.447	0.542				
0.9	5.66e−017	0.047	0.094	0.143	0.195	0.252	0.317	0.389	0.472	0.566			
1.0	5.45e−017	0.041	0.082	0.125	0.171	0.221	0.277	0.340	0.412	0.494	0.588		
1.1	5.24e−017	0.036	0.072	0.110	0.150	0.194	0.243	0.298	0.360	0.432	0.513	0.607	
1.2	5.04e−017	0.032	0.064	0.097	0.132	0.171	0.214	0.262	0.317	0.379	0.450	0.531	0.625

According Gerber [14] and Albrecher et al. [13], we choose the starting parameters for $V_1(u, b)$ in the following way:

$$c_0 = \frac{1}{\rho}, \qquad s_0 = -\rho, \qquad r_{1,0} = \rho,$$

$$r_{2,0} = \frac{-(\sigma^2 + 2c + \sigma^2\rho) + 2\sqrt{(\sigma^2 + 2c + \sigma^2\rho)^2 - \frac{(a\rho+\delta)\sigma^4}{\rho}}}{\sigma^4},$$

$$r_{3,0} = \frac{-(\sigma^2 + 2c + \sigma^2\rho) - 2\sqrt{(\sigma^2 + 2c + \sigma^2\rho)^2 - \frac{(a\rho+\delta)\sigma^4}{\rho}}}{\sigma^4},$$

$$s_k + r_{2,k} = s_{k+1} + r_{1,k+1}, \qquad s_k + r_{3,k} = s_{k+1} + r_{2,k+1}, \quad k \geq 0,$$

$$r_{1,k} + r_{2,k} + r_{3,k} = -\frac{a - c - \frac{\sigma^2}{2}}{\frac{\sigma^2}{2}}, \qquad r_{1,k} r_{2,k} r_{3,k} = -\frac{\delta}{\frac{\sigma^2}{2}}, \quad k \geq 0,$$

$$c_k = c_{k-1} \frac{r_{2,k-1}}{r_{1,k}} \frac{(1 + r_{2,k-1})(r_{1,k-1} - r_{3,k-1})}{(1 + r_{1,k-1})(r_{2,k-1} - r_{3,k-1})}, \quad k \geq 1,$$

where $r = -\rho$ is the negative solution of

$$\frac{\sigma^2}{2} r^3 + \left(a - c - \frac{\sigma^2}{2}\right) + (c - a - \lambda - \delta) r + \delta = 0.$$

Example. Assume that the claim amounts are exponentially distributed with parameter $\lambda = 1$, for $\sigma = 2$, $\delta = 0.1$, Tables 1 and 2 show the results for the expected sum of discounted dividends $V_1(u, b)$ given by (4.11), where $a = 1.1$ with $c = 1.5$, $a = 1.3$ with $c = 1.5$, respectively.

For $u = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2$, and $b = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2$, Tables 1 and 2 summarize the results for $V_1(u, b)$ for $a = 1.1$ and $a = 1.3$, respectively. As expected, the numbers show that the higher the initial surplus of the insurance company, the higher the expected sum of discounted dividend prior to the time of ruin for fixed a and b. Furthermore, from the values of $V_1(u, b)$ listed in Tables 1 and 2, we observe that $V_1(u, b)$ is decreasing with respect to b for fixed a and u. If one wants to maximize the expected sum of dividend until ruin, then in this model, a and b should be chosen appropriately.

5. The discounted penalty function

Theorem 5.1. *When $0 < u < b$, $m(u, b)$ satisfies the following partial integro-differential equation:*

$$\frac{\sigma^2}{2}\frac{\partial^2 m}{\partial u^2}(u,b) + c\frac{\partial m}{\partial u}(u,b) + a\frac{\partial m}{\partial b}(u,b) - (\lambda+\delta)m(u,b) + \lambda\int_0^u m(u-z,b)\mathrm{d}F(z) + \lambda\int_u^\infty \omega(u, z-u)\mathrm{d}F(z) = 0 \tag{5.1}$$

with boundary conditions

$$\left.\frac{\partial m}{\partial u}(u,b)\right|_{u=b} = 0, \tag{5.2}$$

$$\lim_{b\to\infty} m(u,b) = m(u), \tag{5.3}$$

where $m(u)$ is the discounted penalty without barrier.

Proof. Consider the infinitesimal interval from 0 to t, we obtain

$$m(u,b) = (1-\lambda t)\mathrm{e}^{-\delta t} m(u+ct+\sigma W(t), b+at) + \left[\lambda t\int_0^{u+ct+\sigma W(t)} m(u+ct+\sigma W(t)-z, b+at)\mathrm{d}F(z) + \lambda t\int_{u+ct+\sigma W(t)}^\infty \omega(u+ct+\sigma W(t), z-u-ct-\sigma W(t))\mathrm{d}F(z)\right]\mathrm{e}^{-\delta t} + o(t). \tag{5.4}$$

We adopt a similar approach to Theorem 3.1, (5.1) can be obtained.

If $b \to \infty$, no dividend is paid, So the boundary condition (5.3) is correct.

When $u = b$,

$$m(b,b) = (1-\lambda t)\mathrm{e}^{-\delta t} m(b+at+\sigma W(t), b+at) + \left[\lambda t\int_0^{b+at+\sigma W(t)} m(b+at+\sigma W(t)-z, b+at)\mathrm{d}F(z) + \lambda t\int_{b+at+\sigma W(t)}^\infty \omega(b+at+\sigma W(t), z-b-at-\sigma W(t))\mathrm{d}F(z)\right]\mathrm{e}^{-\delta t} + o(t). \tag{5.5}$$

Using a similar approach to (3.4), we can obtain

$$\frac{\sigma^2}{2}\frac{\partial^2 m}{\partial b^2}(b,b) + a\frac{\partial m}{\partial b}(b,b) + a\frac{\partial m}{\partial b}(b,b) - (\lambda+\delta)m(b,b) + \lambda\int_0^b m(b-z,b)\mathrm{d}F(z) + \lambda\int_b^\infty \omega(b, z-b)\mathrm{d}F(z) = 0. \tag{5.6}$$

Setting $u = b$ in (5.1), we can get (5.3). □

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