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Prime decompositions of knots in $T^2 \times I$

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ABSTRACT

The famous H. Schubert theorem (1949) states that any nontrivial knot in S^3 admits a decomposition into connected sum of prime factors, which are unique up to order. We prove a similar result for knots in $T \times I$, where T is a two-dimensional torus. However, we only consider knots of geometric degree one, use a different type of connected summation, and take into account the order of prime factors.

1. Introduction

Let T be a two-dimensional torus and $I = [0, 1]$. By a *thick torus* we mean a 3-manifold homeomorphic to the product $T \times I$ equipped with a fixed orientation.

Definition 1. A knot in $T \times I$ is an oriented simple closed curve $K \subset \mathrm{Int}(T \times I)$. Two knots $K_i \subset T_i \times I$, $i = 1, 2$, are *equivalent* if there is a homeomorphism of pairs $h : (T_1 \times I, K_1) \to (T_2 \times I, K_2)$ which takes $T_1 \times \{0\}$ to $T_2 \times \{0\}$ and preserves orientations of the thick tori and knots.

Definition 2. Let $K \subset T \times I$ be a knot. A proper annulus $A \subset T \times I$ is called *vertical* if it is isotopic to an annulus of the type $c \times I$, where c is a nontrivial simple closed curve in T. A vertical annulus $A \subset T \times I$ is *admissible* (with respect to K) if K intersects A transversally at one point. By a vertical *multi-annulus* in $T \times I$ we mean the disjoint union of $n \geqslant 1$ vertical annuli. $\mathbb{A}$ is admissible if so are all A_i.

Definition 3. We shall say that a knot $K \subset T \times I$ is *of degree one* if $T \times I$ contains an admissible annulus.

Let $K_i \subset T_i \times I$, $0 \leqslant i \leqslant n-1$, be a collection of $n \geqslant 2$ degree one knots in thick tori. Choose admissible annuli $A_i \subset T_i \times I$. For each i we cut $T_i \times I$ along A_i and get a *thick annulus* $M_i \approx A_i \times [0, 1]$ with two copies $A_i' = A_i \times \{0\}$, $A_i'' = A_i \times \{1\}$ of A_i in ∂M_i. The annuli are joined by the oriented arc $l_i = K \cap M_i$. We assume that the initial and terminal points of l_i lie in A_i' and A_i'' respectively. For each i choose a homeomorphism $h_i : A_i'' \to A_{i+1}'$ which reverses the induced orientations of the annuli, takes $A_i'' \cap (T \times \{0\})$ to $A_i' \cap (T \times \{0\})$, and takes the terminal point of l_i to the initial point of l_{i+1} (indices are taken modulo n).

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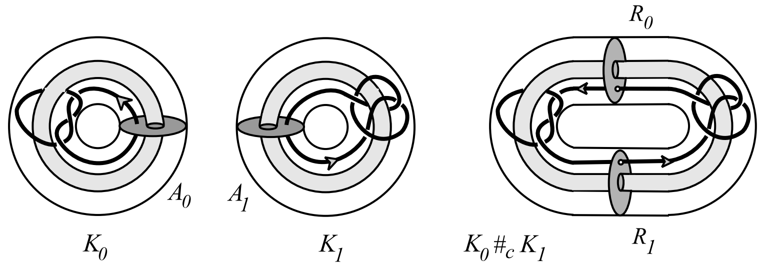

Fig. 1. Circular connected sum.

Definition 4. The knot $K = K_0\#_cK_1\#_c\dots\#_cK_{n-1} \subset T \times I$ obtained by gluing together the pairs (M_i, l_i) along h_i is called a *circular connected sum* of K_i. Admissible annuli in $T \times I$ obtained by identifying A_i'' with A_{i+1}' are denoted R_i, $0 \leqslant i \leqslant n-1$. See Fig. 1 for $n = 2$.

The circular connected sum of degree one knots may depend on the choice of the annuli $A_i \subset T_i \times I$ used for the construction. However, if A_i are fixed, then K and the admissible multi-annulus $\mathbb{R} = R_0 \cup R_1 \cup \dots \cup R_{n-1}$ are uniquely determined. In turn, K and $\mathbb{R}$ determine K_i and A_i. Suppose we are considering a circular connected sum $K_0\#_cK_1$ of two knots such that one of them is *horizontal* (i.e., isotopic to a simple closed curve in a middle torus $T_i \times \{*\}$). Then the sum $K_0\#_cK_1$ is equivalent to the second knot. Such a summation is called *trivial*.

Definition 5. A nonhorizontal degree one knot $K \subset T \times I$ is called *prime* if it cannot be represented as a nontrivial circular connected sum of two other knots.

Let K be a degree one knot in $T \times I$. Suppose there is a 3-ball $B \subset T \times I$ such that $l = K \cap B$ is a knotted arc in B. Replacing l by an unknotted arc $l_1 \subset B$ with the same endpoints, we get a new degree one knot $K_1 \subset T \times I$.

Definition 6. We shall say that K_1 is obtained from K by *cutting off a local knot* and that K is obtained from K_1 by *inserting a local knot*.

Note that the exact place for the insertion, i.e., a ball $B \subset T \times I$ such that $l_1 = K_1 \cap B$ is an unknotted arc, is not important. Indeed, B can be moved by an isotopy of pairs $h_t : (T \times I, K_1) \to (T \times I, K_1)$ to any other position along K_1. This fact is well known in the classical knot theory, where it is used for proving commutativity of the connected sum operation.

Definition 7. A nonhorizontal degree one knot K in $T \times I$ is called *essential* if it does not contain local knots. K is called *almost horizontal* if it can be obtained from a horizontal knot in $T \times I$ by inserting local knots.

One can easily see that inserting local knots is equivalent to taking circular connected sums with the corresponding almost horizontal knots. It follows that almost horizontal summands of a circular connected sum can be shifted to any position, for example, one may write them at the end of the sum.

Theorem 1. *Any nonhorizontal degree one knot K can be represented as a circular connected sum*

$$K = K_0\#_cK_1\#_c\dots\#_cK_{n-1}\#_cL_0\#_cL_1\#_c\dots\#_cL_{m-1},$$

where K_i are essential and L_j are almost horizontal prime knots. The summands K_i are uniquely determined up to cyclic permutation while the summands L_j are uniquely determined up to any permutation.

For knotted theta-curves in S^3 and in arbitrary 3-manifolds similar prime decomposition theorems, which take into account the order of prime factors, can be found in [4,2].

2. Properties of admissible annuli

Definition 8. Let $K \subset T \times I$ be a degree one knot and let $\mathbb{R} = R_0 \cup R_1 \cup \dots \cup R_{n-1} \subset T \times I$, $n \geqslant 2$, be an admissible multi-annulus. Then a vertical multi-annulus $\mathbb{C} \subset T \times I$ is called *tight* (with respect to $\mathbb{R}$) if either $\mathbb{C} \cap \mathbb{R} = \emptyset$ or the following conditions hold:

1. $\mathbb{C} \cap \mathbb{R}$ consists of radial arcs of the annuli.
2. These arcs decompose $\mathbb{C}$ into strips (embedded rectangles) such that the lateral sides of each strip lie in different annuli of $\mathbb{R}$.

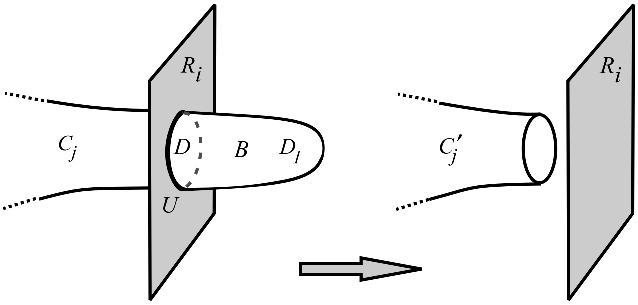

Fig. 2. Removing trivial circles.

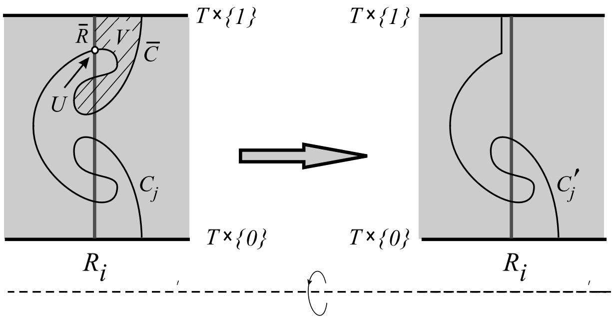

Fig. 3. Removing nontrivial circles.

Definition 9. Let $K \subset T \times I$ be a degree one knot and let $\mathbb{C} \subset T \times I$ be a vertical multi-annulus such that K intersects $\mathbb{C}$ transversally. Then the *weight* $w(\mathbb{C})$ of $\mathbb{C}$ is the number of points in $K \cap \mathbb{C}$.

Lemma 1. *Let $K \subset T \times I$ be a degree one knot and let $\mathbb{R} = R_0 \cup R_1 \cup \cdots \cup R_{n-1}$, $n \geqslant 2$, be an admissible multi-annulus. Then for any vertical multi-annulus $\mathbb{C} \subset T \times I$ there is an isotopy $h_t : T \times I \to T \times I$, $0 \leqslant t \leqslant 1$, such that $h_0(\mathbb{C}) = \mathbb{C}$, the multi-annulus $\mathbb{C}' = h_1(\mathbb{C})$ is tight, and $w(\mathbb{C}) \geqslant w(\mathbb{C}')$. Moreover, if K is essential and $\mathbb{C}$ is admissible, h_t may be chosen so as to be invariant on K, i.e., $h_t(K) = K$ for all t.*

Proof. We may assume that $\mathbb{C}$ and $\mathbb{R}$ are in general position. Then any connected component of $\mathbb{C} \cap \mathbb{R}$ is one of the following curves: a trivial circle, a trivial arc, a nontrivial circle, or a radial arc. Our goal is to remove all curves of the first three types and some curves of the last type.

Step 1. Suppose $\mathbb{C} \cap \mathbb{R}$ contains a trivial circle $U \subset R_i$. Using an innermost disc argument, we may assume that U bounds a disc $D \subset R_i \subset \mathbb{R}$ such that $D \cap \mathbb{C} = U$. Denote by D_1 the disc bounded by U in $C_j \subset \mathbb{C}$. Then $D \cup D_1$ is a sphere bounding a ball $B \subset T \times I$. We use B for constructing an isotopy h_t which moves D_1 to the other side of R_i and $\mathbb{C}$ to a new multi-annulus $\mathbb{C}'$, thus annihilating U and maybe some other circles in $\mathbb{C} \cap \mathbb{R}$ (see Fig. 2). Since R_i is admissible, $K \cap D$ is either empty or consists of one point. In the latter case $K \cap D_1 \neq \emptyset$. It follows that in both cases $w(\mathbb{C}') \leqslant w(\mathbb{C})$.

Suppose $\mathbb{C}$ is admissible. Then either $l = K \cap B$ is empty or l is an arc. If K is essential, then l is unknotted. Therefore h_t may be chosen so as to be invariant on K. Further on we will assume that K contains no trivial circles.

Step 2. All trivial arcs in $\mathbb{C} \cap \mathbb{R}$ can be removed just in the same way as above, using an outermost arc argument and half-discs bounded by trivial arcs and arcs in $\partial(T \times I)$ instead of discs. Further we assume that $\mathbb{C} \cap \mathbb{R}$ contains no trivial arc.

Step 3. Suppose that $\mathbb{C}$ intersects an annulus R_i of $\mathbb{R}$ along nontrivial circles, which decompose it into smaller annuli. Then R_i contains two outermost annuli, each bounded by a circle in $\mathbb{C} \cap R_i$ and a circle in $\partial(T \times I)$. Since R_i is admissible, at least one of them (denote it $\bar{R}$) has no common points with K. The circle $U = \bar{R} \cap \mathbb{C}$ cuts off an annulus $\bar{C} \subset C_j \subset \mathbb{C}$ having a boundary circle in the same torus of $\partial(T \times I)$ as $\bar{R}$. Then $\bar{R} \cup \bar{C}$ together with an annulus in $\partial(T \times I)$ bound in $T \times I$ a solid torus V. We use V for constructing an isotopy $h_t : T \times I \to T \times I$ which moves $\bar{C}$ to the other side of $\bar{R}$ and $\mathbb{C}$ to a new multi-annulus $\mathbb{C}'$, thus annihilating U and maybe some other circles in $\mathbb{C} \cap \mathbb{R}$. Clearly $w(\mathbb{C}') \leqslant w(\mathbb{C})$. If $\mathbb{C}$ is admissible, then $V \cap K = \emptyset$, since $\bar{R} \cap K = \emptyset$. Therefore we may construct h_t such that it keeps K fixed (see Fig. 3). In order to get a 3-dimensional illustration, rotate it around the axis shown at the bottom of the figure.

Step 4. Suppose that $\mathbb{C} \cap \mathbb{R}$ consists of radial arcs. They decompose $\mathbb{C}$ and $\mathbb{R}$ into strips. If $\mathbb{C}$ is not tight, then there are strips $P \subset C_j \subset \mathbb{C}$ and $Q \subset R_i \subset \mathbb{R}$ such that they have common lateral sides and $P \cup Q$ cuts off a 3-ball B from $T \times I$. We use B for constructing an isotopy of $T \times I$ which moves P to the other side of R_i and C_j to a new annulus C'_j, thus annihilating two or more radial arcs of $\mathbb{C} \cap \mathbb{R}$, see Fig. 4. Clearly $w(\mathbb{C}') \leqslant w(\mathbb{C})$. If K is essential and $\mathbb{C}$ is admissible, we use the same argument as in Step 1 for constructing an isotopy which is invariant on K. $\square$

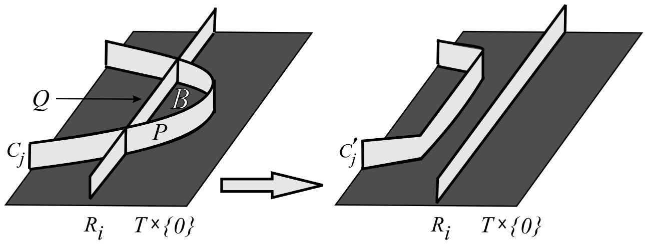

Fig. 4. Removing radial arcs.

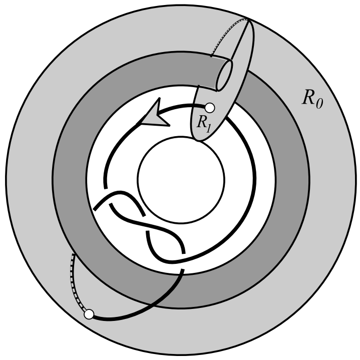

Fig. 5. Two nonisotopic admissible annuli.

Let $K \subset T \times I$ be a degree one knot and let $R, R' \subset T \times I$ be disjoint admissible annuli. They decompose $T \times I$ into two parts $M_i \approx R \times [0, 1]$, $i = 1, 2$. We shall say that R, R' are *parallel in* $(T \times I, K)$ if for at least one i the arc $K \cap M_i$ is trivial in M_i, i.e., has the form $\{*\} \times [0, 1]$.

Lemma 2. *Let $K \subset T \times I$ be a degree one essential knot and let $\mathbb{R} = R_0 \cup R_1 \cup \cdots \cup R_{n-1}$, $n \geqslant 2$, be an admissible multi-annulus such that at least two annuli of $\mathbb{R}$ are not parallel in $(T \times I, K)$. Then for any admissible multi-annulus $\mathbb{C} \subset T \times I$ there is an isotopy $h_t : T \times I \to T \times I$, $0 \leqslant t \leqslant 1$, such that $h_0(\mathbb{C}) = \mathbb{C}$, h_t is invariant on K, and the multi-annulus $\mathbb{C}' = h_1(\mathbb{C})$ is disjoint with $\mathbb{R}$.*

Proof. By Lemma 1 we may assume that $\mathbb{C}$ is tight. We claim that $\mathbb{C} \cap \mathbb{R} = \emptyset$. On the contrary, assume that $\mathbb{C}$ intersects $\mathbb{R}$. Then $\mathbb{C}$ consists of strips such that each strip P joins two neighboring annuli $R_i, R_{i+1} \subset \mathbb{R}$ and lies in the thick annulus M_i between them. Note that P cuts M_i into a ball. If $K \cap P = \emptyset$, then the arc $l_i = K \cap M_i$ is contained in this ball. Since K is essential, l_i is unknotted. Thus R_i, R_{i+1} are parallel and the pair (M_i, l_i) is trivial, i.e., homeomorphic to $(R_i \times [0, 1], \{*\} \times [0, 1])$.

Recall that $\mathbb{C}$ is admissible. It follows that K intersects only one strip. Therefore, only one thick annulus between neighboring annuli may be nontrivial, but then its complement in $T \times I$ consists of trivial regions and thus is also trivial. This contradicts our assumption that $\mathbb{R}$ contains nonparallel annuli. □

Remark 1. Suppose that an essential knot $K \subset T \times I$ is nonprime. Then any two admissible annuli in $T \times I$ are isotopic in $T \times I$. Indeed, since K is nonprime, $T \times I$ contains a pair of disjoint admissible annuli R', R'' which are not parallel in $(T \times I, K)$. In fact we can take any pair of annuli decomposing K into a nontrivial circular connected sum. Let $C \subset T \times I$ be another admissible annulus. By Lemma 2 it is isotopic in $(T \times I, K)$ to an annulus which is disjoint with R' and thus is isotopic to R'. Note that the assumption that K be nonprime is essential. See Fig. 5 for a knot having two nonisotopic admissible annuli R_0, R_1.

Lemma 3. *Let $K \subset T \times I$ be an essential knot and let $\mathbb{R} = R_0 \cup R_1 \cup \cdots \cup R_{n-1}$, $n \geqslant 2$, be an admissible multi-annulus in $T \times I$ such that no two annuli of $\mathbb{R}$ are parallel in $(T \times I, K)$. Suppose that an annulus $C \subset T \times I$ intersects K transversally and the base circles $C \cap (T \times \{0\})$ and $\mathbb{R} \cap (T \times \{0\})$ of C and $\mathbb{R}$ are not homotopic in $T \times \{0\}$. Then $n \leqslant w(C)$.*

Proof. By Lemma 1 we may transform C into a tight position without increasing its weight. Since the base circles of C and $\mathbb{R}$ are not homotopic, $C \cap \mathbb{R}$ is a nonempty collection of radial arc, which decompose C and $\mathbb{R}$ into strips. Note that K must intersect each strip of C in any region of $T \times I$ between two neighboring annuli. Otherwise the region would be trivial and the annuli parallel. Since any region contains at least one strip intersecting K, we may conclude that $n \leqslant w(\mathbb{C})$. □

3. Proof of the main theorem

Let a nonhorizontal degree one knot K be given. First we cut off all local knots. By [3] and Theorem 7 of [1], any knot K in a 3-manifold without nonseparating 2-spheres contains only finitely many local knots, which are uniquely determined by K. Therefore the set of almost horizontal summands L_j of K is finite and these summands are unique up to order. Further on we shall assume that K does not contain local knots, i.e., is essential.

Let us prove that a prime decomposition of K does exist. If K is prime, we are done. Suppose K is not prime. Then among all decompositions of K into circular connected sums we take a decomposition $K = K_0 \#_c K_1 \#_c \ldots \#_c K_{n-1}$ having the maximal number n of summands. Clearly $n \geqslant 2$.

We claim that all K_i are prime. Let $\mathbb{R} = R_0 \cup R_1 \cup \cdots \cup R_{n-1} \subset T \times I$ be the admissible multi-annulus corresponding to that decomposition. The annuli of $\mathbb{R}$ split $T \times I$ into thick annuli $M_i \approx R_i \times [0,1]$. On the contrary, suppose that for some i the knot $K_i \subset T_i \times I$ is not prime. Then $T_i \times I$ contains a pair of disjoint admissible annuli R', R'' such that they are not parallel in $(T_i \times I, K_i)$. Consider the admissible multi-annulus $R' \cup R'' \subset T_i \times I$ and the annulus $A_i \subset T_i \times I$ used for constructing the circular connected sum. According to Lemma 2 we may assume that R', R'' are disjoint with A_i and thus can be considered as annuli in the thick annulus M_i between R_i and R_{i+1}. Since R', R'' are not parallel in $(T_i \times I, K_i)$, at least one of them is not parallel to R_i or R_{i+1}. This contradicts our assumption that n is maximal.

Let us prove that the summands of a prime decomposition of K into a circular connected sum are unique up to cyclic permutation. Let $K = K_0 \#_c K_1 \#_c \ldots \#_c K_{n-1}$, $K = K'_0 \#_c K'_1 \#_c \ldots \#_c K'_{m-1}$ be two representations and let $\mathbb{R} = R_0 \cup R_1 \cup \cdots \cup R_{n-1}$, $\mathbb{R}' = R'_0 \cup R'_1 \cup \cdots \cup R'_{m-1}$ be the corresponding admissible multi-annuli in $T \times I$. By Lemma 2 we may assume that $\mathbb{R} \cap \mathbb{R}' = \emptyset$. It follows that any annulus R'_j lies in a thick annulus M_i between two neighboring annuli R_i and R_{i+1}. Since K_i is prime, R'_j must be parallel to exactly one of them. Similarly, any annulus of $\mathbb{R}$ is parallel to exactly one annulus of $\mathbb{R}'$. We may conclude that $m = n$ and that after an appropriate isotopic deformation of $\mathbb{R}$ we get $\mathbb{R} = \mathbb{R}'$. Therefore both decompositions have the same set of prime summands. Their orderings are determined by K, so may differ only by a cyclic permutation.

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