JOURNAL OF DIFFERENTIAL EQUATIONS **62**, 73–94 (1986)

Asymptotic Behavior of Solutions to a Class of Nonlinear Evolution Equations

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Received July 14, 1982; revised June 8, 1983

INTRODUCTION

This work is motivated by the study of the heat equations with temperature controls through the interior or the boundary of the domain. For instance, we consider the following type of parabolic variational problems with obstacles: Find a function $u = u(x, t)$ on $[0, 1] \times [0, \infty)$ satisfying

$$
\text{(V)} \quad \begin{cases} u \in C([0, \infty); L^2(0, 1)) \cap W^{1,2}(0, T; L^2(0, 1)) \quad \text{for every } T > 0, \\ u(\cdot, 0) = u_0, \\ u(\cdot, t) \in K(t) \equiv \{z \in W^{1,2}(0, 1); 0 \leqslant z \leqslant g(\cdot, t) \text{ on } [0, 1]\} \text{ for all } t \geqslant 0, \\ \displaystyle\int_0^1 (u_t(x, t) - f(x, t))(u(x, t) - z(x))\, dx \\ \displaystyle\qquad + \int_0^1 u_x(x, t)(u_x(x, t) - z_x(x))\, dx \leqslant 0 \\ \qquad \text{for all } z \in K(t) \text{ and a.e. } t \geqslant 0, \end{cases}
$$

where u_0 is a function on $[0, 1]$ and g, f are functions on $[0, 1] \times [0, \infty)$. This kind of variational problems has been discussed by many authors (for instance, see [3, 11]) from the viewpoint of the theory of nonlinear evolution equations governed by time-dependent subdifferential operators

0022-0396/86 $3.00

in Hilbert spaces; in fact, problem (V) is written as a Cauchy problem in the space $H = L^2(0, 1)$ of the form

$$(\mathrm{CP}; u_0) \qquad -u'(t) \in \partial\phi^t(u(t)), \qquad 0 < t < \infty, \quad u(0) = u_0,$$

where $\phi^t: H \to (-\infty, \infty]$, $\not\equiv \infty$, is a l.s.c. (lower semi-continuous) convex function given by

$$\begin{aligned} \phi^t(z) &= \frac{1}{2}\int_0^t |z_x(x)|^2\,dx - \int_0^1 f(x,t)\,z(x)\,dx \qquad &&\text{if} \quad z \in K(t), \\ &= \infty &&\text{otherwise,} \end{aligned}$$

for each $t \geq 0$, and $\partial\phi^t$ denotes its subdifferential. Our interest in problem (V) is the asymptotic behavior of solutions.

In [1, 2], Ball and Peletier dealt with the heat equation $u_t = u_{xx}$ with a class of nonlinear boundary conditions, and discussed the asymptotic behavior of solutions. Their approach is based upon the general results which have been developed in the theory of dynamical systems and of compact or uniform processes. For example, see Dafermos [6, 7] and Slemrod [14]. In problem (V), the solution u is, however, constrained by the moving convex sets $K(t)$, say $u(\cdot, t) \in K(t)$ holds for all $t \geq 0$, and this restraint does not easily allow us the direct use of general results in the theory of compact or uniform processes. Indeed, it seems to be difficult to verify that problem (V) generates a process in some of classes which have been treated so far. It would, of course, be very interesting and important to be able to make it possible to use the theory of processes extensively in order to investigate the asymptotic behavior of solutions to problem (V) and more generally to problems having the same kind of time-dependence. But we intend to discuss this question elsewhere.

In this paper we study the asymptotic behavior of solutions to Cauchy problems of the form $(\mathrm{CP}; u_0)$ formulated in abstract Hilbert spaces. In the case when ϕ^t is independent of t, namely $\phi^t = \phi$ for $t \geq 0$, the asymptotic behavior of the solution u to

$$-u'(t) \in \partial\phi(u(t)), \qquad 0 < t < \infty, \quad u(0) = u_0,$$

was investigated in detail, for example, in Brézis [4], Bruck [5], Dafermos and Slemrod [8] and Pazy [13]. But, in the time-dependent case, particularly in such a case as $\overline{D(\phi^s)} \neq \overline{D(\phi^t)}$ $(s \neq t)$, which happens in problem (V), we have not noted any general results on the asymptotic behavior of solutions. Under some assumptions imposed on the family $\{\phi^t; 0 \leq t < \infty\}$ and the limit ϕ^∞ of ϕ^t as $t \to \infty$ in a sense, we shall show that the solution

$u(t)$ of $(CP; u_0)$ converges to an equilibrium point u_∞ as $t \to \infty$ in the sense that

(a) $\lim_{t\to\infty} \phi^t(u(t))$ exists and equals $\min \phi^\infty$ (the minimum of ϕ^∞), and

(b) $u(t) \to u_\infty$ weakly (or strongly) in H as $t \to \infty$ and ϕ^∞ attains $\min \phi^\infty$ at u_∞ (*i.e.*, $0 \in \partial\phi^\infty(u_\infty)$).

Notations. In general, for a (real) Banach space X, we denote the norm by $|\cdot|_X$. Throughout this paper, let H be a Hilbert space with inner product $(\cdot, \cdot)_H$. For a proper l.s.c. convex function ϕ on H, $D(\phi)$ is the effective domain of ϕ, $\partial\phi$ is the subdifferential operator of ϕ and $D(\partial\phi)$ is the domain of $\partial\phi$. We refer to the book of Brézis [4] for their definitions and general properties.

1. Statement of Main Results

Let $\{\phi^t\} = \{\phi^t; 0 \leqslant t < \infty\}$ be a family of proper l.s.c. convex functions ϕ^t on H, and consider the Cauchy problem

$$(\mathrm{CP}; u_0) \quad -u'(t) \in \partial\phi^t(u(t)), \qquad 0 < t < \infty, \quad u(0) = u_0,$$

where u_0 is given in $D(\phi^0)$ and the unknown u is an H-valued function on $[0, \infty)$; $u'(t)$ $(=(d/dt)\, u(t))$ denotes the strong derivative of $u(t)$ in H. By a solution of (CP; u_0) we mean a function u: $[0, \infty) \to H$ such that

(i) $u \in W^{1,2}(0, T; H)$ for every $0 < T < \infty$ and $u(0) = u_0$,

(ii) the function $t \to \phi^t(u(t))$ is locally bounded on $[0, \infty)$, and

(iii) $-u'(t) \in \partial\phi^t(u(t))$ for a.e. $t \geqslant 0$.

We consider the following assumptions (A) and (B) on $\{\phi^t\}$.

(A) For each $r \geqslant 0$ there are absolutely continuous real-valued functions a_r, b_r on $[0, \infty)$ such that

(a1) $a_r' \in L^2(0, \infty)$ and $b_r' \in L^1(0, \infty)$, and

(a2) for each s, $t \in [0, \infty)$ with $s \leqslant t$ and each $z \in D(\phi^s)$ with $|z|_H \leqslant r$ there exists $\tilde{z} \in D(\phi^t)$ satisfying

$$|\tilde{z} - z|_H \leqslant |a_r(t) - a_r(s)|(1 + |\phi^s(z)|^{1/2})$$

and

$$\phi^t(\tilde{z}) - \phi^s(z) \leqslant |b_r(t) - b_r(s)|(1 + |\phi^s(z)|).$$

(B) ϕ^t converges to a proper l.s.c. convex function ϕ^∞ on H as $t \to \infty$ in the sense of Mosco [12], that is, the conditions (b1) and (b2) below hold:

(b1) if w: $[0, \infty) \to H$ and $w(t) \to z$ weakly in H as $t \to \infty$, then

$$\liminf_{t \to \infty} \phi^t(w(t)) \geqslant \phi^\infty(z);$$

(b2) for each $z \in D(\phi^\infty)$ there is a function w: $[0, \infty) \to H$ such that $w(t) \to z$ strongly in H and $\phi^t(w(t)) \to \phi^\infty(z)$ as $t \to \infty$.

According to the results in [11, Chap. 1], under condition (A), (CP; u_0) admits a unique solution u for each $u_0 \in D(\phi^0)$, and we then define two sets $\Omega_s(u_0)$ and $\Omega_w(u_0)$ by

$$\Omega_s(u_0) = \{z \in H; u(t_n) \to z \text{ strongly in } H \text{ for some sequence } \{t_n\} \text{ with } t_n \to \infty\}$$

and

$$\Omega_w(u_0) = \{z \in H; u(t_n) \to z \text{ weakly in } H \text{ for some sequence } \{t_n\} \text{ with } t_n \to \infty\}.$$

Also, we set

$$F(\phi^\infty) = \{z \in H; \phi^\infty(z) = \min \phi^\infty\} \qquad (= \{z \in H; 0 \in \partial\phi^\infty(z)\}).$$

With these notations, the first main result is stated as follows.

THEOREM 1. *Suppose* (A) *and* (B) *hold. Let* $u_0 \in D(\phi^0)$ *and* u *be the solution to* (CP; u_0), *and suppose* u *is bounded on* $[0, \infty)$. *Then we have*:

(i) $\lim_{t \to \infty} \phi^t(u(t)) = \min \phi^\infty$.

(ii) $\Omega_w(u_0) \neq \emptyset$ and $\Omega_w(u_0) \subset F(\phi^\infty)$.

COROLLARY 1. *In addition to all the assumptions of Theorem* 1, *suppose that* $F(\phi^\infty)$ *is singleton, say* $F(\phi^\infty) = \{u_\infty\}$. *Then* $u(t) \to u_\infty$ *weakly in* H *as* $t \to \infty$.

COROLLARY 2. *In addition to all the assumptions of Theorem* 1, *suppose the following condition* (C) *holds*:

(C) *There is a family* $\{S_r\} = \{S_r; 0 \leqslant r < \infty\}$ *of compact subsets of* H *such that* $\{z \in H; |z|_H \leqslant r, |\phi^t(z)| \leqslant r\} \subset S_r$ *for all* $t \geqslant 0$ *and* $r \geqslant 0$.

Then we have $\Omega_s(u_0) \neq \emptyset$ *and* $\Omega_s(u_0) \subset F(\phi^\infty)$.

The above two corollaries follow immediately from Theorem 1.

Next, in order to show that the solution $u(t)$ of (CP; u_0) converges weakly in H as $t \to \infty$, we propose the following condition (B)′, which is stronger than (B).

(B)′ ϕ^t converges to ϕ^∞ on H as $t \to \infty$ in the sense that (b1) of (B) and (b2)′ below hold:

(b2)′ For each $z \in D(\phi^\infty)$ there exist w: $[0, \infty) \to H$, α: $[0, \infty) \to \mathbb{R}$ and β: $[0, \infty) \to \mathbb{R}$ such that $\alpha \in L^2(0, \infty)$, $\beta \in L^1(0, \infty)$,

$$\alpha(t) \to 0, \qquad \beta(t) \to 0, \qquad \text{as} \quad t \to \infty$$

and

$$|w(t) - z|_H \leqslant \alpha(t), \qquad \phi^t(x(t)) - \phi^\infty(z) \leqslant \beta(t) \qquad \text{for all } t \geqslant 0.$$

The second main result is then stated as follows.

THEOREM 2. *Suppose* (A) *and* (B)′ *hold, and suppose that for each* $r \geqslant 0$ *there is a non-negative function* η_r *in* $L^1(0, \infty)$ *such that*

$$\phi^t(z) + \eta_r(t)(1 + |\phi^t(z)|) \geqslant \inf\{\phi^\infty(h); h \in H\}$$
$$\textit{for all } z \in H \textit{ with } |z|_H \leqslant r \textit{ and a.e. } t \geqslant 0. \quad (1.1)$$

Then the following statements are equivalent:

(a) *For every* $u_0 \in D(\phi^0)$ *the solution* $u(t)$ of (CP; u_0) *converges weakly in* H *as* $t \to \infty$.

(b) $F(\phi^\infty) \neq \varnothing$.

(c) *For every* $u_0 \in D(\phi^0)$ *the solution* u *of* (CP; u_0) *is bounded on* $[0, \infty)$.

In Theorem 2 the assertion (c) → (b) is a consequence of Theorem 1 and (a) → (c) is trivial. Therefore we shall give the proofs of assertions (b) → (c) and (c) → (a).

Remark 1. Let u and v be respectively the solutions to (CP; u_0) and (CP; v_0) with u_0, v_0 in $D(\phi^0)$. Then we have (cf. [11, Chap. 1, Sect. 1.2])

$$|u(t) - v(t)|_H \leqslant |u(s) - v(s)|_H \qquad \text{for all } s, t \in [0, \infty) \text{ with } s \leqslant t.$$

Taking this fact into account, we can replace (c) of Theorem 2 by the following (c)′:

(c)′ There is at least one element u_0 in $D(\phi^0)$ such that the solution u to (CP; u_0) is bounded on $[0, \infty)$.

2. Proof of Theorem 1

Throughout this section we fix u_0 in $D(\phi^0)$ and let u be the solution to (CP; u_0).

PROPOSITION 1. *Suppose* (A) *holds with the following condition* (D):

(D) *There is a constant* $c_0 \geqslant 0$ *such that*

$$\phi^t(z) + c_0(|z|_H + 1) \geqslant 0 \qquad \text{for all } z \in H \text{ and } t \geqslant 0.$$

Further suppose u is bounded on $[0, \infty)$. *Then we have*:

(i) $\lim_{t \to \infty} \phi^t(u(t))$ *exists and is finite.*

(ii) $u' \in L^2(0, \infty; H)$.

We now recall (see [11, Chap. 1]) the following type of energy inequality for the solution u, which plays an important role in our argument and holds under (A) and (D): if $|u(t)|_H < r$ for $0 \leqslant t \leqslant T$ ($< \infty$), then

$$\phi^t(u(t)) - \phi^s(u(s)) + \frac{1}{2}\int_s^t |u'(\tau)|_H^2 \, d\tau$$
$$\leqslant \int_s^t \{k_{r,1}(\tau)\, \phi^\tau(u(\tau)) + k_{r,2}(\tau)\} \, d\tau \tag{2.1}$$

for every $0 \leqslant s \leqslant t \leqslant T$, where

$$k_{r,1}(\tau) = 4\,|a_r'(\tau)|^2 + |b_r'(\tau)| \qquad \text{and} \qquad k_{r,2}(\tau) = k_{r,1}(\tau)\{1 + 4c_0(r+1)\}$$

with functions a_r, b_r in (A); note that $k_{r,i} \in L^1(0, \infty)$, $i = 1, 2$.

Proof of Proposition 1. Suppose $|u(t)|_H < r$ for all $t \geqslant 0$. Then it follows from (2.1) that

$$\phi^t(u(t)) \leqslant \phi^s(u(s)) + \int_s^t \{k_{r,1}(\tau)\, \phi^\tau(u(\tau)) + k_{r,2}(\tau)\} \, d\tau$$

for every $0 \leqslant s \leqslant t < \infty$. Therefore, by Gronwall's inequality,

$$\phi^t(u(t)) \leqslant \phi^s(u(s)) \exp\left(\int_s^t k_{r,1}(\tau)\, d\tau\right)$$
$$+ \int_s^t k_{r,2}(\tau) \exp\left(\int_\tau^t k_{r,1}(\sigma)\, d\sigma\right) d\tau \tag{2.2}$$

for every $0 \leqslant s \leqslant t < \infty$. Taking $s = 0$ in (2.2), we obtain

$$\phi^t(u(t)) \leqslant (|\phi^0(u_0)| + |k_{r,2}|_{L^1(0,\infty)}) \exp |k_{r,1}|_{L^1(0,\infty)} \equiv M < \infty$$

for every $t \geqslant 0$. From this inequality with (D) we infer that

$$|\phi^t(u(t))| \leqslant \phi^t(u(t)) + 2c_0(r+1) \leqslant M + 2c_0(r+1) \equiv M_1$$

for every $t \geqslant 0$. Thus $t \to \phi^t(u(t))$ is bounded on $[0, \infty)$. Next, from (2.1) again it follows that

$$\frac{1}{2}\int_0^t |u'(\tau)|_H^2 \, d\tau \leqslant 2M_1 + M_1 |k_{r,1}|_{L^1(0,\infty)} + |k_{r,2}|_{L^1(0,\infty)}$$

for every $t \geqslant 0$, which implies $u' \in L^2(0, \infty; H)$. Putting for each $t \geqslant 0$

$$J(t) = \phi^t(u(t)) + \frac{1}{2}\int_0^t |u'(\tau)|_H^2 \, d\tau - \int_0^t \{k_{r,1}(\tau)\, \phi^\tau(u(\tau)) + k_{r,2}(\tau)\} \, d\tau,$$

we see from (2.1) that $J(t)$ is non-increasing in t, so that $\lim_{t \to \infty} J(t)$ exists and is finite. Therefore, so is $\lim_{t \to \infty} \phi^t(u(t))$. Q.E.D.

LEMMA 1. *Conditions* (A) *and* (B) *imply* (D).

Proof. As is already known (cf. [11, Chap. 1, Sect. 1.5]), under (A) there is a constant c_T for each $T > 0$ such that

$$\phi^t(z) + c_T(|z|_H + 1) \geqslant 0 \qquad \text{for all } z \in H \text{ and } 0 \leqslant t \leqslant T.$$

Also, according to a result of Fujiyama and Watanabe [9], under (B) there are constants $T \geqslant 0$ and $c'_T \geqslant 0$ satisfying

$$\phi^t(z) + c'_T(|z|_H + 1) \geqslant 0 \qquad \text{for all } z \in H \text{ and } t \geqslant T.$$

From these inequalities condition (D) is derived. Q.E.D.

Proof of Theorem 1. We note that Proposition 1 is valid under our assumptions by Lemma 1; hence

$$m_0 \equiv \lim_{t \to \infty} \phi^t(u(t))$$

exists and is finite. We have $\Omega_w(u_0) \neq \emptyset$ by the boundedness of u on $[0, \infty)$. Now, let z_0 be any point of $\Omega_w(u_0)$ and let $\{t_n\}$ be the sequence such that $t_n \uparrow \infty$ and $u(t_n) \to z_0$ weakly in H. Then, by Proposition 1 and (b1) of (B), we have

$$\phi^\infty(z_0) \leqslant \liminf_{n \to \infty} \phi^{t_n}(u(t_n)) = m_0 < \infty, \tag{2.3}$$

and hence $z_0 \in D(\phi^\infty)$. Next, choose a sequence $\{s_n\}$ with $s_n \uparrow \infty$ such that $-u'(s_n) \in \partial\phi^{s_n}(u(s_n))$ and $u'(s_n) \to 0$ strongly in H. Let z be any point of $D(\phi^\infty)$, and by (b2) of (B) take a sequence $\{z_n\} \subset H$ such that $z_n \to z$ strongly in H and $\phi^{s_n}(z_n) \to \phi^\infty(z)$. Then, by the definition of subdifferential, we see that

$$(-u'(s_n), z_n - u(s_n))_H \leqslant \phi^{s_n}(z_n) - \phi^{s_n}(u(s_n)).$$

Passing to the limit in n yields

$$m_0 = \lim_{n \to \infty} \phi^{s_n}(u(s_n)) \leqslant \lim_{n \to \infty} \phi^{s_n}(z_n) = \phi^\infty(z). \tag{2.4}$$

In particular, if $z = z_0$, then (2.4) implies $m_0 \leqslant \phi^\infty(z_0)$. From this with (2.3) and (2.4) we infer that $\phi^\infty(z_0) = \min \phi^\infty$. Thus the theorem is proved. Q.E.D.

3. Proof of Theorem 2

We give only the proofs of (c) → (a) and (b) → (c).

Proof of (c) → (a). Let u_0 be any element of $D(\phi^0)$ and suppose the solution u to (CP; u_0) is bounded on $[0, \infty)$. Then, according to Proposition 1 and Theorem 1, we have $u' \in L^2(0, \infty; H)$, $\Omega_w(u_0) \neq \emptyset$, $\Omega_w(u_0) \subset F(\phi^\infty)$ and $\lim_{t \to \infty} \phi^t(u(t)) = \min \phi^\infty$. In order to show the weak convergence of $u(t)$ in H as $t \to \infty$ it suffices to prove that $\Omega_w(u_0)$ is a singleton. Now, let u_∞ and $\tilde{u}_\infty$ be in $\Omega_w(u_0)$. Correspondingly to the element u_∞, by using (b2)′ and (B)′ we find w: $[0, \infty) \to H$, $\alpha \in L^2(0, \infty)$, and $\beta \in L^1(0, \infty)$ such that

$$|w(t) - u_\infty|_H \leqslant \alpha(t), \qquad \phi^t(w(t)) - \min \phi^\infty \leqslant \beta(t) \qquad \text{for} \quad t \geqslant 0. \tag{3.1}$$

We then observe that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u(t) - u_\infty|_H^2 \\
&\quad = (-u'(t), u_\infty - u(t))_H \\
&\quad = (-u'(t), w(t) - u(t))_H + (-u'(t), u_\infty - w(t))_H \\
&\quad \leqslant \phi^t(w(t)) - \phi^t(u(t)) + |u'(t)|_H\, |w(t) - u_\infty|_H \\
&\quad = \phi^t(w(t)) - \min \phi^\infty + \min \phi^\infty - \phi^t(u(t)) + |u'(t)|_H\, |w(t) - u_\infty|_H
\end{aligned}$$

for a.e. $t \geqslant 0$. Here, using (3.1) and (1.1) with $r > \sup\{|u(t)|_H; 0 \leqslant t < \infty\}$, we obtain that

$$\frac{1}{2}\frac{d}{dt}|u(t)-u_\infty|_H^2 \leqslant \beta(t)+\eta_r(t)(1+|\phi^t(u(t))|)+|u'(t)|_H\,\alpha(t) \equiv \rho(t) \quad (3.2)$$

for a.e. $t \geqslant 0$. Since $\rho \in L^1(0, \infty)$, it follows that

$$|u(t)-u_\infty|_H^2 \leqslant |u(s)-u_\infty|_H^2 + 2\int_s^t \rho(\tau)\,d\tau \qquad \text{for every } 0 \leqslant s \leqslant t < \infty,$$

and hence $l \equiv \lim_{t\to\infty} |u(t)-u_\infty|_H^2$ exists. Similarly $\tilde{l} \equiv \lim_{t\to\infty} |u(t)-\tilde{u}_\infty|_H^2$ exists. Therefore,

$$l-\tilde{l} = \lim_{t\to\infty} \{-2(u(t), u_\infty - \tilde{u}_\infty)_H + |u_\infty|_H^2 - |\tilde{u}_\infty|_H^2\}.$$

Since $u(t_n) \to u_\infty$ weakly in H for some sequence $\{t_n\}$ with $t_n \uparrow \infty$, we obtain that $l-\tilde{l} = -|u_\infty - \tilde{u}_\infty|_H^2$. Similarly $\tilde{l}-l = -|u_\infty - \tilde{u}_\infty|_H^2$. This implies $|u_\infty - \tilde{u}_\infty|_H = 0$, i.e., $u_\infty = \tilde{u}_\infty$. Thus $\Omega_w(u_0)$ is a singleton. Q.E.D.

Proof of (b) → (c). By assumption, $F(\phi^\infty) \neq \emptyset$. Let $z_\infty \in F(\phi^\infty)$, and by using (b2)′ of (B)′ take functions $w: [0, \infty) \to H$, $\alpha \in L^2(0, \infty)$ and $\beta \in L^1(0, \infty)$ such that

$$\alpha(t) \to 0, \qquad \beta(t) \to 0 \qquad \text{as} \quad t \to \infty$$

and

$$|w(t)-z_\infty|_H \leqslant \alpha(t), \qquad \phi^t(w(t)) - \min \phi^\infty \leqslant \beta(t) \qquad \text{for} \quad t \geqslant 0.$$

We fix a sequence $\{t_n\}$ with $t_n \uparrow \infty$, and denote by u_n for each n the solution to

$$(\mathrm{CP})_n \qquad -u_n'(t) \in \partial\phi^t(u_n(t)), \qquad t_n < t < \infty, \qquad u_n(t_n) = z_n,$$

where $z_n = w(t_n)$. We take a positive number L such that $|z_n|_H \leqslant L$ and $|\phi^{t_n}(z_n)| \leqslant L$ for all n. Also, let r_0 be a fixed positive number with $r_0 > L + |z_\infty|_H$ and define for each n

$$T_n = \sup\{T; t_n \leqslant T < \infty, |u_n(t) - z_\infty|_H \leqslant r_0 \text{ for } t_n \leqslant t \leqslant T\};$$

note that $T_n > t_n$. We want to show that $T_n = \infty$ for a certain n, which implies the boundedness of u_n on $[t_n, \infty)$. Now, for the contrary suppose that $T_n < \infty$ for all n. Then, putting

$$r = |z_\infty|_H + r_0 + 1$$

and noting that $|u_n(t)|_H \leqslant |z_\infty|_H + r_0 < r$ for all $t \in J_n \equiv [t_n, T_n]$, we obtain by (2.1)

$$\phi^t(u_n(t)) - \phi^s(u_n(s)) + \frac{1}{2}\int_s^t |u_n'(\tau)|_H^2 \, d\tau$$

$$\leqslant \int_s^t \{k_{r,1}(\tau)\, \phi^\tau(u_n(\tau)) + k_{r,2}(\tau)\} \, d\tau \tag{3.3}$$

for every s, $t \in J_n$ with $s \leqslant t$.

Now, we show that there is a constant $L_1 > 0$ such that

$$|\phi^t(u_n(t))| \leqslant L_1 \qquad \text{for all } t \in J_n \text{ and } n = 1, 2, ..., \tag{3.4}$$

and

$$|u_n'|_{L^2(J_n;H)} \leqslant L_1 \qquad \text{for} \quad n = 1, 2, ... \tag{3.5}$$

Indeed, just as in the proof of Proposition 1, it follows from (3.3) that

$$\phi^t(u_n(t) \leqslant (|\phi^{t_n}(z_n)| + |k_{r,2}|_{L^1(0,\infty)}) \exp |k_{r,1}|_{L^1(0,\infty)}$$
$$\leqslant (L + |k_{r,2}|_{L^1(0,\infty)}) \exp |k_{r,1}|_{L^1(0,\infty)} \equiv M$$

for all $t \in J_n$ and $n = 1, 2, ...$. Hence, the inequality in condition (D) yields

$$|\phi^t(u_n(t))| \leqslant M + 2c_0(r+1) \equiv M_1 \qquad \text{for all } t \in J_n \text{ and } n = 1, 2, ...,$$

and subsequently

$$|u_n'|^2_{L^2(J_n;H)} \leqslant 4M_1 + 2M_1\, |k_{r,1}|_{L^1(0,\infty)} + 2\, |k_{r,2}|_{L^1(0,\infty)} \equiv M_2$$

for all $t \in J_n$ and $n = 1, 2, ...$. Accordingly, if we put

$$L_1 = M_1 + M_2^{1/2},$$

then (3.4) and (3.5) hold.

Next, we show that there is a sequence $\{A_n\}$ of non-negative numbers such that $A_n \to 0$ and

$$|u_n(T_n) - z_\infty|_H^2 \leqslant A_n \qquad \text{for} \quad n = 1, 2, ... \tag{3.6}$$

In fact, just as (3.2) in the proof of (b) $\to$ (c), it holds that

$$\frac{1}{2}\frac{d}{dt}\, |u_n(t) - z_\infty|_H^2 \leqslant \beta(t) + \eta_r(t)(1 + |\phi^t(u_n(t))|) + |u_n'(t)|_H\, \alpha(t)$$

for a.e. $t \in J_n$ and $n = 1, 2, \ldots$, where η_r is as in (1.1). Integrating this inequality over J_n and using (3.4) and (3.5), we see that

$$|u_n(T_n) - z_\infty|_H^2 \leqslant \alpha(t_n)^2 + 2\int_{t_n}^{\infty} \{\beta(\tau) + (1 + L_1)\,\eta_r(\tau)\}\,d\tau$$
$$+ 2L_1 \left(\int_{t_n}^{\infty} \alpha(\tau)^2\,d\tau\right)^{1/2}$$

for all n. Therefore we can take as A_n the right-hand side of the above inequality.

From (3.6) it follows that

$$|u_n(T_n) - z_\infty|_H < r_0 \qquad \text{for large } n,$$

which contradicts the definition of T_n. Consequently, it holds that $T_n = \infty$ for a certain n, and hence u_n is bounded on $[t_n, \infty)$ for such an integer n. Combining this with the facts in Remark 1, we see immediately that any solution to (CP; u_0) with u_0 in $D(\phi^0)$ is bounded on $[0, \infty)$. Thus the proof of (b) $\to$ (c) is complete. Q.E.D.

4. Applications

In this section we give two applications. For simplicity we set

$$H = L^2(0, 1) \qquad \text{and} \qquad X = W^{1,2}(0, 1)(\subset C([0, 1])).$$

Application 1. We first consider the problem (V) mentioned in the Introduction. Let $g = g(x, t)$, $f = f(x, t)$ be functions on $[0, 1] \times [0, \infty)$ such that $g(\cdot, t) \in X$ and $f(\cdot, t) \in H$ for every $t \geqslant 0$, and suppose the following conditions (i)–(iii) are satisfied:

(i) $c_1 \leqslant g \leqslant c_2$ on $[0, 1] \times [0, \infty)$ for positive constant c_1, c_2.

(ii) There are functions $g_0 \in W^{1,1}(0, \infty) \cap W^{1,2}(0, \infty)$ and $g_1 \in W^{1,1}(0, \infty)$ such that

$$|g(x, t) - g(x, s)| \leqslant |g_0(t) - g_0(s)| \qquad \text{for all } x \in [0, 1] \text{ and } s, t \in [0, \infty).$$

and

$$|g_x(\cdot, t) - g_x(\cdot, s)|_H \leqslant |g_1(t) - g_1(s)| \qquad \text{for all } s, t \in [0, \infty).$$

(iii) There is a function $f_0 \in W^{1,1}(0, \infty)$ such that

$$|f(\cdot, t) - f(\cdot, s)|_H \leqslant |f_0(t) - f_0(s)| \qquad \text{for all } s, t \in [0, \infty).$$

Under these conditions we easily see that

$$g(\cdot, t) \to g_\infty \qquad \text{strongly in } X \text{ as } t \to \infty$$

and

$$f(\cdot, t) \to f_\infty \qquad \text{strongly in } H \text{ as } t \to \infty$$

for some $g_\infty \in X$ and $f_\infty \in H$. We set

$$K(t) = \{z \in X; 0 \leqslant z \leqslant g(\cdot, t) \text{ on } [0, 1]\}, \qquad 0 \leqslant t < \infty,$$

and

$$K_\infty = \{z \in X; 0 \leqslant z \leqslant g_\infty \text{ on } [0, 1]\}.$$

Also, for each $t \geqslant 0$ we define ϕ^t: $H \to (-\infty, \infty]$ by

$$\begin{aligned} \phi^t(z) &= \tfrac{1}{2} |z_x|_H^2 - (f(\cdot, t), z)_H \qquad &&\text{if} \quad z \in K(t), \\ &= \infty &&\text{otherwise,} \end{aligned}$$

and ϕ^∞: $H \to (-\infty, \infty]$ by

$$\begin{aligned} \phi^\infty(z) &= \tfrac{1}{2} |z_x|_H^2 - (f_\infty, z)_H \qquad &&\text{if} \quad z \in K_\infty, \\ &= \infty &&\text{otherwise.} \end{aligned}$$

Evidently ϕ^t, $0 \leqslant t \leqslant \infty$, is proper l.s.c. and convex on H, and $D(\phi^t) = K(t)$ $(0 \leqslant t < \infty)$ and $D(\phi^\infty) = K_\infty$.

LEMMA 2. *Let s, $t \in [0, \infty)$ and $z \in K(s)$. Then the function*

$$\tilde{z}(x) = \frac{g(x, t)}{g(x, s)} z(x)$$

belongs to $K(t)$ and satisfies that

$$|\tilde{z} - z|_H \leqslant |g_0(t) - g_0(s)| \tag{4.1}$$

and

$$\begin{aligned} |\phi^t(\tilde{z}) - \phi^s(z)| \leqslant C\{&|g_0(t) - g_0(s)| + |g_1(t) - g_1(s)| \\ &+ |f_0(t) - f_0(s)|\}(1 + |\phi^s(z)|), \end{aligned} \tag{4.2}$$

where C is a positive constant independent of s, t and z. Hence $\{\phi^t\}$ satisfies condition (A).

Proof. We easily see $\tilde{z} \in K(t)$ and (4.1). Also, we observe that

$$
\begin{aligned}
&|\tilde{z}_x(x)|^2 - |z_x(x)|^2 \\
&= \left| z_x(x) \frac{g(x,t)}{g(x,s)} + \frac{z(x)}{g(x,s)} \left(g_x(x,t) - g_x(x,s) \frac{g(x,t)}{g(x,s)} \right) \right|^2 - |z_x(x)|^2 \\
&= \left(\left| \frac{g(x,t)}{g(x,s)} \right|^2 - 1 \right) |z_x(x)|^2 + \left| \frac{z(x)}{g(x,s)} \right|^2 \\
&\quad \times \left| g_x(x,t) - g_x(x,s) \frac{g(x,t)}{g(x,s)} \right|^2 \\
&\quad + \frac{2z(x)\, z_x(x)\, g(x,t)}{g(x,s)^2} \left(g_x(x,t) - g_x(x,s) \frac{g(x,t)}{g(x,s)} \right)
\end{aligned}
$$

for a.e. $x \in [0, 1]$, so that by our conditions

$$
\begin{aligned}
||\tilde{z}_x(x)|^2 - |z_x(x)|^2| \leqslant M_1 \{ & |g_0(t) - g_0(s)|\, |z_x(x)|^2 + |g_x(x,t) - g_x(x,s)|^2 \\
& + |g_x(x,s)|^2 |g_0(t) - g_0(s)|^2 \\
& + |g_x(x,t) - g_x(x,s)|\, |z_x(x)| \\
& + |g_0(t) - g_0(s)|\, |g_x(x,s)|\, |z_x(x)| \}
\end{aligned}
$$

for a.e. $x \in [0, 1]$, where M_1 is a positive constant independent of s, t and z. By integrating this over $[0, 1]$ we get an inequality of the form

$$
||\tilde{z}_x|_H^2 - |z_x|_H^2| \leqslant M_2(|g_0(t) - g_0(s)| + |g_1(t) - g_1(s)|)(1 + |z_x|_H^2), \tag{4.3}
$$

where M_2 is a positive constant independent of s, t and z. Besides, we have

$$
\begin{aligned}
&|(f(\cdot,t), \tilde{z})_H - (f(\cdot,s), z)_H| \\
&\quad \leqslant |f(\cdot,t)|_H\, |\tilde{z} - z|_H + |f(\cdot,t) - f(\cdot,s)|_H\, |z|_H \\
&\quad \leqslant M_3(|g_0(t) - g_0(s)| + |f_0(t) - f_0(s)|),
\end{aligned} \tag{4.4}
$$

where a constant $M_3 \geqslant 0$ is also independent of s, t and z. From (4.3) and (4.4) we easily infer (4.2) with a suitable constant $C \geqslant 0$ independent of s, t and z. If we take $a_r(t) = g_0(t)$ and

$$
b_r(t) = C \int_0^t \{ |g_0'(\tau)| + |g_1'(\tau)| + |f_0'(\tau)| \}\, d\tau,
$$

then condition (A) holds.

LEMMA 3. *$\{\phi^t\}$ satisfies condition* (B)′.

Proof. It is easy to verify (b1) of (B). Let z be any point in $D(\phi^\infty)$, and set

$$w(x, t) = \frac{g(x, t)}{g_\infty(x)} z(x) \qquad \text{for } (x, t) \in [0, 1] \times [0, \infty).$$

Then $w(\cdot, t) \in K(t)$ and $w(x, t) = g(x, t)\, w(x, s)/g(x, s)$ for any s, $t \geq 0$. Hence by Lemma 2 we have

$$|w(\cdot, t) - w(\cdot, s)|_H \leqslant |g_0(t) - g_0(s)|$$

and

$$|\phi^t(w(\cdot, t)) - \phi^s(w(\cdot, s))| \leqslant C\{|g_0(t) - g_0(s)| + |g_1(t) - g_1(s)| + |f_0(t) - f_0(s)|\}(1 + |\phi^s(w(\cdot, s))|).$$

Now, letting $s \to \infty$ yields that

$$|w(\cdot, t) - z|_H \leqslant |g_0(t)| \equiv \alpha(t)$$

and

$$|\phi^t(w(\cdot, t)) - \phi^\infty(z)| \leqslant C\{|g_0(t)| + |g_1(t)| + |f_0(t)|\}(1 + |\phi^\infty(z)|) \equiv \beta(t)$$

for all $t \geqslant 0$. Clearly w, α and β satisfy the desired properties in (b2)′.

Q.E.D.

LEMMA 4. $\phi^t(z) + C\{|g_0(t)| + |g_1(t)| + |f_0(t)|\}(1 + |\phi^t(z)|) \geqslant \min \phi^\infty$ *for all* $t \geqslant 0$ *and* $z \in H$.

Proof. It is easy to see that ϕ^∞ has $\min \phi^\infty$. Now, let $z \in K(t)$ and put

$$w(x, s) = \frac{g(x, s)}{g(x, t)} z(x), \qquad 0 \leqslant x \leqslant 1, \quad s \geqslant 0.$$

Clearly $w(\cdot, s) \to z_\infty$ strongly in X as $s \to \infty$ for some $z_\infty \in K_\infty$. From (4.2) or Lemma 2 it follows that

$$|\phi^s(w(\cdot, s)) - \phi^t(z)| \leqslant C\{|g_0(s) - g_0(t)| + |g_1(s) - g_1(t)| + |f_0(s) - f_0(t)|\}(1 + |\phi^t(z)|),$$

so that letting $s \to \infty$ gives that

$$|\phi^\infty(z_\infty) - \phi^t(z)| \leqslant C\{|g_0(t)| + |g_1(t)| + |f_0(t)|\}(1 + |\phi^t(z)|).$$

Thus

$$\phi^t(z) + C\{|g_0(t) + |g_1(t)| + |f_0(t)|\}(1 + |\phi^t(z)|)$$
$$\geqslant \phi^\infty(z_\infty) \geqslant \min \phi^\infty. \qquad \text{Q.E.D.}$$

PROPOSITION 2. *Let* u_0 *be in* $K(0)$ $(=D(\phi^0))$ *and* $u = u(x, t)$ *be the solution of problem* (V). *Then* $u(\cdot, t)$ *converges strongly in* X *as* $t \to \infty$ *and the limit* u_∞ *is a solution of the variational inequality*

$$(\mathrm{V})_\infty \qquad u_\infty \in K_\infty, \quad (u_{\infty,x}, u_{\infty,x} - z_x)_H \leqslant (f_\infty, u_\infty - z)_H \quad \textit{for all } z \in K_\infty.$$

Proof. As is seen from the definition of subdifferential $\partial\phi^t$, problem (V) is equivalent to Cauchy problem (CP; u_0). In view of Lemmas 2, 3 and 4 all the assumptions of Theorem 2 are satisfied. It is easy to see that $F(\phi^\infty) \neq \varnothing$. Hence the applications of Theorems 1 and 2 give

$$u(\cdot, t) \to u_\infty \qquad \text{weakly in } H \text{ as } t \to \infty \tag{4.5}$$

for some $u_\infty \in F(\phi^\infty)$ and

$$\phi^t(u(\cdot, t)) \to \phi^\infty(u_\infty(u_\infty)) = \min \phi^\infty \qquad \text{as} \quad t \to \infty. \tag{4.6}$$

From (4.5) and (4.6) it immediately follows that $u(\cdot, t) \to u_\infty$ weakly in X and $|u(\cdot, t)|_X \to |u_\infty|_X$ as $t \to \infty$, which implies $u(\cdot, t) \to u_\infty$ strongly in X as $t \to \infty$. Since the relation $0 \in \partial\phi^\infty(u_\infty)$ is equivalent to $(\mathrm{V})_\infty$, the proof of the proposition is thus complete. Q.E.D.

Application 2. As another application we consider the heat equation with temperature controls through the boundary of the following type:

$$u_t - u_{xx} = f \qquad \text{on } (0, 1) \times (0, \infty), \tag{4.7}$$

$$u(x, 0) = u_0(x) \qquad \text{for } 0 \leqslant x \leqslant 1, \tag{4.8}$$

$$\begin{cases} g_0(t) \leqslant u(0, t) \leqslant h_0(t) & \text{for } 0 \leqslant t < \infty, \\ u_x(0+, t) \leqslant \gamma_0(g_0(t)) & \text{if } u(0, t) = g_0(t), \\ u_x(0+, t) = \gamma_0(u(0, t)) & \text{if } g_0(t) < u(0, t) < h_0(t), \\ u_x(0+, t) \geqslant \gamma_0(h_0(t)) & \text{if } u(0, t) = h_0(t), \end{cases} \tag{4.9}$$

$$\begin{cases} g_1(t) \leqslant u(0, t) \leqslant h_1(t) & \text{for } 0 \leqslant t < \infty, \\ -u_x(1-, t) \leqslant \gamma_1(g_1(t)) & \text{if } u(1, t) = g_1(t), \\ -u_x(1-, t) = \gamma_1(u(1, t)) & \text{if } g_1(t) < u(1, t) < h_1(t), \\ -u_x(1-, t) \geqslant \gamma_1(h_1(t)) & \text{if } u(1, t) = h_1(t), \end{cases} \tag{4.10}$$

where u_0 is a function on $[0, 1]$, f on $[0, 1] \times [0, \infty)$ and g_i, h_i $(i=0, 1)$ are absolutely continuous functions on $[0, \infty)$; γ_i $(i=0, 1)$ are functions on $\mathbb{R}$. We assume the following conditions (i)–(iv) are fulfilled:

(i) $f(\cdot, t) \in H$ for any $t \geqslant 0$ and there is $f_0 \in W^{1,1}(0, \infty)$ such that

$$|f(\cdot, t) - f(\cdot, s)|_H \leqslant |f_0(t) - f_0(s)| \qquad \text{for all } s, t \in [0, \infty).$$

(ii) g_i', $h_i' \in L^1(0, \infty) \cap L^2(0, \infty)$, $g_i - g_{i,\infty} \in L^1(0, \infty) \cap L^2(0, \infty)$ with $g_{i,\infty} = \lim_{t \to \infty} g_i(t)$, and $h_i - h_{i,\infty} \in L^1(0, \infty) \cap L^2(0, \infty)$ with $h_{i,\infty} = \lim_{t \to \infty} h_i(t)$, $i=0, 1$.

(iii) $g_i < h_i$ on $[0, \infty)$ and $g_{i,\infty} < h_{i,\infty}$, $i=0, 1$.

(iv) γ_i is continuous and non-decreasing on $\mathbb{R}$, $i=0, 1$.

In order to reformulate the problem (4.7)–(4.10) as a Cauchy problem of the form (CP; u_0) in H, we use proper l.s.c. convex functions ϕ^t on H for $0 \leqslant t < \infty$ which are given by

$$\phi^t(z) = \tfrac{1}{2} |z_x|_H^2 - (f(\cdot, t), z)_H + \Gamma_0(z(0)) + \Gamma_1(z(1)) \qquad \text{if } \; z \in K(t),$$
$$= \infty \qquad \text{otherwise},$$

where Γ_i is the primitive of γ_i satisfying $\Gamma_i(0) = 0$ $(i=0, 1)$ and

$$K(t) = \{z \in X;\, g_0(t) \leqslant z(0) \leqslant h_0(t),\, g_1(t) \leqslant z(1) \leqslant h_1(t)\}.$$

Also, we define a proper l.s.c. convex function ϕ^∞ on H by

$$\phi^\infty(z) = \tfrac{1}{2} |z_x|_H^2 - (f_\infty, z)_H + \Gamma_0(z(0)) + \Gamma_1(z(1)) \qquad \text{if } \; z \in K_\infty,$$
$$= \infty \qquad \text{otherwise},$$

where f_∞ is the strong limit of $f(\cdot, t)$ in H as $t \to \infty$ and

$$K_\infty = \{z \in X;\, g_{0,\infty} \leqslant z(0) \leqslant h_{0,\infty},\, g_{1,\infty} \leqslant z(1) \leqslant h_{1,\infty}\}.$$

LEMMA 5. *Let z^*, $z \in H$. Then the relation $z^* \in \partial\phi^t(z)$ is equivalent to the following system:*

$$z^* = -z_{xx} - f(\cdot, t) \qquad \textit{on} \quad (0, 1) \quad (\textit{in the distribution sense}), \tag{4.11}$$

$$\begin{cases} g_0(t) \leqslant z(0) \leqslant h_0(t), & \\ z_x(0+) \leqslant \gamma_0(g_0(t)) & \textit{if} \quad z(0) = g_0(t), \\ z_x(0+) = \gamma_0(z(0)) & \textit{if} \quad g_0(t) < z(0) < h_0(t), \\ z_x(0+) \geqslant \gamma_0(h_0(t)) & \textit{if} \quad z(0) = h_0(t), \end{cases} \tag{4.12}$$

$$\begin{cases} g_1(t) \leqslant z(1) \leqslant h_1(t), \\ -z_x(1-) \leqslant \gamma_1(g_1(t)) & \text{if} \quad z(1) = g_1(t), \\ -z_x(1-) = \gamma_1(z(1)) & \text{if} \quad g_1(t) < z(1) < h_1(t), \\ -z_x(1-) \geqslant \gamma_1(h_1(t) & \text{if} \quad z(1) = h_1(t). \end{cases} \tag{4.13}$$

Proof. First assume $z^* \in \partial\phi^t(z)$. Then by the definition of $\partial\phi^t$ we have $g_i(t) \leqslant z(i) \leqslant h_i(t)$ $(i = 0, 1)$ and

$$(z^*, y - z)_H \leqslant \phi^t(y) - \phi^t(z) \qquad \text{for all } y \in K(t).$$

Now, taking as y the function $r\rho + (1 - r)z$ with $\rho \in K(t)$ and $0 < r < 1$, and letting $r \downarrow 0$, we obtain

$$\begin{aligned}(z^*, \rho - z)_H \leqslant {} & (z_x, \rho_x - z_x)_H - (f(\cdot, t), \rho - z)_H \\ & + \gamma_0(z(0))(\rho(0) - z(0)) + \gamma_1(z(1))(\rho(1) - z(1)).\end{aligned} \tag{4.14}$$

If ρ has the form $\pm\eta + z$ with $\eta \in \mathscr{D}(0, 1)$, then (4.14) implies

$$\pm(z^*, \eta)_H \leqslant \pm(z_x, \eta_x)_H \mp (f(\cdot, t), \eta)_H.$$

Therefore $(z^*, \eta)_H = (z_x, \eta_x)_H - (f(\cdot, t), \eta)_H$, which shows (4.11). By using integration by parts in (4.14) and noting (4.11) we get

$$\begin{aligned}0 \leqslant {} & \{\gamma_0(z(0)) - z_x(0+)\}(\rho(0) - z(0)) \\ & + \{\gamma_1(z(1-) + z_x(1-)\}(\rho(1) - z(1)).\end{aligned} \tag{4.15}$$

In the case where $z(0) = g_0(t)$, choose as ρ the function $\varepsilon(1 - x) + z(x)$ for a small $\varepsilon > 0$. Then it follows from (4.15) that $z_x(0+) \leqslant \gamma_0(z(0)) = \gamma_0(g_0(t))$. Similarly, if $z(0) = h_0(t)$, then $z_x(0+) \geqslant \gamma_0(z(0)) = \gamma_0(t))$. Also, in the case where $g_0(t) < z(0) < h_0(t)$, choose as ρ the function $\pm\varepsilon(1 - x) + z(x)$ for a small $\varepsilon > 0$. Then from (4.15) we derive $z_x(0+) = \gamma_0(z(0))$. Thus (4.12) holds, and (4.13) can be similarly shown.

Conversely, assume (4.11)–(4.13) hold. Then for any $y \in K(t)$

$$\begin{aligned}(z^*, y - z)_H &= -(z_{xx}, y - z)_H - (f(\cdot, t), y - z)_H \\ &= (z_x, y_x - z_x)_H - (f(\cdot, t), y - z)_H \\ &\quad + z_x(0+)(y(0) - z(0)) - z_x(1-)(y(1) - z(1)) \\ &\leqslant (z_x, y_x - z_x)_H - (f(\cdot, t), y - z)_H \\ &\quad + \gamma_0(z(0))(y(0) - z(0)) + \gamma_1(z(1))(y(1) - z(1)) \\ &\leqslant \phi^t(y) - \phi^t(z),\end{aligned}$$

so that $z^* \in \partial\phi^t(z)$. Q.E.D.

In view of Lemma 5 it is easily understood that problem (CP; u_0) is a variational formulation of problem (4.7)–(4.10).

LEMMA 6. *Let $s, t \in [0, \infty)$ and $z \in K(s)$, and let θ_i, $i=0, 1$, be numbers in $[0, 1]$ satisfying*

$$z(i)=\theta_i g_i(s)+(1-\theta_i) h_i(s).$$

Then the function

$$\tilde{z}(x)=z(x)+(1-x)\{\theta_0(g_0(t)-g_0(s))+(1-\theta_0)(h_0(t)-h_0(s))\}$$
$$+x\{\theta_1(g_1(t)-g_1(s))+(1-\theta_1)(h_1(t)-h_1(s))\}$$

belongs to $K(t)$ and satisfies

$$|\tilde{z}-z|_H \leqslant |g_0(t)-g_0(s)|+|h_0(t)-h_0(s)|$$
$$+|g_1(t)-g_1(s)|+|h_1(t)-h_1(s)| \tag{4.16}$$

and

$$|\phi^t(\tilde{z})-\phi^s(z)| \leqslant C\{|g_0(t)-g_0(s)|+|h_0(t)-h_0(s)|+|g_1(t)-g_1(s)|$$
$$+|h_1(t)-h_1(s)|+|f_0(t)-f_0(s)|\}(1+|\phi^s(z)|), \tag{4.17}$$

where C is a positive constant independent of s, t and z. Hence $\{\phi^t\}$ satisfies condition (A).

Proof. Clearly $\tilde{z} \in K(t)$ and (4.16) holds. For simplicity we denote by $G(t, s)$ the right-hand side of (4.16). By an elementary computation we get

$$||\tilde{z}_x|_H^2-|z_x|_H^2| \leqslant C'G(t, s)(1+|z_x|_H) \tag{4.18}$$

with a constant $C'>0$ independent of s, t and z. Also, putting

$$L_i=\sup\{|\gamma_i(r)|;\, g_i(t) \leqslant r \leqslant h_i(t), 0 \leqslant t<\infty\}, \qquad i=0, 1,$$

we observe that

$$|\Gamma_i(\tilde{z}(i))-\Gamma_i(z(i))|$$
$$=|\Gamma_i(\theta_i g_i(t)+(1-\theta_i) h_i(t))-\Gamma_i(\theta_i g_i(s)+(1-\theta_i) h_i(s))|$$
$$\leqslant L_i\{|g_i(t)-g_i(s)|+|h_i(t)-h_i(s)|\}, \qquad i=0, 1. \tag{4.19}$$

Besides we have

$$|(f(\cdot, t), \tilde{z})_H-(f(\cdot, s), z)_H|$$
$$\leqslant |f(\cdot, t)|_H |\tilde{z}-z|_H+|f(\cdot, t)-f(\cdot, s)|_H |z|_H$$
$$\leqslant C''\{G(t, s)+|f_0(t)-f_0(s)|\, |z|_H\} \tag{4.20}$$

with a constant C'' independent of s, t and z. From (4.18)–(4.20) it follows that (4.17) holds with a constant $C>0$ independent of s, t and z. Finally, if we take

$$a_r(t)=b_r(t)=(1+C)\int_0^t \{|g_0'(\tau)|+|h_0'(\tau)| + |g_1'(\tau)|+|h_1'(\tau)|+|f_0'(\tau)|\}\,d\tau,$$

then condition (A) is satisfied. Q.E.D.

LEMMA 7. *$\{\phi^t\}$ satisfies condition* (B)$'$.

Proof. It is easy to verify (b1). Now, let z be any point of K_∞, and let

$$z(i)=\theta_i g_{i,\infty}+(1-\theta_i)\,h_{i,\infty}, \qquad 0\leqslant\theta_i\leqslant 1,\ i=0,1.$$

We put $w(x,t)=z(x)+(1-x)\{\theta_0(g_0(t)-g_{0,\infty})+(1-\theta_0)(h_0(t)-h_{0,\infty})\}+x\{\theta_1(g_1(t)-g_{1,\infty})+(1-\theta_i)(h_1(t)-h_{1,\infty})\}$. Clearly $w(\cdot,t)\in K(t)$ and $w(x,t)=w(x,s)+(1-x)\{\theta_0(g_0(t)-g_0(s))+(1-\theta_0)(h_0(t)-h_0(s))\}+x\{\theta_1(g_1(t)-g_1(s))+(1-\theta_1)(h_1(t)-h_1(s))\}$ for any s, $t\in[0,\infty)$. Therefore, just as in the proof of Lemma 3, by making use of Lemma 6 we can show that the functions w: $[0,\infty)\to H$,

$$\alpha(t)\equiv|g_0(t)-g_{0,\infty}|+|h_0(t)-h_{0,\infty}|+|g_1(t)-g_{1,\infty}|+|h_1(t)-h_{1,\infty}|$$

and

$$\beta(t)\equiv C(\alpha(t)+|f_0(t)|)(1+|\phi^\infty(z)|)$$

have the required properties in (b2)$'$. Q.E.D.

LEMMA 8.

$$\phi^t(z)+C\{|g_0(t)-g_{0,\infty}|+|h_0(t)-h_{0,\infty}|+|g_1(t)-g_{1,\infty}| + |h_1(t)-h_{1,\infty}|+|f_0(t)|\}(1+|\phi^t(z)|)\geqslant \min\phi^\infty$$

for all $t\geqslant 0$ and $z\in H$.

Proof. The inequality of the lemma can be derived by making use of Lemma 6, just as that of Lemma 4. Q.E.D.

By virtue of Lemmas 6–8 we can apply our abstract results to obtain the following proposition.

PROPOSITION 3. *Let u_0 be in $K(0)(=D(\phi^0))$ and u be the solution to*

(CP; u_0). *Then* $u(\cdot, t)$ *converges strongly in* X *as* $t \to \infty$ *and the limit* u_∞ *is a solution of the system*

$$-u_{\infty,xx} = f_\infty \qquad \text{on } (0, 1),$$

$$\begin{cases} g_{0,\infty} \leqslant u_\infty(0) \leqslant h_{0,\infty}, & \\ u_{\infty,x}(0+) \leqslant \gamma_0(g_{0,\infty}) & \text{if } u_\infty(0) = g_{0,\infty}, \\ u_{\infty,x}(0+) = \gamma_0(u_\infty(0)) & \text{if } g_{0,\infty} < u_\infty(0) < h_{0,\infty}, \\ u_{\infty,x}(0+) \geqslant \gamma_0(h_{0,\infty}) & \text{if } u_\infty(0) = h_{0,\infty}, \end{cases}$$

$$\begin{cases} g_{1,\infty} \leqslant u_\infty(1) \leqslant h_{1,\infty}, & \\ -u_{\infty,x}(1-) \leqslant \gamma_1(g_{1,\infty}) & \text{if } u_\infty(1) = g_{1,\infty}, \\ -u_{\infty,x}(1-) = \gamma_1(u_\infty(1)) & \text{if } g_{1,\infty} < u_\infty(1) < h_{1,\infty}, \\ -u_{\infty,x}(1-) \geqslant \gamma_1(h_{1,\infty}) & \text{if } u_\infty(1) = h_{1,\infty}. \end{cases}$$

5. A Generalization

In this section we give a generalization of Theorem 1 to the case of Cauchy problem

$$(\text{CP}; B, u_0) \qquad -u'(t) \in \partial\phi^t(Bu(t)), \qquad 0 < t < \infty,\ u(0) = u_0,$$

where B is a single-valued operator from $D(B) = H$ into itself such that

$$(Bz - Bz_1, z - z_1)_H \geqslant M\,|Bz - Bz_1|_H^2,$$
$$|Bz - Bz_1|_H \geqslant M'\,|z - z_1|_H \qquad \text{for } z, z_1 \in H,$$

for positive constants M, M', and B is supposed to be the subdifferential of a finite continuous convex function on H. By a result of [10] (or [11, Chap. 2, Sect. 2.8]), under conditions (A) and (C) in Corollary 2 to Theorem 1, (CP; B, u_0) admits at least one solution u for each $u_0 \in H$ with Bu_0 in $D(\phi^0)$; by a solution of (CP; B, u_0) we mean a function u: $[0, \infty) \to H$ such that $u \in W^{1,2}(0, T; H)$ for every $0 < T < \infty$, $u(0) = u_0$, the function $t \to \phi^t(Bu(t))$ is locally bounded on $[0, \infty)$ and $-u'(t) \in \partial\phi^t(Bu(t))$ for a.e. $t \in [0, \infty)$.

Proposition 4. *Suppose* (A), (B) *and* (C) *hold, and let* $u_0 \in H$ *with* $Bu_0 \in D(\phi^0)$. *Let* u *be a solution to* (CP; B, u_0) *which is bounded on* $[0, \infty)$. *Then we have*:

(i) $\lim_{t\to\infty} \phi^t(Bu(t)) = \min \phi^\infty$.

(ii) *The set of all strong cluster points of $Bu(t)$ as $t \to \infty$ is not empty and contained in $F(\phi^\infty)$.*

In this case, instead of (2.1), we have the following type of energy inequality (cf. [10] or [11, Chap. 2, Sect. 2.8]) for a solution u to (CP; B, u_0): if $|Bu(t))|_H < r$ for $0 \leqslant t \leqslant T$ $(<\infty)$, then

$$\phi^t(Bu(t)) - \phi^s(Bu(s)) + \frac{M}{2} \int_s^t \left| \frac{d}{d\tau} Bu(\tau) \right|_H^2 d\tau$$

$$\leqslant \int_s^t \{\tilde{k}_{r,1}(\tau)\, \phi^\tau(Bu(\tau)) + \tilde{k}_{r,2}(\tau)\}\, d\tau \tag{5.1}$$

for every $0 \leqslant s \leqslant t \leqslant T$, where

$$\tilde{k}_{r,1}(\tau) = \frac{4}{MM'} |a'_r(\tau)|^2 + |b'_r(\tau)|, \qquad \tilde{k}_{r,2}(\tau) = \tilde{k}_{r,1}(\tau)\{1 + 4c_0(r+1)\}$$

with the same constant c_0 of condition (D). By using (5.1) we can prove Proposition 4 just as Theorem 1 and its Corollary 2.

Acknowledgments

The authors wish to thank the referee for his helpful comments. Also, the authors are indebted to him for informing them of the papers [1, 2, 6–8, 14].

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